Collisional equilibrium, particle production and the inflationary universe

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August 20, 1996

Abstract

Particle production processes in the expanding universe are described within a simple kinetic model. The equilibrium conditions for a Maxwell-Boltzmann gas with variable particle number are investigated. We find that radiation and nonrelativistic matter may be in equilibrium at the same temperature provided the matter particles are created at a rate that is half the expansion rate. Using the fact that the creation of particles is dynamically equivalent to a nonvanishing bulk pressure we calculate the backreaction of this process on the cosmological dynamics. It turns out that the ‘adiabatic’ creation of massive particles with an equilibrium distribution for the latter necessarily implies power-law inflation. Exponential inflation in this context is shown to become inconsistent with the second law of thermodynamics after a time interval of the order of the Hubble time.

PACS numbers: 98.80.Hw, 05.20.Dd, 04.40.Nr, 05.70.Ln

1 Introduction

Particle production processes in the early universe are supposed to have influenced the cosmological history considerably. These processes are quantum in nature \cite{1}. Their backreaction on the cosmological dynamics, however, are frequently studied phenomenologically. As was observed by Zel’dovich \cite{2} and Murphy \cite{3} and lateron confirmed by Hu \cite{4} a nonvanishing particle production rate is equivalent to a bulk viscous pressure in fluid cosmology. This equivalence has been used recently in a series of papers \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{14}, discussing different aspects of this effective viscous pressure
approach. Taking into account that the fluid dynamics of a gas may be derived from Boltzmann’s equation for the one-particle distribution function, the problem has been addressed whether the effective viscous pressure approach is compatible with the kinetic theory of a gas with varying particle number [15]. A modified Boltzmann equation was proposed in which an additional source term describes the change of the one-particle distribution function due to particle number nonconserving processes, supposedly of quantum origin. For a simple rate approximation of this source term and under the assumption that the characteristic time scale for interactions between the particles is much smaller than the time scale on which the number of particles changes, it turned out that the effective viscous pressure approach is compatible with kinetic theory in homogeneous spacetimes but not in inhomogeneous ones [15].

A modification of Boltzmann’s equation implies modified conditions for collisional equilibrium. In the standard relativistic kinetic theory, i.e., without the mentioned additional source term, it is a well established result [16], [17] that a simple gas can only be in ‘global’ equilibrium in spacetimes that admit a timelike Killing-vector, i.e., in stationary spacetimes. As an immediate consequence, it is impossible to characterize such a gas by a ‘global’ equilibrium distribution function in an expanding universe. The only exception is a gas of massless \((m = 0)\) particles (radiation), where the corresponding equilibrium condition is less restrictive and requires only the existence of a conformal Killing-vector.

As was shown in [15], the conditions for the one-particle distribution function to have a structure characteristic for collisional equilibrium are weaker in the case of a modified Boltzmann equation, i.e., a Boltzmann equation with an additional source term, than in the standard case. More specifically, for the mentioned effective rate model of the source term the one-particle distribution function takes its collisional equilibrium structure if the corresponding spacetime admits a conformal Killing-vector for massive particles as well. A gas of particles with \(m > 0\) in the expanding universe may also be characterized by a distribution function with equilibrium structure, provided the number of particles is not fixed but changes according to a specific rate, depending on the detailed equations of state.

The present paper provides a detailed discussion of the conditions for collisional equilibrium of a simple gas with nonconserved particle number. Our main objective is to consider the possible implications of these solutions of the modified Boltzmann equation for the cosmological dynamics.

Of course, the applicability of a kinetic approach to the early stages of the cosmological evolution is not straightforward. A gas, however, is the only system for which the correspondence between microscopic variables, governed by a distribution function, and phenomenological fluid quantities is sufficiently well understood. All the considerations of this paper refer to a model universe for which the kinetic approach is assumed to be valid. We hope that this idealized model nevertheless shares some basic features with our real universe.

Under these prepositions we shall establish apparently surprising links between the equilibrium properties of a relativistic gas with nonconserved particle number and the spacetime geometry of the universe. The most striking feature is the circumstance that the collisional equilibrium of a classical Boltzmann gas
with varying particle number in some cases requires a power-law inflationary universe.

The conditions for collisional equilibrium do not fully fix the particle production rate. The remaining freedom may be used to impose additional physical requirements. We shall discuss here two different cases and explore the implications of each of them for the cosmological evolution.

The first case will be that of ‘adiabatic’ particle production, characterized by a constant entropy per particle. The particles are assumed to be created with a fixed entropy. This additional assumption entirely determines the creation rate that is necessary for given equations of state to keep the particles at equilibrium. This rate is the higher the more massive the particles are. For nonrelativistic particles it is just half the expansion rate.

Now, a given creation rate is equivalent to a given effective viscous pressure. With the help of the latter it is possible to calculate the backreaction of the production process on the cosmological dynamics. For nonrelativistic particles this backreaction leads to power-law inflation with a behaviour of the scale factor \( R \) like \( R \propto t^{4/3} \) in a homogeneous, isotropic and spatially flat universe. Despite of the equilibrium structure of the distribution function the generation of particles is accompanied by a nonvanishing entropy production. For massive particles the comoving entropy grows as \( t^2 \).

The second case is connected with exponential inflation. There have been suggestions in the literature that a sufficiently high production rate of particles or strings might give rise to a de Sitter phase ([19], [6], [9], [10], [12]). We shall consider here the question whether this suggestion is compatible with the kinetic theory of a gas with variable particle number. To this end we use the above mentioned freedom in the production rate to impose the condition of a constant Hubble rate and clarify the consequences of this assumption. We show that exponential inflation becomes inconsistent with a non-negative entropy production, as required by the second law of thermodynamics, after a time of the order of the Hubble time.

