Duality in supersymmetric gauge theories

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Abstract

In these lectures we discuss various aspects of gauge theories with extended $N = 2$ and $N = 4$ supersymmetry that are at the basis of recently found exact results. These results include the exact calculation of the low energy effective action for the light degrees of freedom in the $N = 2$ super Yang-Mills theory and the conjecture, supported by some checks, that the $N = 4$ super Yang-Mills theory is dual in the sense of Montonen-Olive.

1 Introduction

In the last few years a number of exact results have been obtained both in four-dimensional supersymmetric gauge theories and in supersymmetric string theories in various dimensions. They are all based on one hand on supersymmetry and on the other hand on the exploitation of a duality symmetry that is the generalization of the duality present in free electromagnetism and that allows one to relate theories, which at the first sight look very different, in much the same way as in two dimensions Sine-Gordon is related to the Thirring model.

In these lectures we will not discuss string theories, but we will limit ourselves to the study of supersymmetric Yang-Mills theories with extended supersymmetry $N = 2$ and $N = 4$. In particular we will discuss in some detail the background that is needed for obtaining the exact results found by Seiberg and Witten [1] for $N = 2$ supersymmetric Yang-Mills theory, that will be also shortly reviewed, and for formulating the Montonen-Olive [2] duality conjectured for the $N = 4$ super Yang-Mills theory.

Several recent reviews [3, 4, 5] have appeared in the last year. In particular the beautiful review by David Olive [3] has strongly inspired these lectures.

The plan of the lectures is as follows.

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In section (2) we will discuss duality in electromagnetism and the Dirac quantization condition. In section (3) the ’t Hooft-Polyakov magnetic monopole and the Julia-Zee dyon solutions in the Georgi-Glashow model are discussed in some detail. These classical solutions correspond to particles in the quantum theory. This implies that the particle spectrum does not only contain the particles corresponding to the fields present in the classical Lagrangian and having a constant or small mass in the weak coupling limit, but also contains other particles corresponding to the soliton solutions and having a big mass in the weak coupling limit. Those additional particles are also required by certain duality properties that extend the duality of electromagnetism discussed in section (2). The discussion of the soliton solutions in sect. (3) brings us in section (4) to the Montonen-Olive duality conjecture. After its formulation it became very soon clear that this duality property cannot be satisfied in the Georgi-Glashow model where the quantum corrections invalidate the conclusions based on semiclassical considerations. The theories that have a chance to realize it were those in which the semiclassical properties are not destroyed by quantum corrections and those are the supersymmetric gauge theories. That is why in sect. (5), as an introduction to them, we discuss the representations of supersymmetry algebra with and without central charges and in sect. (6) we construct the supersymmetric Yang-Mills actions from dimensional reduction from $D = 10$. In sect. (7) we present the semiclassical analysis of the $N = 2$ super Yang-Mills theory, that can also be easily extended to the $N = 4$ case, showing that this theory has magnetic monopoles and dyon solutions as the Georgi-Glashow model discussed in sect. (3) and presents all the main features that brought to the Montonen-Olive duality conjecture for the Georgi-Glashow model. In particular in the cases with extended supersymmetry the mass formula for the BPS states is not just a property of the classical theory, but is a consequence of the supersymmetry algebra and as long as supersymmetry is not broken, is an exact quantum formula. In section (8) we write the supersymmetry algebra for the case $N = 2$ and we derive from it an expression for the mass of the BPS states. From the previous considerations it appears that both the $N = 2$ and $N = 4$ super Yang-Mills theories are perfectly good candidates for the realization of the Montonen-Olive duality conjecture. Actually it turns out that the $N = 2$ theory cannot verify it. We discuss this issue in sect. (9) where we bring various arguments that select the $N = 4$ theory as the best candidate for the Montonen-Olive duality conjecture. In sect. (10) we discuss the Schwinger-Zwanziger quantization condition and from it and other very general assumptions we show that the electric and magnetic charges of the physical states must lie on a two-dimensional lattice. Then, using the mass formula for the BPS-states, we give a characterization of the single particle stable states. In sect. (11), following very closely Ref. [3], we riformulate the Montonen-Olive duality conjecture adapted to the $N = 4$ super Yang-Mills theory and we show that the various formulations are related to each other by the action of the modular group $SL(2, Z)$. The last three sections provide a short summary of the beautiful paper by Seiberg and Witten [1] where the low energy effective action for the light degrees of freedom is constructed.
In particular in sect. (12) we describe the global parametrization of the moduli space, in sect (13) we discuss its singularities and finally in sect. (14) we give the explicit solution found in Ref. [1].

## 2 Electromagnetic duality

The free Maxwell equations

\[
\nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0 \\
\n\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \nabla \wedge \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0
\]

are not only invariant under Lorentz and conformal transformations. They are also invariant under a duality transformation:

\[
\vec{E} \rightarrow \cos \phi \vec{E} - \sin \phi \vec{B} \\
\vec{B} \rightarrow \cos \phi \vec{B} + \sin \phi \vec{E}
\]

In particular if we take \( \phi = -\pi/2 \) one obtains from eq. (2) a discrete duality transformation:

\[
\vec{E} \rightarrow \vec{B} \quad \vec{B} \rightarrow -\vec{E}
\]

that is generated by the duality matrix acting on the two-vector consisting of the electric and magnetic fields as

\[
\begin{pmatrix}
\vec{E} \\
\vec{B}
\end{pmatrix} \rightarrow 
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\vec{E} \\
\vec{B}
\end{pmatrix}
\]

In terms of the complex vector \( \vec{E} + i\vec{B} \) the duality transformation in eq. (2) becomes

\[
\vec{E} + i\vec{B} \rightarrow e^{i\phi} (\vec{E} + i\vec{B})
\]

Notice that the energy and momentum density of the electromagnetic field given respectively by

\[
\frac{1}{2} |\vec{E} + i\vec{B}|^2 = \frac{1}{2} (\vec{E}^2 + \vec{B}^2)
\]

and

\[
\frac{1}{2i} (\vec{E} + i\vec{B})^* \wedge (\vec{E} + i\vec{B}) = \vec{E} \wedge \vec{B}
\]

are invariant under the duality transformation in eq. (2), while the real and imaginary part of

\[
\frac{1}{2} (\vec{E} + i\vec{B})^2 = \frac{1}{2} (\vec{E}^2 - \vec{B}^2) + i\vec{E} \cdot \vec{B}
\]

that are respectively the Lagrangian of the electromagnetic field and the topological charge density, transform as a doublet under the duality group.
If we perform a discrete duality transformations twice, we get

$$(\vec{E}, \vec{B}) \rightarrow (-\vec{E}, -\vec{B}) \quad (9)$$

that corresponds to the charge conjugation operation.

The reason why this beautiful duality property of the free electromagnetic field is not even mentioned in the courses on electromagnetism is due to the fact that it is lost when we introduce the interaction of the electromagnetic field with matter by just adding in the right hand side of the Maxwell equations an electric current $\vec{j}_e$ and an electric charge density $\rho_e$. If we want to keep duality we must also introduce a magnetic current $\vec{j}_m$ and a magnetic charge density $\rho_m$ together with their electric counterparts. If we do so the Maxwell equations given in eq. (1) and written in complex notations become:

$$\vec{\nabla} \cdot (\vec{E} + i\vec{B}) = \rho_e + i\rho_m \quad (10)$$

and

$$\vec{\nabla} \wedge (\vec{E} + i\vec{B}) = i \frac{\partial}{\partial t}(\vec{E} + i\vec{B}) + j_e + i j_m \quad (11)$$

The previous equations are invariant under the duality transformation given in eq. (5) if the electric and magnetic currents and densities transform as

$$\rho_e + i\rho_m \rightarrow e^{i\phi} (\rho_e + i\rho_m) \quad (12)$$

and

$$j_e + i j_m \rightarrow e^{i\phi} (j_e + i j_m) \quad (13)$$

In particular if we have only pointlike particles with both electric and magnetic charge, then duality implies the following transformation:

$$q + ig \rightarrow e^{i\phi} (q + ig) \quad (14)$$

Particles with magnetic charge are not introduced in usual electromagnetism for the very simple reason that they are not observed in the experiments. If we include them we must either think that their mass is higher than the presently available energy or find other reasons for their absence. However, if we insist in preserving duality also in the presence of interaction, as shown by Dirac [6], a theory with both electric and magnetic charges $q_i$ and $g_j$ can be consistently quantized only if the Dirac quantization condition is satisfied

$$q_i g_j = 2\pi \hbar n_{ij} \quad (15)$$

where $n_{ij}$ are arbitrary integers. A complete discussion of the Dirac quantization condition can be found in the beautiful review by Goddard and Olive [7].

The Dirac quantization condition is clearly not invariant under the duality transformation in eq. (14). It is only invariant under the discrete transformation obtained from eq. (14) for $\phi = -\frac{\pi}{2}$:

$$q \rightarrow g \quad g \rightarrow -q \quad (16)$$
3 The 't Hooft-Polyakov monopole

In this section we will discuss the monopole solution found by 't Hooft [8] and Polyakov [9] in the Georgi-Glashow model. Many details are here omitted. The reader interested in them is recommended to consult Ref. [7].

