Normalization of modes in an open universe

Juan García-Bellido
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH, United Kingdom

Andrew R. Liddle
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH, United Kingdom

David H. Lyth
School of Physics and Materials, University of Lancaster, Lancaster LA1 4YB, United Kingdom

David Wands
School of Mathematical Studies, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom

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We discuss the appropriate normalization of modes required to generate a homogeneous random field in an open Friedmann-Robertson-Walker universe. We consider scalar random fields and certain tensor random fields that can be obtained by covariantly differentiating a scalar. Modes of interest fall into three categories: the familiar sub-curvature modes, the more recently discussed super-curvature modes, and a set of discrete modes with positive eigenvalues which can be used to generate homogeneous tensor random fields even though the underlying scalar field is not homogeneous. A particular example of the last case which has been discussed in the literature is the bubble wall fluctuation in open inflationary universes.

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I. INTRODUCTION

Recently, considerable attention has been directed to models of structure formation in an open universe [1]. The description of perturbations in an open universe is more subtle than in the spatially flat case, because one has to make a distinction between the concept of a function and that of a random field. The latter can be thought of as an ensemble of functions with a probability assigned to each one. In cosmology we are interested in some finite region around us, possibly much bigger than the Hubble distance. A given perturbation about the homogeneous background, say the density perturbation, is some function of space-time in this region, but the specific form of the function is not thought to be very important. Rather, we make the hypothesis that it is a typical realization of a random field. It is assumed that the field is homogeneous with respect to the translations and rotations of the coordinate system. It is usually also assumed to be Gaussian, which means that there exists an expansion into mode functions with independent Gaussian probability distributions for the coefficients. From now on we usually take the homogeneity and Gaussianity as read, referring simply to a ‘random field’.

Let us first recall the situation for a scalar field, such as the density perturbation. In order to generate a random field (and also to decouple the time dependence of the modes in the linear approximation) the mode functions will be eigenfunctions of the Laplacian in comoving coordinates. In a flat universe, the most general square-integrable function may be expanded in terms of the eigenfunctions with \( k^2 > 0 \) (as usual \( -k^2 \) is the eigenvalue of the Laplacian). In that case these same functions also give the most general scalar random field. In an open universe the situation is different. In units of the curvature scale, the most general square-integrable function may be expanded using only the \( k^2 > 1 \) modes (sub-curvature modes). In particular a perturbation defined in any finite region can be expanded, which is all we need for any cosmological application. But now the situation for a random field is different; to obtain the most general scalar random field one needs, in addition, the \( 0 < k^2 \leq 1 \) modes (super-curvature modes). This fact has been known to mathematicians for half a century [2,3], though it has only recently been brought to the attention of the astrophysics community [4]. One of the objects of the present paper is to derive, in a simple way, the normalization of the mode functions which will lead to a homogeneous random field (with a natural choice for the variances of the coefficients).

In cosmology we might also be interested in vector and (second rank or higher) tensor functions, and the corresponding random fields. In general such objects have to be expanded in terms of different mode functions, which we shall not consider. But an important special case arises when the vector or tensor is the spatial derivative of a scalar. The central purpose of this paper is to treat this case, and focus on a fact concerning the open universe which does not seem to have been discussed in the mathematics literature. Namely, that a homogeneous vector or tensor random field can be constructed by differentiating an inhomogeneous scalar random field. We
The normalization of the conjecture that the most general random field derived from a scalar can be expanded in terms of the continuum modes plus some new mode functions with \( k^2 = 1 - n^2 \) (discrete modes) where \( n \) is an integer. We support our conjecture by displaying an explicit normalization of the mode functions for the cases \( k^2 = 0 \) and \( k^2 = -3 \) which is suitable for generating a tensor random field. Examples of both cases exist in the recent literature. A discrete mode with \( k^2 = 0 \) is generated by fluctuations a massless scalar field in de Sitter space-time [5] and the effect on the microwave background of such a mode was evaluated in Ref. [6]. A discrete \( k^2 = -3 \) mode can be generated by fluctuations of the bubble wall [7,8], in single-bubble models of open universe inflation [9].