The paper is organized as follows. In section 2 we recall the previously developed kinetic theory for particle production and discuss the general conditions for collisional equilibrium. Section 3 is devoted to ‘adiabatic’ particle production and its relation to power-law inflation. In section 4 we check an inflationary scenario where a de Sitter phase is generated by ‘nonadiabatic’ production of gas particles with a Maxwell-Boltzmann equilibrium distribution function and show that it violates the second law of thermodynamics after about one Hubble time. The final section 5 summarizes the main results of the paper.

Units have been chosen so that \( c = k_B = \hbar = 1 \).

2 Kinetic theory for particle production

2.1 General kinetic theory

According to the discussion in [15] we assume that a change in the number of particles of a relativistic gas should manifest itself in a source term \( H \) on the level of kinetic theory.
The corresponding one-particle distribution function \( f = f(x, p) \) of a relativistic gas with varying particle number is supposed to obey the equation

\[
L[f] \equiv p^i f_{,i} - \Gamma^i_{kl} p^k p^l \frac{\partial f}{\partial p^i} = C[f] + H(x, p),
\]

where \( f(x, p) p^k n_k d\Sigma dP \) is the number of particles whose world lines intersect the hypersurface element \( n_k d\Sigma \) around \( x \), having 4-momenta in the range \( (p, p + dp) \); \( i, k, l \ldots = 0, 1, 2, 3 \).

\[
dP = A(p) \delta (p^i p_i + m^2) dP_4 \quad \text{is the volume element on the mass shell } p^i p_i = -m^2 \quad \text{in the momentum space. } A(p) = 2, \text{if } p^i \text{ is future directed and } A(p) = 0 \text{ otherwise; } dP_4 = \sqrt{-g} dp^1 dp^2 dp^3 dp^4.
\]

\( C[f] \) is the Boltzmann collision term. Its specific structure discussed e.g. by Ehlers [16] will not be relevant for our considerations. Following Israel and Stewart [17] we shall only require that (i) \( C \) is a local function of the distribution function, i.e., independent of derivatives of \( f \), (ii) \( C \) is consistent with conservation of 4-momentum and number of particles, and (iii) \( C \) yields a non-negative expression for the entropy production and does not vanish unless \( f \) has the form of a local equilibrium distribution (see (7)).

The term \( H(x, p) \) on the r.h.s. of (1) takes into account the fact that the number of particles whose world lines intersect a given hypersurface element within a certain range of momenta may additionally change due to creation or decay processes, supposedly of quantum origin. On the level of classical kinetic theory we shall regard this term as a given input quantity. Later we shall give an example for the possible functional structure of \( H(x, p) \).

By the splitting of the r.h.s of eq.(1) into \( C \) and \( H(x, p) \) we have separated the collisional from the creation (decay) events. In this setting collisions are not accompanied by creation or annihilation processes. In other words, once created, the interactions between the particles are both energy-momentum and number preserving. For a vanishing \( H(x, p) \) eq.(1) reduces to the familiar Boltzmann equation (see, e.g., [16], [18], [17], [20]).

The particle number flow 4-vector \( N^i \) and the energy momentum tensor \( T^{ik} \) are defined in a standard way (see, e.g., [16]) as

\[
N^i = \int dP p^i f(x, p), \quad T^{ik} = \int dP p^i p^k f(x, p).
\]

The integrals in (2) and throughout the paper are integrals over the entire mass shell \( p^i p_i = -m^2 \). The entropy flow vector \( S^a \) is given by [16], [17]

\[
S^a = - \int p^a [f \ln f - f] dP,
\]

where we have restricted ourselves to the case of classical Maxwell-Boltzmann particles.

Using the general relationship [18]

\[
\left[ \int p^{a_1} \ldots p^{a_n} p^b f dP \right]_{,b} = \int p^{a_1} \ldots p^{a_n} L[f] dP
\]

(4)
and eq. (1) we find

\[ N^a_{;a} = \int (C \{f\} + H) \, dP \, , \quad T^a_{;b} = \int p^a (C \{f\} + H) \, dP \, , \]

and

\[ S^a_{;a} = -\int \ln f \,(C \{f\} + H) \, dP \, . \]

In collisional equilibrium, which we shall assume from now on, \( \ln f \) in (6) is a linear combination of the collision invariants \( 1 \) and \( p^a \). The corresponding equilibrium distribution function becomes (see, e.g., [16])

\[ f^0 (x, p) = \exp \left[ \alpha + \beta_a p^a \right] \, , \]

where \( \alpha = \alpha (x) \) and \( \beta_a \) is a timelike vector that depends on \( x \) only.

Inserting the equilibrium function into eq. (1) one gets

\[ \left[ p^a \alpha_{;a} + \beta_{(a;b)} p^a p^b \right] f^0 = H (x, p) \, . \]

It is well known ([16], [17]) that for \( H (x, p) = 0 \) this equation, which characterizes the ‘global equilibrium’, admits solutions only for very special cases in which \( \alpha = \text{const} \) and \( \beta_a \) is a timelike Killing-vector. It will be the main objective of this paper to clarify the consequences of (8) for a specific choice of the source term \( H (x, p) \) to be discussed below.

With (7), the balances (5) reduce to

\[ N^a_{;a} = \int H \, dP \, , \quad T^a_{;b} = \int p^a H \, dP \, . \]

In collisional equilibrium there is entropy production only due to the source term \( H \). From eq. (6) we obtain

\[ S^a_{;a} = -\int H (x, p) \ln f^0 \, dP \, , \]

implying \( S^a_{;a} = -\alpha N^a_{;a} - \beta_a T^a_{;b} \).

With \( f \) replaced by \( f^0 \) in (2) and (3), \( N^a \), \( T^{ab} \) and \( S^a \) may be split with respect to the unique 4-velocity \( u^a \) according to

\[ N^a = nu^a \, , \quad T^{ab} = \rho u^a u^b + p h^{ab} \, , \quad S^a = nsu^a \, , \]

where \( h^{ab} \) is the spatial projection tensor \( h^{ab} = g^{ab} + u^a u^b \), \( n \) is the particle number density, \( \rho \) is the energy density, \( p \) is the equilibrium pressure and \( s \) is the entropy per particle. The exact integral expressions for \( n \), \( \rho \), \( p \) and \( s \) are given by the formulae (177) - (180) in [16].