Let us consider the Georgi-Glashow model

\[
L = -\frac{1}{4} F_{a \mu \nu} F^{a \mu \nu} + \frac{1}{2} (D_\mu \Phi)_a (D^\mu \Phi)_a - V(\Phi)
\]

where

\[
(D^\mu \Phi)_a = \partial^\mu \Phi_a - e\epsilon_{abc} A^\mu_b \Phi_c
\]

and

\[
F_{a \mu \nu} = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - e\epsilon_{abc} A^\mu_b A^\nu_c
\]

The gauge group is SU(2) and the potential \( V \) is equal to

\[
V(\Phi) = \frac{\lambda}{4} \left( \Phi^2 - a^2 \right)^2
\]

The classical equations of motion, that follow from \( L \), are

\[
(D_\nu F^{\mu \nu})_a = -e\epsilon_{abc} \Phi_b (D^\mu \Phi)_c
\]

\[
(D_\mu D^\mu \Phi)_a = -\lambda \Phi_a (\Phi^2 - a^2)
\]

They must be considered together with the Bianchi identity

\[
D_\mu * F^{\mu \nu} = 0 \quad * F^{\mu \nu} \equiv \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma}
\]

\( \epsilon^{\mu \nu \rho \sigma} \) is the antisymmetric Levi Civita tensor with \( \epsilon^{0123} = 1 \).

The energy is given by

\[
E \equiv \int d^3 x \; \theta_{00} = \int d^3 x \left\{ \frac{1}{2} \left[ (B^i_a)^2 + (E^i_a)^2 + \Pi_a^2 + (D^i \Phi)^2 \right] + V(\Phi) \right\}
\]

where

\[
\Pi_a = (D^0 \Phi)_a \quad F^{i0}_a = E^i_a \quad F_{a \ ij} = -\epsilon_{ijk} B^k_a
\]

The energy is positive semi-definite. It vanishes if and only if

\[
F_{a \mu \nu} = (D^\mu \Phi)_a = 0 \quad V(\Phi) = 0
\]

These conditions are satisfied by taking

\[
\Phi_a = a \delta_{a3} \quad A^\mu_a = 0
\]

or equivalently any gauge rotated version of them.
This field configuration corresponds to the vacuum of our model and obviously satisfies the equations of motions and the Bianchi identity in eqs. (21,22,23).

It is easy to see that, if \( a \neq 0 \), the \( SU(2) \) gauge group is broken to \( U(1) \). With the v.e.v of the Higgs field taken along the third direction (\( \Phi_a = a \delta_{a3} \)) the \( U(1) \) gauge field \( A_{3}^{\mu} \) remains massless, while the two charged fields

\[
W_{\pm} = \frac{1}{\sqrt{2}} (A_{1}^{\mu} \pm i A_{2}^{\mu})
\]

get a mass equal to

\[
M_{\pm} = a|q|
\]

where \( q \) is their electric charge. They are charged with respect to the unbroken \( U(1) \) corresponding to the generator of the gauge group \( SU(2) \) that leaves invariant the v.e.v of the scalar field

\[
Q = \frac{e}{a} T_a \Phi_a = e T_3 h
\]

Finally from the Higgs mechanism one gets also a neutral Higgs scalar particle with mass equal to

\[
M_H = \sqrt{2 \lambda} a \ h
\]

\( h \) has been explicitly written in some of the previous formulas, while has been put equal to 1 in most cases.

In addition to the constant vacuum solution of eq. (27) the equations of motion admit also static (time independent) solutions. The simplest of them can be obtained starting with a radially symmetric ansatz:

\[
\Phi_a = \frac{r^a}{e r^2} H(a er) \quad A_a^0 = 0 \quad A_a^i = -\epsilon_{aij} \frac{r^j}{e r^2} [1 - K(a er)]
\]

Inserting this ansatz into the energy one gets:

\[
E = \frac{4 \pi a e}{e} \int_{0}^{\infty} \frac{d\xi}{\xi^2} \left[ \xi^2 \left( \frac{dK}{d\xi} \right)^2 + K^2 H^2 + \frac{1}{2} \left( \xi \frac{dH}{d\xi} - H \right)^2 + \frac{1}{2} (K^2 - 1)^2 + \frac{\lambda}{4 e^2} (H^2 - \xi^2)^2 \right]
\]

where \( \xi \equiv a er \) is a dimensionless quantity. The insertion of the ansatz in the equations of motions (21,22) gives a system of coupled differential equations for the radial functions \( H \) and \( K \):

\[
\xi^2 \frac{d^2 K}{d\xi^2} = KH^2 + K(K^2 - 1)
\]

and

\[
\xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2)
\]
In order to have a finite energy solution one must also impose boundary conditions for both $\xi = 0$ and $\xi \to \infty$. It can be shown that the previous system of equation admits a finite energy solution. However, in general, it is not possible to write it down explicitly unless one takes the parameter $\lambda$ of the potential of the Higgs field equal to 0. This corresponds to the so called Bogomolny [10], Prasad, Sommerfield [11] (BPS) limit. In this limit one obtains:

$$K(\xi) = \frac{\xi}{\sinh \xi}, \quad H(\xi) = \frac{\xi}{\tanh \xi} - 1$$ (36)

In order to have a better understanding of this limiting case let us rewrite the sum of the two terms appearing in the energy density that involve the square of the non abelian magnetic field and the square of the space components of the covariant derivative of the Higgs field as follows

$$\left( B^i_a \right)^2 + \left( D^i \Phi \right)^2_a = \left[ B^i_a \pm \left( D^i \Phi \right)_a \right]^2 \pm 2B^i_a \left( D^i \Phi \right)_a$$ (37)

When we insert it in the energy (see eq. (24)) we see that all terms are positive except the last one in the r.h.s of eq. (37). We get therefore a lower bound for the energy

$$E \geq \pm \int d^3 x B^i_a \left( D^i \Phi \right)_a$$ (38)

that, after a partial integration and the use of the Bianchi identity in eq. (23), becomes

$$E \geq \pm \int d^3 x \partial^i \left[ B^i_a \Phi_a \right]$$ (39)

The equality sign is obtained if and only if the following equations are satisfied:

$$E^i_a = 0 \quad \Pi_a = 0 \quad \lambda = 0 \quad B^i_a \pm \left( D^i \Phi \right)_a = 0$$ (40)

They are first order equations that imply the validity of the second order equations of motion (21,22). It can be checked that, if we insert the ansatz in eq. (32) in the first order equations (40) one obtains the solution in eq. (36).

Inserting the static classical solution into the energy density given by the integrand in the r.h.s of eq. (24), one can see that it is concentrated in a small region around the origin and goes to zero exponentially as $r$ goes to infinity.

In the quantum theory the classical solution corresponds to a new particle of the spectrum that is an extended object (with size $\sim 1/a$) located in the region where the energy density is appreciably different from zero. In this way we see that the Georgi-Glashow model does not contain only perturbative states as a photon, a massless Higgs field in the BPS limit and a couple of charged bosons, all corresponding to the fields present in the Lagrangian in eq. (17) and having either a zero mass or a mass proportional to the gauge coupling constant. It contains also additional particles that are soliton solutions of the classical equations of motion whose mass is instead
proportional to the inverse of the gauge coupling constant as follows from eq. (33) and therefore are very massive in the weak coupling limit (small $\epsilon$).

We want now to show that the soliton solution given by the ansatz in eq. (32) is actually a magnetic monopole with respect to the unbroken $U(1)$ group.

It can be seen that for large enough values of $r$ the following equations are satisfied:

$$D_\mu \Phi = 0 \quad \Phi^2 = a^2$$

(41)

apart from a small exponential correction.

Corrigan et al. [12] have shown that the most general solution of the previous equations corresponds to a vector field given by:

$$A_a^\mu = \frac{1}{a^2 e} \epsilon_{abc} \Phi_b \partial^\mu \Phi_c + \frac{1}{a} \Phi_a B^\mu$$

(42)

where $B^\mu$ is arbitrary. The corresponding field strength is equal to

$$F_a^{\mu\nu} = \frac{1}{a} \Phi_a F^{\mu\nu}$$

(43)

where

$$F^{\mu\nu} = \frac{1}{ae^3} \epsilon_{abc} \Phi_a \partial^\mu \Phi_b \partial^\nu \Phi_c + \partial^\mu B^\nu - \partial^\nu B^\mu$$

(44)

It satisfies the free Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0 \quad \partial_\mu \ast F^{\mu\nu} = 0$$

(45)

if the eqs. of motion in eqs. (21), (22) and (23) are satisfied.

We see that outside the region where the extended particle is located the non abelian field strength is aligned along the direction of the Higgs field $\Phi_a$ and is proportional to an abelian field strength $F^{\mu\nu}$ that can be interpreted as the field strength of the unbroken $U(1)$ electromagnetic.

From eq. (43) we can compute the non abelian magnetic field that is equal to

$$B^i_a = -\frac{1}{2} \epsilon_{ijk} F_a^{\ jk} = -\frac{1}{2ae^3} \Phi_a \epsilon_{ijk} \epsilon^{bcd} \partial^b \Phi^c \partial^d \Phi^d$$

(46)

Inserting it in eq. (39) one gets:

$$E \geq \pm \frac{4\pi a}{e} T$$

(47)

where the topological charge $T$ is given by

$$T = \int d^3 x \ K_0$$

(48)

with

$$K_\mu = \frac{1}{8\pi a^3} \epsilon_{\mu\nu\rho\sigma} \epsilon^{abc} \partial^\nu \Phi^a \partial^\rho \Phi^b \partial^\sigma \Phi^c$$

(49)
We call it topological current because, unlike a Noether current, it is conserved independently from the equations of motion as it can be trivially checked. It can also be seen (see Ref. [7]) that the topological charge $T$ is an integer since it counts the times that the two-sphere, defined by the second equation in eq. (41), is covered when the two-sphere at infinity in space is covered once.

In conclusion we get

$$E \geq \mp ag$$

(50)

where $g$ is the magnetic charge of the soliton solution that is obtained by integrating the equation:

$$\partial^i B^i = \frac{4\pi}{e} K_0$$

that follows from eq. (44). One gets:

$$g = \int d^3x \partial^i B^i = \frac{4\pi}{e} T$$

(52)

In the case of the static solution corresponding to the ansatz in eq. (32) it is easy to see that $T = \mp 1$ in such a way that $E \geq a|g|$.

The value obtained for the magnetic charge $g = \mp \frac{4\pi}{e}$ is consistent with the Dirac quantization condition (with n=2) given in eq. (15)

$$qg = 4\pi \hbar$$

(53)

where $q$ is the charge of the W-boson given in eq. (30) for $T_3 = \pm 1$. The fact that we obtain $n = 2$ is a consequence of the fact that the gauge bosons transform according to the triplet representation of the gauge group $SU(2)$. The value $n = 1$ would have been obtained with matter fields transforming according to the fundamental doublet representation.

The soliton solution following from the ansatz in eq. (32) has a non vanishing magnetic charge, but has zero electric charge. In order to also have a solution with a non vanishing electric charge [13] we must allow for a non vanishing electric potential of the type

$$A^0_a = \frac{\gamma^a}{er^2} J(r)$$

(54)

instead of the vanishing ansatz given in eq. (32). With this ansatz the equations of motion in eqs. (34,35) are modified as follows:

$$\xi^2 \frac{d^2 K}{d\xi^2} = K \left[K^2 + H^2 - J^2 - 1\right]$$

(55)

and

$$\xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2) \quad \xi^2 \frac{d^2 J}{d\xi^2} = 2K^2 J$$

(56)
In the BPS limit where \( \lambda = 0 \) one can obtain an analytical solution given by:

\[
H(\xi) = \cosh \gamma \left[ \frac{\xi}{\tanh \xi} - 1 \right] \quad K(\xi) = \frac{\xi}{\sinh \xi} \quad (57)
\]

and

\[
J(\xi) = \sinh \gamma \left[ \frac{\xi}{\tanh \xi} - 1 \right] \quad (58)
\]

where \( \gamma \) is an arbitrary constant.

This solution corresponds to an extended object that has both electric and magnetic charge given by

\[
q \equiv \int d^3x \partial^i E^i = \frac{4\pi}{e} \sinh \gamma \quad g = \frac{4\pi}{e} T \quad (59)
\]

while its mass is given by:

\[
M = \frac{4\pi}{e} a \cosh^2 \gamma \quad (60)
\]

Using the asymptotic behaviour for large \( r \) of the Higgs field

\[
\Phi^a = \frac{r^a}{r} H(ear) \rightarrow \frac{r^a}{r} v \quad (61)
\]

where \( v \equiv a \cosh \gamma \), together with the expressions for the electric and magnetic charges given in eq. (59) we obtain the mass of the dyon:

\[
M = v \sqrt{q^2 + g^2} \quad (62)
\]

This formula has been deduced for the dyon soliton solution in the BPS limit, but it is actually valid for any particle of the spectrum. Notice that it is invariant under the duality transformation in eq. (14).

In the classical theory the electric charge of the dyon can get any value given by the formula in eq. (59). When one semiclassically quantizes the dyon solution one discovers that the values for the electric charges are quantized [14] and given by

\[
q = en_e \quad (63)
\]

where \( n_e \) is an integer.

### 4 Montonen-Olive duality

Leaving aside for a moment the dyon solution discussed at the end of last section we have found that the semiclassical spectrum of the Georgi-Glashow model in the BPS limit consists of a massless photon and Higgs particle, of an electrically charged \( W \) boson with charge equal to \( q_0 = \pm e \hbar \) and of a magnetic monopole with magnetic charge equal to \( g_0 = \pm \frac{4\pi e}{90} = \frac{4\pi \hbar}{90} \).
If there is duality invariance as suggested for instance by the formula in eq. (62) we can make a duality transformation with angle $\phi = -\frac{\pi}{2}$ such that

$$q_0 \rightarrow g_0 \quad g_0 \rightarrow -q_0$$

(64)

This transformation implies that

$$q_0 \rightarrow \frac{4\pi \hbar}{q_0}$$

(65)

Based on this observation Montonen and Olive [2] suggested that there are two equivalent formulations of the same theory dual to each other. In the first one, that we call electric, the $W$'s are elementary particles while the magnetic monopoles are solitons. In the second one, that we call magnetic, the elementary particles are instead the magnetic monopoles while the $W$ bosons are solitons. They also suggested that the two formulations had essentially the same Lagrangian. The only important difference between them is that the electric theory is weakly coupled when $q_0 \rightarrow 0 \quad (e \rightarrow 0)$ while the magnetic theory is weakly coupled when $g_0 \rightarrow 0$ corresponding to $e \rightarrow \infty$. They brought the following arguments in support of their duality conjecture.

1. The mass formula in eq. (62), valid for all particles of the theory, is duality invariant.

2. Since there is no interaction between two monopoles, while there is a non zero interaction between a monopole and an antimonopole, if duality is correct, one must expect that the interaction between equal charge $W$-bosons must be zero while that between opposite charged $W$-bosons must be non vanishing. This is actually verified in the BPS limit because in this limit the Higgs field is also massless and contributes with opposite sign with respect to the photon for equal charge $W$, while it contributes with opposite sign for opposite charge $W$.

The Montonen-Olive duality proposal, leaves, however many unanswered questions that we list:

1. The elementary $W^\pm$ bosons have spin equal to 1. If the magnetic monopoles are dual to them they must also have spin equal to 1. But how can this happen?

2. The previous considerations are based on a mass formula that is only valid classically. How are the quantum corrections going to modify it?

3. In the previous considerations we have neglected the dyons. What is their role in the all picture?
The previous questions do not have an answer in the framework of the Georgi-Glashow model discussed in the previous section since the quantum Georgi-Glashow model is, actually, not duality invariant. But it was soon recognized [15] that, in order to have a theory with Montonen-Olive duality, one must include supersymmetry since in a supersymmetric theory the quantum corrections coming from the bosons and the fermions tend to cancel each others preserving the structure of the classical mass formula. Actually the argument used in Ref. [15] for the \( N = 2 \) super Yang-Mills theory is too naive and in fact wrong as pointed out in Refs. [16, 17, 18, 19, 20] because this theory is not ultraviolet finite. In order to have a classical mass formula that is not modified by quantum corrections one must consider the \( N = 4 \) super Yang-Mills theory that is free from ultraviolet divergences [21, 22], as it was done by Osborn [23] who made also the important observation that in this case magnetic monopoles and dyons have also supersymmetric partners with spin equal to 1. The introduction of the \( N = 4 \) theory open the way to the solution of the first two puzzles discussed above. In the meantime Witten and Olive [24] found out that the structure of the duality invariant mass formula in eq. (62) for a BPS state in the \( N = 2 \) theory is a direct consequence of the supersymmetry algebra opening the way to the quantum exact determination of the mass of the BPS states. This observation is playing an essential role also in recent developments in string theories.

5 Representations of supersymmetry algebra

As in the case of the Poincaré group the representations of the supersymmetry algebra for massive particles are different from those for massless particles. The supersymmetry algebra is given in both cases by:

\[
\{ Q^i_\alpha, \bar{Q}^j_{\dot{\alpha}} \} = 2 \sigma_\mu P^\mu \delta^{ij} \quad i, j = 1 \ldots N
\]  
\( (66) \)

\[
\{ Q^i_\alpha, Q^j_\beta \} = 0
\]
\( (67) \)

The difference between the two cases is due to the fact that in the massive case one can always choose a center of mass frame where \( P_\mu = (M, \vec{0}) \), while this is not possible in the massless case.

In the massive case in the center of mass frame one gets the following algebra:

\[
\{ a^i_\alpha, (a^j_\beta)^+ \} = \delta_{\alpha\beta} \delta^{ij}
\]  
\( (68) \)

and

\[
\{ a^i_\alpha, a^j_\beta \} = \{ (a^i_\alpha)^+, (a^j_\beta)^+ \} = 0
\]  
\( (69) \)

where

\[
a^i_\alpha = \frac{1}{\sqrt{2M}} Q^i_\alpha \quad (a^j_\beta)^+ = \frac{1}{\sqrt{2M}} \bar{Q}^i_{\dot{\alpha}}
\]  
\( (70) \)
The representation of the fermionic harmonic oscillator algebra is constructed starting from a vacuum state $|0\rangle$ satisfying the equation:

$$a^i_\alpha |0\rangle = 0$$  \hspace{1cm} (71)

and acting on it with the creation operators:

$$\frac{1}{\sqrt{n!}}(a^i_{\alpha_1})^+(a^j_{\alpha_2})^+ \cdots (a^n_{\alpha_n})^+ |0\rangle \hspace{1cm} n = 0, 1 \ldots 2N$$  \hspace{1cm} (72)

The number of states in eq. (72) is equal to $\binom{2N}{n}$. Since $n$ runs from 0 to $2N$, the total number of states in the representation of the massive supersymmetry algebra is equal to:

$$\sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N}$$  \hspace{1cm} (73)

The states in the representation have a maximum helicity gap $\Delta \lambda = N$. Half of them are fermions and the other half are bosons.