Using (11) and defining

\[ \Gamma \equiv \frac{1}{n} \int H (x, p) \, dP \, , \]

the first eq. (9) becomes

\[ \dot{n} + \Theta n = n \Gamma \, . \]
It is obvious that $\Gamma$ is the particle production rate. Similarly, with the decomposition (11) and the abbreviation

$$t^a \equiv - \int p^a H \left( x, p \right) dP ,$$

(14)

the energy balance, following from the second equation (9), may be written as

$$\dot{\rho} + \Theta \left( \rho + p \right) - u_a t^a = 0 ,$$

(15)

where $\Theta = u^a_a$ is the fluid expansion.

Introducing the quantity

$$\pi \equiv - \frac{u_a t^a}{\Theta} ,$$

(16)

it is possible to rewrite the energy balance (15) as

$$\dot{\rho} + \Theta \left( \rho + p + \pi \right) = 0 ,$$

(17)

i.e., $\pi$ enters the energy balance in the same way as a bulk viscous pressure does. A rewriting like this corresponds to the introduction of an effectively conserved energy-momentum tensor

$$\hat{T}^{ik} = \rho u^i u^k + \left( p + \pi \right) h^{ik} ,$$

(18)

instead of the nonconserved quantity $T^{ik}$ of (11). This interpretation of the source term (14) as an effective bulk pressure was shown to be consistent for homogeneous spacetimes [15].

### 2.2 Effective rate approximation

The quantity $H \left( x, p \right)$ is an input quantity on the level of classical kinetic theory. It supposedly represents the net effect of certain quantum processes with variable particle numbers (see, e.g., [1]) at the interface to the classical (non-quantum) level of description. Lacking a better understanding of these processes it was assumed in [15] that the influence of these processes on the distribution function $f \left( x, p \right)$ may be approximately described by a linear coupling to the latter:

$$H \left( x, p \right) = \zeta \left( x, p \right) f^0 \left( x, p \right) .$$

(19)

Moreover, $\zeta$ was supposed to depend on the momenta $p^a$ only linearly:

$$\zeta = - \frac{u_a p^a}{\tau \left( x \right)} + \nu \left( x \right) .$$

(20)

$\zeta$, or equivalently $\nu$ and $\tau$, characterize the rate of change of the distribution function due to the underlying processes with variable particle numbers. The restriction to a linear dependence of $\zeta$ on the momentum is equivalent to the requirement that these processes couple to the particle number flow vector and to the energy momentum tensor in the balances for these quantities only, but
not to higher moments of the distribution function. This ‘effective rate approximation’ is modeled after the relaxation time approximations for the Boltzmann collision term ([22], [23], [24]). While the physical situations in both approaches are very different, their common feature is the simplified description of nonequilibrium phenomena by a linear equation for the distribution function in terms of some effective functions of space and time that characterize the relevant scales of the process under consideration. In the relaxation time approximation this process is determined by the rate at which the system relaxes to an equilibrium state. In the present case the corresponding quantity is the rate by which the number of particles changes.

With (19) and (20) for \( H(x, p) \) the source terms \( n\Gamma \) and \( t^a \) in (13) and (15) are given by

\[
n\Gamma = -\frac{u_a}{\tau} N^a + \nu(x) M(x) = \frac{n}{\tau} + \nu M, \tag{21}
\]

where \( M \) is the zeroth moment of the distribution function, \( M \equiv \int dP f(x, p) \), and

\[
-u_a t^a = -\frac{u_a u_b T \epsilon_{ab}}{\tau(x)} + \nu(x) u_a N^a = -\frac{\rho}{\tau} - \nu n, \tag{22}
\]

respectively. The expression (20) for \( \zeta \) ensures that the source terms (21) and (22) depend on \( M, N^i \) and \( T^{ik} \) only, but not on higher moments of the distribution function.

Using the specific structure (19) and (20) of the source term \( H(x, p) \) in (8) the condition (8) becomes

\[
p^a \alpha_a + \beta_{(a;b)} p^a p^b = \frac{E}{\tau} + \nu, \tag{23}
\]

where \( E = -u_a p^a \). Decomposing \( p^a \) according to \( p^a = E u^a + \lambda e^a \), where \( e^a \) is a unit spatial vector, i.e., \( e^a e_a = 1 \), \( u^a e_a = 0 \), the mass shell condition \( p^a p_a = -m^2 \) is equivalent to \( \lambda^2 = E^2 - m^2 \). The conditions on \( \alpha \) turn out to be

\[
\dot{\alpha} = \frac{1}{\tau}, \quad h^a_b \alpha_a = 0, \tag{24}
\]

while \( \beta_a \) obeys the equation for a conformal Killing vector

\[
\beta_{(a;b)} = \Psi(x) g_{ab}, \quad m^2 \beta_{(a;b)} u^a u^b = \nu, \tag{25}
\]

implying \( \Psi = -\nu/m^2 \). Different from the case \( H(x, p) = 0 \) where \( \alpha \) has to be constant in space and time, \( \alpha \) is only spatially constant in the present case but changes along the fluid flow lines. From (25) it follows that \( \nu = 0 \) for \( m = 0 \). The condition \( \nu = 0 \) for \( m > 0 \), however, is equivalent to \( \beta_{(a;b)} = 0 \), i.e., the corresponding spacetime is stationary.