In the massless case we can instead choose a frame where $P_\mu = (E, 0, 0, -E)$. In this frame the supersymmetry algebra becomes:

$$\{a^i, (a^j)^+\} = \delta^{ij}$$  \hspace{1cm} (74)

$$\{a^i, a^j\} = \{(a^i)^+, (a^j)^+\} = 0$$  \hspace{1cm} (75)

where

$$a^i = \frac{1}{2\sqrt{E}}Q^i_1 \hspace{1cm} (a^j)^+ = \frac{1}{2\sqrt{E}}\bar{Q}^i_1$$  \hspace{1cm} (76)

The anticommutators involving the generators of the supersymmetry algebra with indices $\alpha = 2$ and $\dot{\alpha} = \dot{2}$ are all vanishing and therefore they can be consistently put equal to zero:

$$Q^i_2 = \bar{Q}^i_2 = 0$$  \hspace{1cm} (77)

Starting again from the vacuum state annihilated by the annihilation operators $a^i$ we can construct the states of the representation acting on it with the creation operators obtaining the state:

$$\frac{1}{\sqrt{n!}}(a^{i_1})^+(a^{i_2})^+ \cdots (a^{i_n})^+ |0\rangle \hspace{1cm} n = 0, 1 \ldots N$$  \hspace{1cm} (78)

that contains $\binom{N}{n}$ states. The total number of states in the massless representation is equal to:

$$\sum_{n=1}^{N} \binom{N}{n} = 2^N$$  \hspace{1cm} (79)
that is smaller than in the case of a massive representation. The maximum helicity in this case is $\Delta \lambda = N/2$. In the case $N = 1$ one gets only one fermionic and one bosonic state. In most cases, however, we must add another multiplet with opposite helicity in order to have a $CPT$ invariant theory ($CPT$ reverses the sign of helicity).

Let us finally consider the representation of the massive $N = 2$ algebra with non vanishing central charges [25]. In this case in the center of mass frame the algebra is

$$\{Q^i, \bar{Q}^j\} = 2M \delta^{ij} \delta_{\alpha\dot{\alpha}}$$

$$\{Q^i, Q^j\} = \epsilon^{ij} \epsilon_{\alpha\beta} Z$$

One can get rid of the phase in $Z$ by a supercharge redefinition. Then one can rewrite the previous algebra in terms of the two quantities:

$$a_\alpha = \frac{1}{\sqrt{2}} \left[ Q^1_\alpha + \epsilon_{\alpha\beta} \bar{Q}^2_\beta \right]$$

$$b_\alpha = \frac{1}{\sqrt{2}} \left[ Q^1_\alpha - \epsilon_{\alpha\beta} \bar{Q}^2_\beta \right]$$

obtaining

$$\{a_\alpha, (a_\beta)^+\} = (2M + |Z|)\delta_{\alpha\beta}$$

$$\{b_\alpha, (b_\beta)^+\} = (2M - |Z|)\delta_{\alpha\beta}$$

while all the other anticommutators are vanishing.

If $2M = |Z|$ all anticommutators involving the oscillators $b$ are vanishing and therefore we can put them equal to zero. We can then use only the oscillators $a$ for constructing the representation, obtaining the same number of states as in the massless case. In the case $N = 2$ here considered we get the following four states:

$$|0 >\quad a_\alpha^+ |0 >\quad a_\alpha^+ a_\beta^+ |0 >$$

instead of the 16 states that we found in the case without central charge (see eq. (73) for $N = 2$).

Extending the previous procedure to the case $N = 4$ we obtain a short representation with 16 states instead of the one with $2^8 = 256$ states obtained without central charge (See eq. (73) for $N = 4$).

The fact that the representations of extended supersymmetry with non vanishing central charges are shorter and have the same dimension of those for the massless case makes it possible to have a consistent supersymmetric Higgs mechanism since in this case one has the same number of degrees of freedom before and after the Higgs mechanism.

6 Supersymmetric actions

In this section we construct the supersymmetric extension of Yang-Mills theory in $D = 4$ by dimensional reduction from higher dimensions [26].
Let me start from the following action in $D$ dimensions

$$S = \int d^D x \left\{ -\frac{1}{4} F_{MN}^a F^{a MN} - \frac{i}{2} \bar{\lambda}^a \Gamma_M (D^M \lambda)^a \right\}$$  \hspace{1cm} (85)$$

If we perform the following supersymmetry transformation

$$\delta A^a_M = \frac{i}{2} \left[ \bar{\lambda}^a \Gamma_M \alpha - \bar{\alpha} \Gamma_M \lambda^a \right]$$  \hspace{1cm} (86)$$

together with

$$\delta \lambda_a = \sigma_{RS} F^a_{RS} \alpha \quad \delta \bar{\lambda}_a = -\bar{\alpha} \sigma_{RS} F^a_{RS}$$  \hspace{1cm} (87)$$

it can be seen, by using the useful identities,

$$\Gamma^M \sigma^{RS} = \frac{1}{2} \left[ g^{MR} \Gamma^S - g^{MS} \Gamma^R - \frac{1}{(D-3)!} \epsilon^{MRSN_1...N_{D-3}} \Gamma_{D+1} \Gamma_{N_1} \cdots \Gamma_{N_{D-3}} \right]$$  \hspace{1cm} (88)$$

and

$$\sigma^{RS} \Gamma^M = \frac{1}{2} \left[ -g^{MR} \Gamma^S + g^{MS} \Gamma^R - \frac{1}{(D-3)!} \epsilon^{MRSN_1...N_{D-3}} \Gamma_{D+1} \Gamma_{N_1} \cdots \Gamma_{N_{D-3}} \right]$$  \hspace{1cm} (89)$$

that the term with the $\epsilon$ tensors cancel using the Bianchi identity for $F_{\mu\nu}$ and the action in eq. (85) transforms as a total derivative

$$\delta S = \frac{i}{2} \int d^D x \partial_M \left[ \bar{\alpha} \sigma_{RS} F^a_{RS} \Gamma^M \lambda_a + F^a_{MN} \left( \bar{\alpha} \Gamma_M \lambda^a - \bar{\lambda}^a \Gamma_M \alpha \right) \right]$$  \hspace{1cm} (90)$$

provided that the following equation is satisfied

$$\left( \bar{\lambda}^a \Gamma_M f^{abc} \right) \left[ \bar{\alpha} \Gamma^M \lambda^b - \bar{\lambda}^b \Gamma^M \alpha \right] = 0$$  \hspace{1cm} (91)$$

$\alpha$ and $\lambda$ are spinors in $D$-dimensions, $\sigma_{MN} = \frac{1}{4} [\Gamma_M, \Gamma_N]$, $\Gamma_{D+1} = \Gamma_0 \cdots \Gamma_D$ and

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M - e f^{abc} A^b M A^c N \quad (D^M \lambda)^a = \partial_M \lambda^a - e f^{abc} A^b M \lambda^c$$  \hspace{1cm} (92)$$

Therefore the action in eq. (85) is $N = 1$ supersymmetric if the equation (91) is satisfied. As shown in Ref. [26] this happens in the following cases:

1. $D=3$, if $\lambda$ is a Majorana spinor
2. $D=4$, if $\lambda$ is a Majorana spinor
3. $D=6$, if $\lambda$ is a a Weyl spinor
4. $D=10$, if $\lambda$ is a Weyl-Majorana spinor
There is a simple way to understand this result by noticing that, in all these cases, the number of on shell bosonic degrees of freedom, that is equal to $D - 2$, is equal to the number of on shell fermionic degrees of freedom that is equal to $2^{[\frac{D}{2}]}$ multiplied with a factor $x = \frac{1}{2}$ if the spinor field $\lambda$ is a Majorana or Weyl spinor and a factor $x = \frac{1}{4}$ if the spinor field is a Weyl-Majorana spinor:

$$D - 2 = x \cdot 2^{\left[ \frac{D}{2} \right]}$$  \hspace{1cm} (93)

where $[\frac{D}{2}] = \frac{D}{2}$ if $D$ is even and $[\frac{D}{2}] = \frac{D-1}{2}$ if $D$ is odd.

In particular the action in eq. (85) is $N = 1$ supersymmetric if $D = 6$ and $10$. This fact can be used to write actions with extended $N = 2, 4$ supersymmetries in four dimensions by the technique of dimensional reduction. Let us divide the $D$ dimensional space-time component $x^M \equiv (x^\mu, x^i)$ in a part $x^\mu$, where the index $\mu$ runs over the four-dimensional space-time, and in part $x^i$, where the index $i$ runs over the compactified $D - 4$ dimensions. We assume that the various fields are independent from the compactified coordinates.

Let us start to compactify the bosonic term in the action (85) containing the non abelian field strenght given in eq. (92). The dimensional reduction of $F_{MN}$ gives respectively:

$$F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e f^{abc} A^b_\mu A^c_\nu$$  \hspace{1cm} (94)

$$F_{\mu i}^a = \partial_\mu A^a_i - e f^{abc} A^b_\mu A^c_i \equiv (D_\mu A_i)^a$$  \hspace{1cm} (95)

and

$$F_{ij}^a = - e f^{abc} A^b_i A^c_j$$  \hspace{1cm} (96)

Using the previous equations one obtains immediately the compactification of the gauge kinetic term

$$-\frac{1}{4} F_{MN}^a F^{aMN} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} (D_\mu A_i)^a (D_\mu A_i)^a - \frac{e^2}{4} f^{abc} A^b_i A^c_j f^{ade} A^d_i A^e_j$$  \hspace{1cm} (97)

where a sum over repeated indices is understood.

In order to perform the compactification of the fermionic term of the action in eq. (85) we have to distinguish the two cases $D = 6$ and $D = 10$.

A representation of the Dirac algebra for $D = 6$ is given by:

$$\Gamma_\mu = \gamma_\mu \otimes 1 \quad \mu = 0, 1, 2, 3$$  \hspace{1cm} (98)

$$\Gamma_4 = \gamma_5 \otimes i\sigma_1 \quad \Gamma_5 = \gamma_5 \otimes i\sigma_2 \quad \Gamma_7 \equiv \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_4 \Gamma_5 = \gamma_5 \otimes \sigma_3$$  \hspace{1cm} (99)

where the $\sigma$-matrices are the Pauli matrices and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

A Weyl spinor in $D = 6$ satisfies the condition:

$$(1 + \Gamma_7)\lambda = 0$$  \hspace{1cm} (100)
that is automatically satisfied if we take

\[
\lambda = \left( \frac{1-\gamma_5}{2\gamma_5} \chi \right) \quad \bar{\lambda} = \left( \chi \left( \frac{1+\gamma_5}{2} \right) \right)
\]  

(101)

where \( \chi \) is a Dirac spinor in four dimensions.

Inserting it in the fermionic term in eq. (85) one gets:

\[
i \left( \bar{\lambda} \right)^a \Gamma^M D_M \lambda^a = i \left( \bar{\chi} \right)^a \Gamma^\mu (D_{\mu} \chi)^a - \varepsilon f^{abc} \bar{\chi}^a A^b_4 \gamma_5 \chi^c + i e f^{abc} \bar{\chi}^a A^b_5 \chi^c
\]

(102)

The Lagrangian of \( N = 2 \) super Yang-Mills is obtained by summing the bosonic contribution in eq. (97) with the indices \( i, j = 1, 2 \) to the fermionic contribution in eq. (102). One gets

\[
L = \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} \sum_{i=1}^{2} (D_{\mu} A_i)^a (D_{\mu} A_i)^a - \frac{e^2}{2} f^{abc} A^b_4 A^c_5 f^{ade} A^d_4 A^e_5 +
\]

\[
- i \left( \bar{\chi} \right)^a \Gamma^\mu (D_{\mu} \chi)^a + e f^{abc} \bar{\chi}^a A^b_4 \gamma_5 \chi^c - i e f^{abc} \bar{\chi}^a A^b_5 \chi^c
\]

(103)

after a redefinition of the Dirac spinor \( \chi \rightarrow \sqrt{2} \chi \).

The \( N = 4 \) super Yang-Mills is instead obtained starting with a Weyl-Majorana spinor in \( D = 10 \). In \( D = 10 \) the Dirac algebra can be represented as follows:

\[
\Gamma^\mu = \gamma^\mu \otimes 1 \otimes \sigma_3 \quad \mu = 0, 1, 2, 3
\]

(104)

\[
\Gamma^{3+i} = 1 \otimes \alpha^i \otimes \sigma_1 \quad \Gamma^{6+i} = \gamma_5 \otimes \beta^i \otimes \sigma_3 \quad i = 1, 2, 3
\]

(105)

where the fourdimensional internal matrices \( \alpha \) and \( \beta \) satisfy the following algebra:

\[
\{ \alpha^i, \alpha^j \} = \{ \beta^i, \beta^j \} = -2 \delta^{ij} \quad [\alpha^i, \beta^j] = 0
\]

(106)

and

\[
[\alpha^i, \alpha^j] = -2 \varepsilon^{ijk} \alpha^k \quad [\beta^i, \beta^j] = -2 \varepsilon^{ijk} \beta^k
\]

(107)

Finally the correspondent of \( \gamma_5 \) in ten dimensions is given by:

\[
\Gamma_{11} = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9 = 1 \otimes 1 \otimes \sigma_2
\]

(108)

A Weyl-Majorana spinor satisfying the condition:

\[
(1 + \Gamma_{11}) \lambda = 0
\]

(109)

can always be written as

\[
\lambda = \psi \otimes \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \end{array} \right) \quad \bar{\lambda} = \bar{\psi} \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \end{array} \right)
\]

(110)
where the Majorana spinor $\psi$ has a four dimensional space-time index on which the Dirac matrices act and another internal four dimensional index on which instead the internal matrices $\alpha$ and $\beta$ act.

Proceeding as in the $N = 2$ case we arrive at the $N = 4$ super Yang-Mills Lagrangian:

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} \sum_{i=1}^{3} (D_\mu A_i)^a (D^\mu A_i)^a + \frac{1}{2} \sum_{j=1}^{3} (D_\mu B_j)^a (D^\mu B_j)^a - V(A_i, B_j) +$$

$$-\frac{i}{2} (\bar{\psi})^a \gamma^\mu (D_\mu \psi)^a - \frac{e}{2} f^{abc} \bar{\psi}^a \gamma^5 A^b \psi^c - i \frac{e}{2} f^{abc} \bar{\psi}^a \beta^j \gamma_5 B^b j \psi^c$$

(111)

where the potential is equal to:

$$V(A_i, B_j) = \frac{e^2}{4} f^{abc} A_i^b A_j^c f^{afg} A_i^f A_g^j + \frac{e^2}{4} f^{abc} B_i^b B_j^c f^{afg} B_i^f B_g^j + \frac{e^2}{2} f^{abc} A_i^b B_j^c f^{afg} A_i^f B_j^g$$

(112)

7 **Semiclassical analysis of super $N = 2$ theory**

The $N = 2$ super Yang-Mills theory described by the Lagrangian in eq. (103) can be rewritten in the following form

$$L = \frac{1}{4\pi} \text{Im} \left\{ \left( \frac{\theta}{2\pi} + i \frac{4\pi}{e^2} \right) \left[ -\frac{1}{4} (F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a F^{a\rho\sigma}) + \right. \right.$$

$$+(D_\mu \Phi)^a (D^\mu \Phi)^a - \frac{1}{2} [f^{abc} \Phi^b \Phi^c]^2 + \text{FERMIONS} \right\}$$

(113)

after a rescaling by a factor $1/e$ of the fields and the introduction of the vacuum angle $\theta$ and of a complex field $\Phi = \frac{A_5 + i A_4}{\sqrt{2}}$.

The structure of the bosonic part of the Lagrangian in eq. (113) is pretty much the same as the one in the Georgi-Glashow model in eq. (17). There is, however, an important difference. Unlike the Georgi-Glashow here the potential given in the case of a $SU(2)$ gauge group by

$$V(\Phi) = \frac{1}{2e^2} [\epsilon^{abc} \Phi^b \Phi^c]^2$$

(114)

does not fix uniquely the vacuum. In fact any field configuration of the type

$$\Phi^a = (0, 0, a)$$

(115)

corresponds to a minimum of the potential with vanishing value (since supersymmetry is not broken) for any value of the complex variable $a$. The set of all values of $a$ is called the classical moduli space of the theory. Actually a better parametrization of the vacua is given in terms of the gauge invariant variable $u = \frac{1}{2} a^2 = Tr(\Phi^2)$.  

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If $a \neq 0$, as in the Georgi-Glashow model, the $SU(2)$ gauge symmetry is broken to $U(1)$ by the Higgs phenomenon and the charged (with respect to the unbroken $U(1)$) components of the gauge fields $W^\pm$ get a non-vanishing mass, while the Higgs and the gauge field of the unbroken $U(1)$ remain massless as in the Georgi-Glashow model in the BPS limit. In the supersymmetric case, however, the BPS limit is obtained without needing to send to zero any piece of the potential as it was necessary in the Georgi-Glashow model.

As in the Georgi-Glashow model, in the $N = 2$ super Yang-Mills theory there are also time-independent solutions [15] of the classical equations of motion corresponding to magnetic monopoles and dyons.

Their mass can again be written in terms of the electric and magnetic charges as in the Georgi-Glashow model

$$M = \sqrt{2}|a|\sqrt{g^2 + q^2}$$  \hspace{1cm} (116)

where $a$ has been defined in eq. (115).

This formula for the mass is again valid for all particles of the semiclassical spectrum.

After semiclassical quantization the electric and magnetic charges of the particles of the spectrum are given by

$$g = \frac{4\pi}{e} n_m \quad q = en_e$$  \hspace{1cm} (117)

where $n_m = \pm 1$, $n_e$ is an integer and their mass is given by [18, 20]

$$M = \sqrt{2}|Z|$$

$$Z = a \left[ n_e + \left( \frac{\theta}{2\pi} + \frac{i}{4\pi} \frac{4\pi}{e^2(\mu)} + \frac{i}{\pi} \log \frac{a^2}{\mu^2 C} \right) n_m \right]$$  \hspace{1cm} (118)

where $\mu$ is the renormalization scale and $C$ is a scheme dependent constant.

As a consequence of the supersymmetric Higgs mechanism the massless spectrum consists of a photon $A_\mu$, that is the gauge field of the unbroken $U(1)$, of a photino and of a complex scalar particle $A$. They belong to a massless $N = 2$ chiral supermultiplet. If we are interested in studying the low-energy dynamics of these fields and therefore we need to restrict ourselves to a Lagrangian with at most two derivatives and with no more than four-fermion couplings, the requirement of $N = 2$ supersymmetry fixes completely its form giving the following Lagrangian [1]

$$L = \frac{1}{4\pi} Im \left\{ \tau(A) \left[ \partial_\mu \bar{A} \partial^A - \frac{1}{4} \left( F^2 - \frac{i}{2} e^{\mu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) + \text{Fermions} \right] \right\}$$  \hspace{1cm} (119)

where

$$\tau(A) = \frac{\partial^2 \mathcal{F}}{\partial A^2}$$  \hspace{1cm} (120)

is given in terms of an arbitrary function $\mathcal{F}(A)$ of the scalar field $A$. This means that the low-energy dynamics is completely determined by the function $\mathcal{F}$ that in general will receive both perturbative and non-perturbative contributions.
Comparing eq. (119) with eq. (113) we see that at the tree level the function $F$ is given by

$$F_{cl} = \frac{1}{2} \tau_{cl} A^2 \quad \tau_{cl} = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2}$$  \hspace{1cm} (121)$$

At one loop one gets [27, 28] instead

$$F_1 = \frac{i}{2\pi} A^2 \log \frac{A^2}{\mu^2}$$  \hspace{1cm} (122)$$

that is consistent with the $U(1)_R$ and scale anomaly [27, 28].

It can also be shown that higher loops do not give any contribution to $F$. Only non perturbative effects, as for instance instantons, can give an additional contribution to $F$ [28]. In the last few sections of these lectures we will review the arguments of Seiberg and Witten that led to the exact determination of $F$.