Using \( \beta_a = u_a / T \) in (25) yields [25], [24], [15]

\[
\frac{\dot{T}}{T} = -\frac{1}{3} \Theta \tag{26}
\]

for the temperature behaviour in an expanding universe. It is remarkable that this relation which is a consequence of the conformal Killing-vector property
for \( \beta \) holds both for \( m = 0 \) with \( \nu = 0 \) and for \( m > 0 \) in the case \( \nu \neq 0 \). The validity of the same equilibrium relation (26) both for radiation with \( H(x, p) = 0 \) and for massive particles with \( H(x, p) \neq 0 \) has the interesting implication that radiation and matter in the expanding Universe may be in equilibrium at the same temperature, provided the particle number of the matter component is allowed to change. It is well known, that for conserved particle numbers, i.e., \( H(x, p) = \nu = \tau^{-1} = 0 \), an equilibrium between both components is impossible [21, 18].

As a consequence of the fact that \( u_a/T \) is a conformal, timelike Killing-vector, the 4-acceleration \( \dot{u}_a \) may be expressed in terms of the spatial temperature gradient according to

\[
\dot{u}_a = -(\ln T)_b h^b_a. \tag{27}
\]

Within the linear theory of irreversible processes this coincides with the condition for the heat flow to vanish (see, e.g. [26]). For a comoving observer in the rotation free case (27) reduces to [15]

\[
(T\sqrt{-g_{00}})_{,\nu} = 0 , \ (\nu = 1, 2, 3). \tag{28}
\]

The latter formula which states that the quantity \( T\sqrt{-g_{00}} \) has to be spatially constant if particles with \( m > 0 \) are produced, replaces Tolman’s relation \( (T\sqrt{-g_{00}})_{,n} = 0 \), according to which \( T\sqrt{-g_{00}} \) is constant both in space and time in case \( u_a/T \) is a Killing-vector.

The conformal Killing-equation (25) provided us with the temperature law (26). On the other hand, the behaviour of the temperature is determined thermodynamically. The fluid equations of state may generally be written as

\[ p = p(n, T) , \ \rho = \rho(n, T) . \tag{29} \]

Differentiating the latter relation and using the balances (13) and (17) one finds

\[
\frac{\dot{T}}{T} = -\Theta \left[ \frac{\partial p/\partial T}{\partial \rho/\partial T} + \frac{\pi}{T \partial \rho/\partial T} \right] + \Gamma \left[ \frac{\partial p/\partial T}{\partial \rho/\partial T} - \frac{\rho + p}{T \partial \rho/\partial T} \right] , \tag{30} \]

where \( \partial p/\partial T \equiv (\partial p/\partial T)_n \) and \( \partial \rho/\partial T \equiv (\partial \rho/\partial T)_n \). Relation (30) was first derived in [7].

In the specific case of a classical gas the equations of state are \( p = nT \) with [20]

\[
n = \frac{4\pi m^2 T}{(2\pi)^3} K_2 \left( \frac{m}{T} \right) \exp [\alpha] , \tag{31} \]

and \( \rho = ne(T) \) with

\[
e = \frac{m K_1 \left( \frac{m}{T} \right)}{K_2 \left( \frac{m}{T} \right)} + 3T = m \frac{K_2 \left( \frac{m}{T} \right)}{K_1 \left( \frac{m}{T} \right)} - T . \tag{32} \]

The quantities \( K_n \) are modified Bessel-functions of the second kind [20]. Employing the well-known differential and recurrence relations for the latter, one gets

\[
\frac{\partial \rho}{\partial p} \equiv \frac{\partial \rho/\partial T}{\partial \rho/\partial T} = z^2 - 1 + \frac{h}{T} - \left( \frac{h}{T} \right)^2 , \tag{33} \]

\[
(\ln T)_b h^b_a . \tag{27}
\]

Within the linear theory of irreversible processes this coincides with the condition for the heat flow to vanish (see, e.g. [26]). For a comoving observer in the rotation free case (27) reduces to [15]
where \( z = m/T \) and

\[
h = e + \frac{p}{n} = m \frac{K_3(z)}{K_2(z)}, \tag{34}
\]
is the enthalpy per particle.

By virtue of (16), (21) and (22), the temperature law (30) may be expressed in terms of \( \nu \) and \( \tau \). Because of \( \frac{\partial p}{\partial T} = \frac{p}{T} \) the \( \tau \)-terms cancel and we obtain

\[
\frac{\dot{T}}{T} = -\Theta \frac{\partial p/\partial T}{\partial \rho/\partial T} + \frac{\nu n}{T \partial p/\partial T} \left[ 1 - \frac{\rho M}{n n} \right]. \tag{35}
\]

Combining the latter expression with (26) yields

\[
\nu = T \frac{1 - \frac{1}{3} \frac{\partial \rho}{\partial p}}{1 - \frac{n M}{n n}} \Theta. \tag{36}
\]

The \( \nu \)-part of the source terms (20), (21) and (22) is fixed by the thermodynamic functions of the gas. The zeroth moment \( M \) of the classical distribution function \( f^0 \) is given by

\[
M = \frac{4 \pi m T}{(2 \pi)^3} K_1 \left( \frac{m}{T} \right) \exp [\alpha]. \tag{37}
\]

There exists an alternative formula for the function \( \nu \) that may be obtained from the Gibbs-Duhem equation

\[
dp = n s dT + n d\mu. \tag{38}
\]

With [20]

\[
n s = \frac{\rho + p}{T} - n \frac{\mu}{T} \tag{39}
\]

one finds

\[
\dot{p} = (\rho + p) \frac{\dot{T}}{T} + n T \left( \frac{\mu}{T} \right). \tag{40}
\]

Using here \( p = n T \) yields

\[
\left( \frac{\mu}{T} \right) \cdot = \frac{\dot{n}}{n} - \frac{\rho \dot{T}}{p T}. \tag{41}
\]

Applying (13) with (21) for \( \dot{n}/n \), eq.(26) for \( \dot{T}/T \) and the first of eqs.(24) for \( \dot{\alpha} \equiv (\mu/T)' \), we arrive at

\[
\nu = \frac{n}{M} \left( 1 - \frac{\rho}{3p} \right) \Theta = \frac{m K_2}{3 K_1} \left[ 4 - \frac{z K_3}{K_2} \right] \Theta. \tag{42}
\]

The equivalence between the expressions (36) and (42) becomes manifest if one uses the recurrence relations for the functions \( K_n \) (see, e.g., [20]) to replace, e.g., \( K_1(z) \) in \( M \) in (36) by \( K_2 \) and \( K_3 \). With \( \nu \) given either by (36) or (42), the conformal factor \( \Psi \) in (25) is fixed as well:

\[
\Psi = -\frac{1}{3 m K_1} \left[ 4 - \frac{z K_3}{K_2} \right] \Theta, \tag{43}
\]
i.e., it is completely determined by the thermodynamic functions of the gas.