We conclude this section by remembering that the $N = 2$ super Yang-Mills theory is an asymptotic free theory with a $\beta$-function $\beta(e) = -\frac{e^2}{4\pi^2}$ (for $SU(2)$) getting, in perturbation theory, only contribution from one-loop diagrams [29] that also generate the function $F$ given in eq. (122). From the sum of the tree and one-loop contributions $F_{cl} + F_1$ one obtains how the running coupling constant varies with the scale

$$\frac{4\pi}{e^2(\mu)} + \frac{1}{\pi} \log \frac{a^2}{\mu^2} \equiv \frac{4\pi}{e^2(a)}$$ \hspace{1cm} (123)$$

In terms of the renormalization invariant parameter $\Lambda$ the previous equation becomes:

$$\frac{e^2(a)}{4\pi} = \frac{\pi}{\log \frac{a^2}{\Lambda^2}} \quad ; \quad \Lambda^2 = \mu^2 e^{-\frac{4\pi^2}{\eta(\mu)}}$$  \hspace{1cm} (124)$$

showing that, because of asymptotic freedom, perturbation theory is good when $a$ is large.

Many of the previous results as the existence of a manifold of inequivalent vacua and the existence of monopole and dyon solutions are also valid for the $N = 4$ super Yang-Mills theory. This theory, being free from ultraviolet divergences [21, 22], has a vanishing $\beta$-function and no chiral anomaly [30].

### 8 Susy algebra in $N = 2$ super Yang-Mills

The $N = 2$ algebra with central charges given in eqs. (80) and (81) can be rewritten in four-dimensional notations obtaining:

$$\{Q_A, \bar{Q}_B\} = 2\gamma^\mu_{AB} P_\mu \delta^{ij} - 2\gamma_5 \epsilon^{ij} V + 2i \epsilon^{ij} \delta_{AB} U$$  \hspace{1cm} (125)$$

where

$$2U = -Im Z \quad \quad 2V = -Re Z$$  \hspace{1cm} (126)$$
Olive and Witten [24] have explicitly computed the central charge in the $N = 2$ super Yang-Mills theory obtaining:

$$U = \int d^3x \partial_i [S^a E^a_i + P^a B^a_i]$$  \hspace{1cm} (127)

and

$$V = \int d^3x \partial_i [P^a E^a_i + S^a B^a_i]$$  \hspace{1cm} (128)

where $S = -A_5$ and $P = A_4$.

From the algebra in eq. (125) applied to a state in the center of mass frame where $P^\mu = (M, \vec{0})$ and from positivity it follows that

$$M \geq \sqrt{U^2 + V^2}$$  \hspace{1cm} (129)

In the asymptotic vacuum given by:

$$<P^a> = 0$$  \hspace{1cm} $$<S^a> = \sqrt{2a} \delta^{a3}$$  \hspace{1cm} (130)

eq. (129) implies

$$M \geq \sqrt{2|a|/q^2 + g^2} = \sqrt{2|a||q + ig|}$$  \hspace{1cm} (131)

where $q$ and $g$ are respectively the electric and magnetic charge of the state.

Eq. (131) is the quantum generalization of the BPS condition found at the classical level. It is a consequence of the supersymmetry algebra and has now therefore a quantum status. For the BPS states, for which the equality sign holds, it is an exact mass formula valid in the full quantum theory. For this reason the introduction of an extended supersymmetry allows one to overcome the difficulty mentioned in the second point toward the end of section (4).

Similar results are also valid in the $N = 4$ super Yang-Mills theory where, however, instead of only two we have 12 central charges [23].

9 $N = 2$ versus $N = 4$

In the previous sections we have seen that supersymmetry is an essential ingredient for having a dual theory in the sense of Montonen-Olive. Restricting ourselves to the case of pure Yang-Mills theories without super matter we have analyzed in some detail the theories with $N = 2$ and $N = 4$. The theory with $N = 2$ has some attractive feature as for instance the fact that the supersymmetry algebra contains only two central charges, the electric and magnetic charges, while the $N = 4$ theory has many more central charges. It has also the nice property that the algebra is left unchanged under a simultaneous chiral transformation of the supercharges and a duality transformation acting on the electric and magnetic charges as in eq. (14).

On the other hand it is known from the work of Ref. [23] that the the monopole solution of the $N = 2$ theory belongs to the hypermultiplet that does not contain a
spin 1, while the $W$-bosons, that have a spin 1, belong to the $N = 2$ chiral multiplet. This shows immediately that the monopoles and the $W$-bosons cannot be dual in the sense of Montonen-Olive ruling out the $N = 2$ theory.

This problem is overcome in $N = 4$ because this theory contains a unique short multiplet containing one state of spin 1, four states of spin $1/2$ and five states with spin 0 and both the $W$-bosons and the monopoles belong to it. Therefore this selects the $N = 4$ theory as the only theory (without super matter) that can be dual in the sense of Montonen and Olive [23]. This theory has also the attractive feature of being free from ultraviolet divergences. Therefore its $\beta$-function is vanishing and the gauge coupling constant is not renormalized. That means that no discussion is needed on which coupling (the renormalized or the unrenormalized) satisfies the Dirac quantization condition [31]. Because of this, in the following, we will restrict ourselves to this theory when discussing duality in the sense of Montonen and Olive. We will see, however, that duality in the sense of providing different parametrizations of the same theory will also play an important role in the $N = 2$ theory.

10 SZ quantization condition and the charge lattice

The Dirac quantization condition in eq. (15) is only valid for particles having either an electric or a magnetic charge. It has been generalized by Zwanziger to the case of particles having both an electric and a magnetic charge. In this case one obtains the so called Schwinger-Zwanziger(SZ) [32, 33] quantization condition, that for two particles with electric and magnetic charges given by $(q_1, g_1)$ and $(q_2, g_2)$ reads:

$$q_1g_2 - q_2g_1 = 2\pi\hbar n$$

$$n = 0, \pm 1, \pm 2 \ldots$$

(132)

Unlike the Dirac quantization condition, that is only invariant under discrete duality, the SZ quantization condition is invariant under the full electromagnetic duality that acts on $q + ig$ as in eq. (14). Consequently, without loss of generality, we can assume that there exists a subset of purely electric states.

Then we assume also that

1. The SZ quantization condition is satisfied.
2. The electric and magnetic charges are conserved and any sum of physical charges is also a physical charge.
3. The TCP-theorem is valid and therefore for each particle with charges $(q, g)$ there exists also a particle with opposite charges $(-q, -g)$.
4. There exists at least a particle with non zero magnetic charge.
Applying the SZ quantization condition in eq. (132) with a purely electric state together with a state with arbitrary electric and non-vanishing magnetic charge, that we have assumed to exist, we get that the allowed charge values of purely electric states are quantized in terms of a fundamental charge $q_0$.

$$q_i = n_i q_0 \quad n_i = 0, \pm 1, \pm 2 \ldots$$  \hspace{1cm} (133)

where $q_0$ is the electric charge of a state that is physically realized and is independent from the magnetic charge of the state that we have used in the SZ quantization condition (see Ref. [7] for details).

Applying then the SZ quantization condition to the case of a purely electric state with electric charge equal to $q_0$ and of an arbitrary state with magnetic charge $g_i$ we get that also the magnetic charge is quantized

$$g_i = n_i g_0 \quad n_i = 0, \pm 1, \pm 2 \ldots$$  \hspace{1cm} (134)

independently of the electric charge of the state, where

$$g_0 = \frac{2\pi \hbar n_0}{q_0}$$  \hspace{1cm} (135)

$n_0$ is an integer depending on the theory under consideration.

Let us consider now two dyons with the same magnetic charge $g_0$ and with electric charges equal to $q_1$ and $q_2$ respectively. The SZ quantization condition implies:

$$q_1 - q_2 = \frac{2\pi \hbar n}{g_0} = \frac{n}{n_0} q_0 = m q_0$$  \hspace{1cm} (136)

where $n$ and $m$ are integers. The last step in the previous equation follows from eq. (133) since the state with charge equal to $(q_1 - q_2, 0)$ is a purely electric state.

Eq. (136) implies that the electric charges $q_1$ and $q_2$ of particles with magnetic charge equal to $g_0$ must be equal to

$$q = q_0 \left( n + \frac{\theta}{2\pi} \right)$$  \hspace{1cm} (137)

where $\theta$ is an arbitrary real parameter that is, in some sense, an angular variable since $\theta \rightarrow \theta + 2\pi$ is equivalent to shifting $n$ by a unit ($n \rightarrow n + 1$).

Applying the SZ quantization condition to two particles with charges equal to $(q_1, m g_0)$ and $(q_2, m g_0)$ respectively one gets:

$$q_1 - q_2 = \frac{2\pi \hbar n}{m g_0} = \frac{n q_0}{m n_0} = p q_0$$  \hspace{1cm} (138)

where $n, m$ and $p$ are all integers. The last step in the previous equation follows again, as in eq. (136), from eq. (133) since the state with charge $(q_1 - q_2, 0)$ is a purely electric state.
The most general solution of eq. (138) is of the form:

\[ q = q_0 \left( n + \frac{\theta}{2\pi} f_m \right) \]  

(139)

where \( f_m \) is an arbitrary number depending only on the magnetic charge of the particle. It can be determined by inserting eq. (139) in the SZ quantization condition applied to two states with charges equal to \((q_1, mg_0)\) and \((q_2, g_0)\) obtaining

\[(q_1 - mq_2)g_0 = 2\pi n_0 \left[ n_1 - m n_2 + \frac{\theta}{2\pi} (f_m - mf_1) \right] = 2\pi \bar{h} n \]  

(140)

This equation is satisfied if we take:

\[ f_m = mf_1 = m \]  

(141)

since \( f_1 = 1 \) from eq. (137).

We conclude therefore that the electric and magnetic charges of a dyon must be given by:

\[ q + ig = q_0 (m\tau + n) \]  

(142)

with integers \( n \) and \( m \), where

\[ \tau = \frac{\theta}{2\pi} + i \frac{2\pi \bar{h} n_0}{q_0^2} \]  

(143)

Eq. (142) implies that the charges of a dyon must lie on a two-dimensional lattice with periods \( q_0 \) and \( q_0\tau \). This follows in a straightforward way from the assumptions made at the beginning of this section. Notice that \( Im\tau > 0 \).

The quantity \( \tau \) contains a parameter \( \theta \) that in the gauge theory arises when one includes, together with the usual kinetic term, a term proportional to the vacuum angle \( \theta \) containing the topological charge density. The formula in eq. (142) includes also the Witten effect [34] since the electric charge of a dyon gets an additional contribution if \( \theta \neq 0 \) due to its magnetic charge. In fact, from eq. (142) one gets the following electric charge of a dyon:

\[ q = q_0 \left( n + \frac{\theta}{2\pi} m \right) \]  

(144)

In conclusion, under the assumptions stated at the beginning of this section, we have shown that the electric and magnetic charges of an arbitrary state are given by the expression in eq. (142). The question now is how to select those states that are single particle states. This can be done if we restrict ourselves to BPS saturated states that, as we have seen in sect. (8), have a mass given by an exact quantum formula:

\[ M = \sqrt{2|a||q + ig|} \]  

(145)
A single particle BPS-saturated state with mass $M$ must be stable and this is the case if it cannot decay into a couple of BPS saturated states with mass $M_1$ and $M_2$, i.e.

$$M < M_1 + M_2$$

(146)

Using for the mass the expression in eq. (145) together with the exact expression for the charge given in eq. (142) one can easily see, by means of the Schwarz inequality, that eq. (146) is satisfied if and only if the integers $(n, m)$ in eq. (142) are coprimes. This implies that the stable states with zero magnetic charge $(n, 0)$ are only the three states with $n = 0, \pm 1$; the states with magnetic charge corresponding to $m = \pm 1$ are all stable states; the states with magnetic charge corresponding to $m = \pm 2$ are only stable if their electric charge corresponds to odd values of $n$; the states with magnetic charge $m = \pm 3$ are stable if $n$ is different from 0 and is not a multiple of 3 and so on.

11 Riformulation of Montonen-Olive duality

We are now in a position to riformulate the Montonen-Olive duality in a way in which the $W$-bosons, the magnetic monopoles and more in general all the dyons of the spectrum are treated in a completely democratic way [3]. We will see that we will not just have an electric and magnetic description, but we will have an infinite number of descriptions depending on which states of the charge lattice we are choosing as fundamental particles. Having seen that, if we limit ourselves to super Yang-Mills theories without super matter, the only theory that could have a duality in the sense of Montonen-Olive is the $N = 4$ super Yang-Mills theory, in the following we will concentrate on this theory.

The usual formulation of the $N = 4$ super Yang-Mills is obtained by considering the states with zero magnetic charge and with electric charge equal to $\pm q_0$ corresponding to the $W$-bosons that get a mass given by the formula in eq. (145) through the Higgs mechanism, together with the massless states at the origin of the charge lattice having vanishing electric and magnetic charges and corresponding to the photon and Higgs particle. Selecting these states we have determined one of the periods of the lattice. The other period is also fixed when we specify the value of the vacuum angle $\theta$. We then ascribe a short $N = 4$ supermultiplet to each of the three states with charge equal to 0, $q_0$ and $-q_0$ and, having fixed the value of $\theta$, we can explicitly write the full Lagrangian of $N = 4$ super Yang-Mills containing only the states of the lattice that we have chosen. If the theory is dual in the sense of Montonen-Olive the other stable states of the charge lattice must appear as solitons or bound states of solitons.

On the other hand if the theory is dual in the sense of Montonen-Olive one could also start from another couple of stable states of the charge lattice corresponding to a certain dyon of the theory with a complex charge given by $\pm q_0'$ and with mass equal to $M = \sqrt{2}|a||q_0'|$, together with the massless photon and Higgs states located
at the origin of the charge lattice and specify the vacuum angle $\theta$ by giving another vector $q'_0 \tau'$ of the lattice that is not aligned with $q'_0$. We can again ascribe a $N = 4$ short multiplet to any of the states previously chosen and write, as before, a $N = 4$ super Yang-Mills Lagrangian containing the states with charges equal to 0 and $\pm q'_0$ and with a specified vacuum angle $\theta$. Also in this case the remaining stable states of the charge lattice will show up as solitons or bound states of solitons of the new Lagrangian. Duality in the sense of Montonen and Olive means that all the theories based on any pair of independent vectors of the charge lattice are equivalent.

Since the vectors $q'_0$ and $q'_0 \tau'$ form an alternative basis of the charge lattice it must be possible to express them in terms of the original vectors $q_0$ and $q_0 \tau$ through the relation:

$$q'_0 \tau' = aq_0 \tau + bq_0 \quad q'_0 = cq_0 \tau + dq_0$$

with $a, b, c$ and $d$ integer numbers.

Since it must also be possible to express $q_0$ and $q_0 \tau$ in terms of $q'_0$ and $q'_0 \tau'$ the integers of the transformation must satisfy the equation:

$$ad - bc = 1$$

Therefore the transformations from a basis to another basis form the modular group $SL(2,\mathbb{Z})$.

Eqs. (147) imply a relation between $\tau$ and $\tau'$ given by

$$\tau' = \frac{a \tau + b}{c \tau + d}$$

that provides a connection between the values of the parameters $(\theta, q_0)$ in the two choices of basis vectors and actions.

The modular group is generated by the two transformations:

$$T : \quad \tau \to \tau + 1 \quad \to \quad \theta \to \theta + 2\pi$$

that is a symmetry of the theory because the physics is periodic when we translate $\theta$ by $2\pi$, and

$$S : \quad \tau \to -\frac{1}{\tau} \quad \to \quad q_0 \to \frac{2\pi \hbar n_0}{q_0} \quad (i f \ \theta = 0)$$

that relates weak coupling with strong coupling (compare with eq. (65), $n_0 = 2$).

The mass of the BPS-saturated states of the theory is proportional to the absolute value of the charge

$$M \sim |q + ig| = |q_0(m \tau + n)|$$

and is left invariant if we transform $\tau$ as in eq. (149) and $q_0$ and the charge vector $\left(\begin{array}{c} m \\ n \end{array}\right)$ as follows

$$q_0 \to q'_0 = q_0(c \tau + d) \quad \left(\begin{array}{c} m \\ n \end{array}\right) \to \left(\begin{array}{c} m' \\ n' \end{array}\right) = \left(\begin{array}{cc} d & -c \\ -b & a \end{array}\right) \left(\begin{array}{c} m \\ n \end{array}\right)$$
with \( ad - bc = 1 \).

The modular group does not only perform a transformation from a system of basis vectors to another one, but acts also on the integer charge vector \( \begin{pmatrix} m \\ n \end{pmatrix} \) rotating it into a new integer charge vector \( \begin{pmatrix} m' \\ n' \end{pmatrix} \). In other words a modular transformation transforms \( q + ig \) expressed in terms of the basis vectors \( q_0 \) and \( q_0 \tau \) and of the integers \( n \) and \( m \) into an expression having the same form in terms of the new basis vectors \( q'_0 \) and \( q'_0 \tau' \) and of the new integers \( n' \) and \( m' \) related to the old ones by eqs. (149) and (153). The invariance under the modular group requires that the existence in the spectrum of a state with a certain pair of integers implies also the existence in the spectrum of the state with other integers obtained from the first ones by the action of a modular transformation as in the second equation of (153).

In particular from eq. (153) it follows that, given the existence in the spectrum of the \( W^+ \)-boson corresponding to \( m = 0 \) and \( n = 1 \), the invariance under the modular group implies also the existence of the transformed state:

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -c \\ a \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

(154)

Since the condition \( ad - bc = 1 \) is equivalent to require that \( c \) and \( a \) are coprimes, the existence of the \( W^+ \)-boson implies the existence in the spectrum of all stable states of the charge lattice as discussed at the end of the previous section. This is a direct consequence of the Montonen-Olive duality.

Let us consider the states with \( c = -1 \). They are of the type \( \begin{pmatrix} 1 \\ a \end{pmatrix} \) where \( a \) is an arbitrary integer. These are the dyons in eq. (117). The next case is \( c = -2 \). In this case we expect the existence of the states \( \begin{pmatrix} 2 \\ a \end{pmatrix} \) where \( a \) is odd. The existence of such states was shown by Sen [35]. Evidence for the existence of stable states with higher values of \( c \) can be found in Ref. [36].

12 Global parametrization of moduli space in \( N = 2 \) theory

In the last few sections we will be shortly describing the beautiful paper of Seiberg and Witten [1] where an exact expression for \( \tau(A) \) (see eq. (119)) in the low energy effective action of the \( N = 2 \) super Yang-Mills theory has been constructed.

Unlike the \( N = 4 \) theory which can be equivalently formulated either in terms of the original fundamental fields or in terms of the monopoles or more in general of the dyons of the single particle spectrum with essentially the same Lagrangian, the
$N = 2$ theory cannot satisfy the Montonen-Olive duality because the fundamental fields and the magnetic monopoles belong to two different $N = 2$ superfields. The first are in the chiral $N = 2$ vector multiplet while the magnetic monopoles and dyons are in the hypermultiplet \cite{23}.