In the limit \( z \gg 1 \) we have \( \nu \approx -mz \Theta/3 \) and a corresponding expression for the conformal factor. While \( \nu \) is uniquely determined by the condition (26), the function \( \tau \) is still arbitrary. We are free to impose a further condition to fix \( \tau \).
3 Adiabatic particle production and power-law inflation

3.1 The creation rate

In [15] as well as in earlier papers ([7], [8], [9], [10], [11], [12]) the particle production was assumed to be ‘adiabatic’. This implies that all particles are amenable to a perfect fluid description immediately after their creation. The particles are produced with a fixed entropy, i.e., the entropy per particle $s$ does not change. The corresponding condition $\dot{s} = 0$ determines $\tau$. From the Gibbs equation

$$ T \dot{s} = \frac{d\rho}{n} + p \frac{d\frac{1}{n}}{n} $$

(44)

together with (13) and (15) we find

$$ nT\dot{s} = u_\alpha t^\alpha - (\rho + p) \Gamma . $$

(45)

According to (45) the condition $\dot{s} = 0$ is generally equivalent to

$$ u_\alpha t^\alpha = (\rho + p) \Gamma , $$

(46)

relating the source term in the energy balance to that in the particle number balance. Using (21) and (22) with (36) in (46) provides us with

$$ \tau^{-1} = \left( 1 - \frac{1}{3} \frac{\partial \rho}{\partial p} \right) \frac{1 - \frac{\rho + p}{n}}{1 - \frac{p}{n}} \Theta . $$

(47)

Because of (21) the particle production rate $\Gamma$ turns out to be

$$ \Gamma = \left[ 1 - \frac{1}{3} \frac{\partial \rho}{\partial p} \right] \Theta $$

(48)

in this case, equivalent to

$$ \pi = -\left[ 1 - \frac{1}{3} \frac{\partial \rho}{\partial p} \right](\rho + p) , $$

(49)

where we have used (16). In fact, expression (48) for the adiabatic particle production rate may be obtained from (26), (30), (46) and (16) without explicitly knowing $\nu$ and $\tau$. A separate knowledge of $\nu$ and $\tau$, however, will be needed in section 4 below.

Inserting into (48) the expression (33) for $\partial \rho/\partial p$ we have

$$ \Gamma = \left[ 1 - \frac{1}{3} \left( z^2 - 1 + 5 \frac{h}{T} - \left( \frac{h}{T} \right)^2 \right) \right] \Theta , $$

(50)

or an equivalent expression for $\pi$. In the limiting case $m = 0$ (radiation) with $h = 4T$, the relation (50) yields $\Gamma = 0$ in agreement with the result of [15] that
the adiabatic production of massless particles is forbidden if $\beta^a$ is a conformal Killing-vector. In the opposite limiting case $z \gg 1$ the enthalpy per particle is

$$h \approx \left[ z + \frac{5}{2} + \frac{15}{8} z^{-1} \right] T, \quad (51)$$

yielding

$$\Gamma \approx \frac{1}{2} \Theta, \quad (52)$$

i.e., the production rate for massive particles in equilibrium is half the expansion rate. Equivalently, since the Hubble factor of the Robertson-Walker metric is $H \equiv \Theta/3$, the characteristic time-scale $\Gamma^{-1}$ for the production of very massive particles, necessary for an equilibrium distribution of the latter, is $2/3$ of the Hubble time $H^{-1}$ in the case $\dot{s} = 0$.

### 3.2 Backreaction on the cosmological dynamics

The main advantage of a fluid approach to particle production is the possibility to calculate the backreaction of this process on the cosmological dynamics. For homogeneous spacetimes this problem is equivalent to studying the dynamics of a bulk viscous fluid universe [15]. While generally the determination of the bulk pressure is an involved problem on its own we are in a much better situation in the present case. The effective bulk pressure here is completely fixed by the conformal Killing-vector conditions (25), the adiabaticity condition $\dot{s} = 0$, and the equations of state. Using (49) in the energy balance (17), the latter takes the form

$$\dot{\rho} = -\Theta \rho + \rho \frac{\partial \rho}{\partial p}, \quad (53)$$

with $\partial \rho/\partial p$ given by (33). While for $m = 0$, equivalent to $p = \rho/3$, with $\pi = 0$ we recover the familiar behaviour for radiation, the opposite limiting case, $z \gg 1$, yields

$$\dot{\rho} = -\frac{3}{2} H \rho, \quad (54)$$

or, with $H = \dot{R}/R$, where $R$ is the scale factor of the Robertson-Walker metric,

$$\rho \propto R^{-3/2}. \quad (55)$$

The energy density decreases less than without particle production ($\rho \propto R^{-3}$). A parallel statement holds for $n$. From (13) with (52) we find $n \propto R^{-3/2}$ instead of $n \propto R^{-3}$ for $\Gamma = 0$.

We recall that the temperature behaviour is given by (26), irrespective of the equations of state.