Nevertheless the $N = 2$ theory can be formulated either in terms of the variables $A, A_\mu$ and $\tau(A)$, as we have done in eq. (119), or in terms of the dual variables $A_D, A_D\mu$ and $\tau_D(A_D)$ in pretty much the same way that free electromagnetism can be formulated either in terms of the vector potential $A_\mu$ related to the field strength by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ or in terms of the dual vector potential $A_D\mu$ related to the dual field strength by $*F_{\mu\nu} = \partial_\mu A_D\nu - \partial A_D\mu$.

In order to explain this let us first summarize some general property of the $N = 2$ super Yang-Mills theory.

In sect. (7) we have seen that in the $N = 2$ theory the low energy effective theory is completely fixed by giving a holomorphic function $F(A)$. In terms of $F$ we can construct the Kähler potential:

$$K(A, \bar{A}) = \text{Im} \left( \frac{\partial F}{\partial A} \bar{A} \right)$$

(155)

and the metric

$$(ds)^2 = \frac{\partial}{\partial A} \frac{\partial}{\partial \bar{A}} K(A, \bar{A}) dA d\bar{A} = \text{Im}(\tau(A)) dA d\bar{A} = \frac{\partial^2 F(A)}{\partial A^2}$$

(156)

We have seen that the moduli space of the $N = 2$ theory is in the semiclassical theory parametrized by the vacuum expectation value of the scalar field that we have denoted by the complex number $a$. However $a$ cannot provide a global description of the moduli space. In fact the metric $\text{Im}(\tau(a))$, that is a positive definite harmonic function divergent for $|a| \to \infty$, must have a minimum. But a globally defined harmonic function cannot have a minimum and consequently the variable $a$ cannot provide a global parametrization of the moduli space.

Therefore in Ref. \cite{1} it was proposed to choose the gauge invariant quantity $u = \frac{1}{2} \text{Tr}(\Phi^2)$ as the one that provides a global parametrization of the moduli space and to regard both $a(u)$ and the dual variable $a_D(u) = \frac{\partial F}{\partial u}$ as functions of $u$. In terms of both $a$ and $a_D$ the metric in eq. (156) assumes the form

$$(ds)^2 = \text{Im} \left( \frac{da_D}{da} \right) \text{Im} (da D da D) = \text{Im} (da_D da_D) = -\frac{i}{2} \left[ da_D da_D - da_D da_D \right]$$

(157)

that is symmetric under the exchange $a \leftrightarrow a_D$.

Introducing the vector $v^\alpha = \begin{pmatrix} a_D \\ a \end{pmatrix}$ we can rewrite the metric in the more compact form:

$$(ds)^2 = -\frac{i}{2} \epsilon_{\alpha\beta} \frac{dv^\alpha}{du} \frac{dv^\beta}{d\bar{u}} dud\bar{u}$$

(158)
that clearly show its invariance under the transformation:

\[ v \rightarrow Mv + c \]  

(159)

where \( M \) is a matrix of \( SL(2, R) \) and \( c \) is a constant vector.

An arbitrary matrix of \( SL(2, R) \) is generated by the action of two independent matrices \( T_b \) and \( S \). The first one \( T_b \)

\[ T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \]  

(160)

leaves \( a \) invariant and transforms \( a_D \) according to

\[ a_D \rightarrow a_D + ba \]  

(161)

This implies that \( \tau(a) \) is just translated

\[ \tau(a) \rightarrow \tau(a) + b \]  

(162)

resulting in a translation for the vacuum angle \( \theta \)

\[ \theta \rightarrow \theta + 2\pi b \]  

(163)

Since physical quantities are invariant when

\[ \theta \rightarrow \theta + 2\pi n \]  

(164)

for any integer \( n \), comparing eqs. (163) and (164) we deduce that \( b = 1 \) and consequently that the transformation associated to the matrix \( T_{b=1} \) is a symmetry of the theory. By selecting \( b = 1 \) we have reduced the original \( SL(2, R) \) symmetry group to \( SL(2, Z) \).

The other independent generator

\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  

(165)

does not correspond to a symmetry of the theory, but provides a transformation between two different parametrizations of the theory. In fact the low energy effective Lagrangian can be represented either in terms of the variables \( (A^\mu, \lambda, A; \tau(A)) \) or in terms of the dual ones \( (A_D^\mu, \lambda_D, A_D; \tau_D(A_D) = -1/\tau(A)) \). In order to more clearly see the relation between the two formulations it is convenient to set the vacuum angle \( \theta = 0 \). Then we see that, if \( Im\tau(A) = \frac{4\pi}{g^2} \), then \( Im\tau_D(A_D) = \frac{4\pi}{g^2} \). Therefore one description may be more suitable for weak coupling, while the other for strong coupling.

In the final part of this section we discuss the exact mass formula proposed in Ref. [1] for the BPS saturated states in the \( N = 2 \) theory. At the semiclassical level the mass of the BPS saturated states is given in eq. (118). Noticing that the
coefficient of \( n_m \) in eq. (118), with a suitable choice of \( C \), is equal to \( a_D \) eq. (118) can be rewritten as follows

\[
Z = a n_e + a_D n_m = \begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \quad M = \sqrt{2 |Z|} \quad (166)
\]

Seiberg and Witten [1] proposed eq. (166) as an exact formula for the BPS states and made several checks for confirming its validity.

In particular \( Z \) is invariant under the transformation

\[
\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow M \begin{pmatrix} a_D \\ a \end{pmatrix} \quad \begin{pmatrix} n_m & n_e \end{pmatrix} \rightarrow \begin{pmatrix} n_m & n_e \end{pmatrix} M^{-1} \quad (167)
\]

where \( M \) is a matrix of \( SL(2, \mathbb{Z}) \) because the vector \( \begin{pmatrix} n_m & n_e \end{pmatrix} \) has integer entries and its transformed must also have integer entries. This is an independent way to derive the reduction of \( SL(2, \mathbb{R}) \) to \( SL(2, \mathbb{Z}) \). Actually this procedure forces also the extra parameter \( c \) in eq. (159) to be equal to zero.

13 Singularity structure of moduli space

In this section we study the singularity structure of \( a \) and \( a_D \) as functions of the variable \( u \), that provides a global parametrization of the moduli space.

In the semiclassical region corresponding to a large value of \( u \) we get

\[
a = \sqrt{2u} \quad a_D = i \frac{\sqrt{2u}}{\pi} \left[ 2 \log \frac{\sqrt{u}}{\Lambda} + 1 \right] \quad (168)
\]

Under a rotation around \( u = \infty \) given by

\[
\log u \rightarrow \log u + 2i\pi \quad (169)
\]

\( a \) and \( a_D \) are not monodromic functions, but transform according to

\[
a \rightarrow -a \quad a_D \rightarrow -a_D + 2a \quad (170)
\]

It is interesting that the asymptotic freedom property of the theory is responsible for this monodromy transformations.

The existence of a singularity requires the existence of at least another singularity. But, if we had only one additional singularity, it is easy to see that \( a \) would have been a good global parameter being the monodromy group an abelian group. Since this is not possible we must require the existence of at least two additional singularities.

Following the example of what is happening in some \( N = 1 \) supersymmetric theories Seiberg and Witten assume that the singularities occur at those points of the moduli space where additional massless particles appear in the spectrum. In the classical theory this occurs for \( a = 0 \) where the \( SU(2) \) symmetry is restored.
and $W^\pm$ become massless. They bring good indications against this possibility in the quantum theory and instead choose the singularities at the points $u_0$ where the monopole with $(n_m, n_e) = (1, 0)$ and $u'_0$ where the dyon with $(n_m, n_e) = (1, -1)$ become massless.

Using the exact formula in eq. (166) it is easy to see that this occurs when $a_D(u_0) = 0$ with $a(u_0) \neq 0$ and when $a_D(u'_0) - a(u'_0) = 0$ with $a(u'_0), a_D(u'_0) \neq 0$ respectively. The existence of a $Z(2)$ symmetry that transforms $u$ in $-u$ suggests to choose $u'_0 = -u_0$. They introduce a new dimensionless variable $u$ obtained from the previous one by dividing it with the square of a mass parameter chosen in such a way that the singularity due to the vanishing of the mass of a magnetic monopole occurs at $u = 1$. Then by the $Z(2)$ symmetry the other singularity due to the dyon occurs at $u = -1$.

The monodromy around the singularity at $u = 1$ can be easily computed by observing that the low energy theory at the point $u = 1$ consists of a ”magnetic” $N = 2$ super QED (the matter has magnetic and non electric charge). This theory is not asymptotically free and the coefficient of the $\beta$-function, besides a sign, has a factor $1/2$ of difference with respect to the $\beta$-function previously used for studying the singularity around $u = \infty$. By taking into account this difference in the $\beta$-function one arrives at the following monodromy transformations around the point $u = 1$:

$$a_D \rightarrow a_D, \quad a \rightarrow a - 2a_D$$

Finally the monodromy around the point $u = -1$ must be consistent with the previous ones and one obtains

$$a_D \rightarrow -a_D + 2a, \quad a \rightarrow -2a_D + 3a$$

### 14 Explicit solution

Having established the singularities and the monodromy transformations of $a$ and $a_D$ Seiberg and Witten were able to construct an explicit solution that is given by

$$a(u) = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - 1}} \quad a_D(u) = \frac{\sqrt{2}}{\pi} \int_{1}^{u} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - 1}}$$

and that is singular at the points $u = \pm 1, \infty$ with the right monodromies.

In terms of the previous functions one can construct the coefficient of the kinetic term of the gauge field

$$\tau(u) = \frac{d a_D}{d u}$$

that satisfies the important property: $\text{Im} \tau > 0$ for any $u$.

The classical vacuum degeneracy is not lifted by quantum effects not even after having taken into account the non-perturbative effects!!
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