Restricting ourselves to a homogeneous, isotropic and spatially flat universe with

$$3 \frac{\dot{R}^2}{R^2} = \kappa \rho, \quad (56)$$

where $\kappa$ is Einstein’s gravitational constant, and

$$\frac{\dot{H}}{H} = \frac{1}{2} \frac{\dot{\rho}}{\rho}, \quad (57)$$
where the latter follows by virtue of $3\ddot{R}/R = -(\kappa/2)(\rho + 3p + 3\pi)$, we find with the help of (54) that the scale factor $R$ behaves like

$$R \propto t^{4/3},$$

(58)

instead of the familiar $R \propto t^{2/3}$ for $\rho \propto R^{-3}$, i.e., for $\Gamma = 0$.

If massive particles are dynamically dominating, the adiabatic production rate for these particles that is necessary to keep them governed by an equilibrium distribution function backreacts on the dynamics of the Universe in a way that implies power-law inflation. In other words, massive particles with nonconserved particle number and fixed entropy per particle are allowed to be in collisional equilibrium only in a power-law inflationary universe.

The backreaction is largest for $z \gg 1$ and it vanishes for $m = 0$. In the latter case the familiar $R \propto t^{1/2}$ behaviour for radiation is recovered. There exists an intermediate equation of state yielding $R \propto t$ with $\ddot{R} = 0$. The obvious condition for $\ddot{R} = 0$ is

$$\rho + 3p + 3\pi = 0.$$  

(59)

Using the expression (49) for $\pi$, this condition reduces to

$$\frac{\partial \rho}{\partial p} = \frac{2\rho}{\rho + p}.$$  

(60)

Inserting here (33), we obtain the following cubic equation for $h/T$:

$$\left(\frac{h}{T}\right)^3 - 5\left(\frac{h}{T}\right)^2 - \left(z^2 - 3\right)\frac{h}{T} - 2 = 0.$$  

(61)

Together with (34) the latter equation determines the critical value $z_{cr}$ that characterizes the case $\ddot{R} = 0$. The numerical result is $z_{cr} \approx 9.55$. We have $\ddot{R} > 0$ for $z > z_{cr}$ while $z < z_{cr}$ corresponds to $\ddot{R} < 0$.

In terms of a $'\gamma'$-law, i.e., $p = (\gamma - 1)\rho$, the corresponding critical $\gamma$ value $\gamma_{cr}$ is

$$\gamma_{cr} = \frac{z_{cr} K_3(z_{cr})}{K_2(z_{cr})} \approx 1.09.$$  

(62)

There is power-law inflation for $1 \leq \gamma < \gamma_{cr}$. The case $\gamma = \gamma_{cr}$ corresponds to $\ddot{R} = 0$.

### 3.3 Entropy production

It is a specific feature of our approach that a gas, kept in equilibrium in the expanding universe by a nonvanishing particle creation rate exhibits a nonvanishing entropy production density

$$S_{\text{a}}^a = n\Gamma s,$$  

(63)

with $\Gamma$ from (48). While there is no entropy production due to conventional dissipative processes, it is the particle production rate itself that is connected with an increase in the entropy.
The comoving entropy $\Sigma$ is defined by $\Sigma = nsR^3$. This quantity may vary either by a change in the entropy per particle $s$, or by a varying number $N \equiv nR^3$ of particles. The first possibility was excluded here by the requirement (46). The second one, an increase in the number of $m > 0$ particles, however, was necessary for the particles to be governed by an equilibrium distribution function. Consequently, this kind of equilibrium even implies an increase in the comoving entropy.

While the latter statement sounds unfamiliar and does never hold indeed for systems with conserved particle numbers, it is a natural outcome for a nonvanishing creation rate $\Gamma$.

A further unfamiliarity, related to the previous one, is that one has nonzero entropy production in the corresponding power-law inflationary phase. In the standard inflationary scenarios the phases of accelerated expansion are not accompanied by an increase in $\Sigma$ and all the entropy is produced during a subsequent reheating period. In our approach one finds from (13), (52) and (58) in the limiting case of very massive particles $z \gg 1$, that $\Sigma \sim t^2$, i.e., the comoving entropy grows quadratically with the cosmic time.

### 4 Exponential inflation

The functions $\nu$ and $\tau$ characterizing the particle production process were considered to be input quantities on the level of classical kinetic theory. Only the specific expression (36) for $\nu$, however, guarantees the fulfillment of the equilibrium conditions (24) - (25). Once the remaining freedom in $\tau$ is fixed by some physical requirement like that of the ‘adiabaticity’ of the creation process in the preceding section, the dynamics of our model universe is uniquely determined. While the ‘adiabaticity’ condition appears ‘natural’ it is not the only possible choice. In the present section we use the freedom in $\tau$ to impose the condition of a constant Hubble rate, i.e., the condition for exponential inflation, and check whether this is consistent with the equilibrium solution of the (modified) Boltzmann equation. As was already mentioned in the introduction, the possibility of a particle- or string-production-driven de Sitter phase has recently attracted some interest ([19], [6], [9], [10], [12]). We shall show here that this feature is not consistent with the kinetic theory of a Maxwell-Boltzmann gas in collisional equilibrium.

It is obvious from (17) and (57) that the condition $H = H_0 = \text{const}$ for exponential inflation is

$$\pi = - (\rho + p) .$$

(64)

On the other hand, by (16) and (22), the effective viscous pressure is given in terms of $\nu$ and $\tau$. Using either (42) or (36) for $\nu$ and combining (16) and (22) fixes $\tau$ to be

$$\tau^{-1} = \left( \gamma + \frac{z nm}{3 \rho} \right) \Theta = \left[ \gamma - \frac{\rho}{\rho 1 - \frac{1 - \frac{1}{3} \frac{\partial p}{\partial \rho}}{1 - \frac{\rho M}{n \rho}}} \right] \Theta .$$

(65)
Solving (45) with (21) and (22) for $\tau^{-1}$ provides us with

$$\tau^{-1} = \left(1 - \frac{1}{3}\frac{\partial \rho}{\partial p}\right) \frac{1 - \frac{\mu + p M}{n \rho M}}{1 - \frac{\rho}{n}} \Theta - \dot{s}, \quad (66)$$

which differs from (47) by the nonvanishing $\dot{s}$-term. Eliminating $\tau^{-1}$ by (65) yields

$$\dot{s} = -\frac{\gamma}{3} \frac{\partial \rho}{\partial p} \Theta \quad (67)$$

for the change of the entropy per particle. The condition for exponential inflation implies a negative $\dot{s}$! As we shall show below, this decrease in the entropy per particle is compatible with the second law of thermodynamics only up to time scales of the order of the Hubble time. For larger times exponential inflation will turn out to be forbidden thermodynamically.

Instead of (63) we now have

$$S_{i\alpha} = n s \Gamma + n \dot{s} \quad (68)$$

for the entropy production rate. The particle production rate $\Gamma$ itself depends on $\dot{s}$ according to

$$\Gamma = \left(1 - \frac{1}{3}\frac{\partial \rho}{\partial p}\right) \Theta - \dot{s} \quad (69)$$

as follows from (21) with (36) and (66). A negative $\dot{s}$ enlarges the production rate $\Gamma$ compared with the case $\dot{s} = 0$. Inserting (67) into (69) it follows

$$\Gamma = \left(1 + \frac{1}{3}\frac{\partial \rho}{\partial p}\right) \Theta \quad (70)$$

The entropy production density becomes

$$S_{i\alpha} = n s \Theta - \frac{n \mu}{T} \frac{p}{3 \rho} \frac{\partial \rho}{\partial p} \Theta \quad , (71)$$

where the expression (39) has been used. $\mu$ is the chemical potential with $\mu/T \equiv \alpha$.

Equations (13) and (70) imply

$$\frac{\dot{n}}{n} = \frac{1}{3} \frac{\partial \rho}{\partial p} \Theta \quad (72)$$

i.e., the number density increases during the period of exponential inflation. This is a natural consequence of the fact that the particle production rate (70) is larger than the expansion rate.

We point out that the considerations of this section are not restricted to massive particles. Different from the adiabatic case of the previous section that did not allow for the production of ultrarelativistic particles with $z \to 0$, there is no corresponding restriction in an exponentially inflating universe. The reason for this is a simple one: Since (35) does not depend on $\tau$, any choice of the latter may be compatible with the equilibrium conditions. It was the adiabaticity condition of the previous section that introduced a proportionality
between $\tau^{-1}$ and $\nu$. Only by virtue of this additional requirement $\tau$ was related to $\nu$. Since $\nu$ necessarily vanishes for $z \to 0$ the condition for adiabatic particle production implies $\tau^{-1} \to 0$ as well. In the present case of a constant Hubble rate the corresponding condition on $\tau$ is independent of $\nu$. Consequently, even for vanishing $\nu$ a nonzero particle production due to $\tau^{-1} > 0$, that does not affect the behaviour (35) of the temperature, is possible.

With $p_r = \rho_r/3$, where the subscript ‘$r$’ refers to radiation, equation (72) yields

$$\frac{\dot{n}_r}{n_r} = H_0,$$  \hspace{1cm} (73)

equivalent to $n_r \propto R$. Since according to (26) we have $T_r \propto R^{-1}$ always, this behaviour is consistent with $\rho_r = 3n_rT_r = \text{const}$ in the de Sitter phase.

With $\nu_r = 0$ for radiation, the entropy per particle changes according to (45) with (21) and (22) as

$$\dot{s}_r = -\frac{1}{\tau_r}.$$  \hspace{1cm} (74)

Taking into account (39) and $\rho_r + p_r = 4n_rT_r$, the expression (74) is compatible with $\alpha_r \equiv \mu_r/T_r$ and the first equation (24). Furthermore, in the ultrarelativistic case the relations

$$\Gamma_r = \frac{1}{\tau_r},$$  \hspace{1cm} (75)

and

$$S_{a;a}^a = n_r/\tau_r (s_r - 1),$$  \hspace{1cm} (76)

are valid. With $\nu_r = 0$ for $p_r = \rho_r/3$ and using (16), (45) and (21), the condition (64) for exponential inflation reduces to

$$\tau_r = \frac{1}{4}H_0^{-1}.$$  \hspace{1cm} (77)

The characteristic time scale for particle production is a quarter of the Hubble time. Combining (39) and (77), the entropy production density (76) may be written as

$$S_{a;a}^a = 4n_rH_0 \left(3 - \frac{\mu_r}{T_r}\right).$$  \hspace{1cm} (78)

The quantity $\mu_r/T_r$ is given by [20]

$$\frac{\mu_r}{T_r} = \ln \left[\frac{\pi^2}{T^3}n_r\right].$$  \hspace{1cm} (79)

While $\mu_r/T_r$ is constant for $\Gamma = 0$, this is no longer true in the present case. From (39), (74) and (77) we have

$$\left(\frac{\mu_r}{T_r}\right)^\cdot = 4H_0,$$  \hspace{1cm} (80)

and, after integration,

$$\frac{\mu_r}{T_r} = \left(\frac{\mu_r}{T_r}\right)_0 + 4H_0(t - t_0).$$  \hspace{1cm} (81)
The first term on the r.h.s. corresponds to the initial value at \( t = t_0 \). The ratio \( \mu_r/T_r \) grows linearly with \( t \). From (81) and (78) it is obvious that the fulfillment of the condition \( S^a_\alpha \geq 0 \) is only guaranteed for

\[
t-t_0 \leq \frac{3}{4} H_0^{-1} \left[ 1 - \frac{1}{3} \left( \frac{\mu_r}{T_r} \right) \right].
\]  

(82)

After a time of the order of the expansion time the second law becomes violated, i.e., the corresponding process is forbidden.

Denoting the opposite limiting case \( z \gg 1 \) for nonrelativistic matter by the subscript ‘\( m \)’, we find from (72) the expression

\[
\frac{\dot{n}_m}{n_m} \approx \frac{1}{2} \frac{T_m}{m} \Theta,
\]

(83)

for the time dependence of the particle number density, i.e., in the limit \( z \gg 1 \) the number density is almost constant. The change in the entropy per particle is

\[
\dot{s}_m = -\frac{\Theta}{2}.
\]

(84)

The quantities \( \tau_m^{-1} \) and \( \nu_m \) are given by

\[
\tau_m^{-1} \approx \left( \frac{z}{3} + \frac{1}{2} \right) \Theta
\]

(85)

and

\[
\nu_m \frac{M}{n_m} \approx - \left( \frac{z}{3} - \frac{1}{2} \right) \Theta,
\]

(86)

respectively. Although \( \tau_m^{-1} \gg \Theta \) and \( |\nu| M/n_m \gg \Theta \), the terms proportional to \( z \) cancel in the expression (21) for \( \Gamma \), yielding

\[
\Gamma_m \approx \Theta,
\]

(87)

in agreement with the corresponding result from (70). Introducing \( \rho_m = n_m m + 3n_m T_m/2 \) and \( p_m = n_m T_m \) into (39), the expressions (84) and (85) are consistent with \( (\mu_m/T_m) \equiv \alpha_m = \tau_m^{-1} \), as required by (24). The growth rate of \( \alpha_m \) is much larger than the particle production rate (87).

The entropy production density in the case \( z \gg 1 \) becomes

\[
S^a_\alpha \approx \left[ \left( z - \frac{\mu_m}{T_m} \right) + \frac{5}{2} \right] n_m \Theta.
\]

(88)

The chemical potential, the number density and \( z \) are related by [20]

\[
\exp \left[ \frac{\mu}{T} \right] = \frac{2\pi^2}{m^2 T K_2(z)} n.
\]

(89)

Using for \( K_2(z) \) the asymptotic representation for \( z \gg 1 \)

\[
K_2(z) \approx \sqrt{\frac{\pi}{2z}} \exp \left[ -z \right],
\]

(90)
For $\Gamma = 0$ one has $n_m \propto R^{-3}$ and $T_m \propto R^{-2}$ and the quantity $z - \mu_m / T_m$ is a constant. In the present case we find from (39) and (84)

$$\dot{s} = \left( z - \frac{\mu_m}{T_m} \right) = -\frac{3}{2} H_0 .$$

Integration yields

$$z - \frac{\mu_m}{T_m} = \left( z - \frac{\mu_m}{T_m}_0 \right) - \frac{3}{2} H_0 (t - t_0) .$$

After a time interval

$$t - t_0 \approx \frac{5}{3} H_0^{-1} \left[ 1 + \frac{2}{5} \left( z - \frac{\mu_m}{T_m}_0 \right) \right]$$

the condition $S^a_{\alpha} \geq 0$ becomes violated, i.e., the production of massive particles is not consistent either with exponential inflation.

These results show explicitly, that the production of Maxwell-Boltzmann particles in equilibrium is compatible with a de Sitter phase only on time scales of the order of the Hubble time.

In order to avoid misunderstandings we point out that this statement does not necessarily imply that all the above mentioned fluid cosmological scenarios ([19], [6], [9], [10], [12]) are wrong. We have only excluded the case that the condition (64) for exponential inflation and the equilibrium condition (26) are fulfilled simultaneously. If the equilibrium conditions (24) - (25) are dropped, the conditions for exponential inflation are less restrictive. Retaining the equilibrium conditions corresponds to the so-called ‘global’ equilibrium, while dropping them and having nevertheless an equilibrium structure of the distribution function corresponds to the ‘local’ equilibrium case ([24], [15]). Taking into account deviations from either of the equilibrium distribution functions implies an imperfect fluid universe which requires separate investigations ([27], [28]).

Instabilities of the de Sitter universe due to quantum processes ([29, 30]) or due to viscous pressures, modelling particle production phenomena ([31]) are well known features. A breakdown of the de Sitter phase due to a violation of the condition $S^a_{\alpha} \geq 0$ as in the present paper may be regarded as thermodynamical instability. If the classical evolution of our model universe is assumed to start in a ‘global’ equilibrium de Sitter phase, our results imply the latter to become thermodynamically unstable after a time of the order of the Hubble time. On the basis of our setting there exist two qualitative possibilities for a subsequent evolution of the universe in accordance with the second law of thermodynamics. According to the first possibility the universe may leave the ‘global’ equilibrium state and stay in a ‘local’ or nonequilibrium de Sitter phase ([27, 28]). Alternatively, the universe may leave the de Sitter phase altogether but remain in ‘global’ equilibrium, being characterized, e.g., by adiabatic particle production, including the possibility of power-law inflation as in section 17.
3 above. Whether one of these scenarios will actually be realized, can only be decided once reliable knowledge about the functions $\nu$ and $\tau$, that characterize the particle production, is available from the underlying quantum physics.

5 Conclusions

Within an ‘effective rate model’ we have studied the equilibrium conditions for a Maxwell-Boltzmann gas with variable particle number. Our main findings can be summarized as: 1. Radiation and nonrelativistic matter may be in collisional equilibrium at the same temperature provided, the number of matter particles increases at a specific rate. 2. Collisional equilibrium for massive particles with adiabatically increasing particle number is only possible in a power-law inflationary universe. 3. Exponential inflation is consistent with the ‘global’ equilibrium conditions for a Maxwell-Boltzmann gas only for a time interval of the order of the Hubble time. For larger times a ‘global’ equilibrium de Sitter phase violates the second law of thermodynamics.

Acknowledgement

This paper was supported by the Deutsche Forschungsgemeinschaft, the Spanish Ministry of Education (grant PB94-0718) and the NATO (grant CRG 940598). J.T. acknowledges support of the FPI grant AP92-39172486.

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