To be more specific, we write down the mode expansion for a scalar random field in terms of eigenfunctions of the spatial Laplacian, \( \nabla^2 Z_{klm} = -k^2 Z_{klm} \):

\[
 f(r, \theta, \phi) = \int dk \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{klm} Z_{klm}(r, \theta, \phi),
\]

where the \( f_{klm} \) are members of an ensemble. The possible values of \( k \) will be discussed in what follows. We will consider both a continuous spectrum of modes and discrete values of \( k \). We work in a spherical coordinate system with line element

\[
 ds^2 = dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2),
\]

corresponding to the homogeneous spatial hypersurfaces in an open Friedmann–Robertson–Walker universe. The normalized eigenfunctions can be written as

\[
 Z_{klm}(x) = \Pi_{kl}(r) Y_{lm}(\theta, \phi),
\]

where \( Y_{lm}(\theta, \phi) \) are the usual spherical harmonics on the two-sphere [10] and \( \Pi_{kl}(r) \) are eigenfunctions of the operator

\[
 \frac{1}{\sinh^2 r} \frac{d}{dr} \left( \sinh^2 r \frac{d}{dr} \right) + \frac{l(l+1)}{\sinh^2 r}.
\]

The normalization of the \( \Pi_{kl}(r) \) is the subject of the present paper.

In a Gaussian random field, if one uses the complex form of the spherical harmonics the magnitudes \( |f_{klm}| \) of each coefficient have independent Gaussian probability distributions for \( m \geq 0 \). The reality condition

\[
 f_{kl,m} \Pi_{kl} = (-1)^m (f_{klm} \Pi_{kl})^*,
\]

which follows from \( Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi) \), fixes the phase of the \( m = 0 \) modes, while the other modes have uniformly distributed random phases subject to the above equation. In what follows we will find it more convenient to use the real form of the \( Y_{lm}(\theta, \phi) \); the coefficients then have fixed phases, \( \arg(f_{klm}) = -\arg(\Pi_{kl}) \), and the magnitudes of \( f_{klm} \) are independent random variables for all \( m \) from \(-l\) to \(+l\).

The variance of the distribution is defined by

\[
 \langle f^*_{klm} f_{k'l'm'} \rangle = A_k P(k) \delta(k-k') \delta_{lm'm'},
\]

where the brackets denote an ensemble average. As we shall discuss in the next subsection, the variance is taken to be independent of \( l \) and \( m \) to allow us to construct a homogeneous random field. Several different conventions exist in the literature concerning the definition of the power spectrum \( P(k) \), corresponding to different choices of the prefactor \( A_k \). In what follows we shall not need to define a particular separation. Since the field is Gaussian, it is completely defined by the two-point correlation function. For the continuous case this is defined by

\[
 \langle f^*(x_1) f(x_2) \rangle = \int dk A_k P(k) \sum_{lm} Z^*_{klm}(x_1) Z_{klm}(x_2),
\]

and in the discrete case the integral over \( k \) is replaced by a sum. Here and throughout we use the abbreviated notation

\[
 \sum_{lm} \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{l}.
\]

\[\text{II. UNITARITY AND HOMOGENEITY}\]

The above procedure generates a scalar Gaussian random field. The field is said to be homogeneous if the correlation function depends only on the geodesic distance between the two points. By covariantly differentiating such a field, one can obtain a tensor random field which is likewise homogeneous if its correlation function is unaffected by a coordinate change (except of course for the transformation that the change induces in the components of the tensor). It is obvious that differentiating a homogeneous scalar random field always gives a homogeneous tensor random field, but we shall demonstrate that in some circumstances one can also obtain a homogeneous tensor field by differentiating an inhomogeneous scalar field.

The central purpose of this paper is to ask what restriction is placed on the normalization of the modes by the requirement that the random field be homogeneous, and to discuss which values of \( k \) are compatible with this requirement. The homogeneity requirement, that the correlation function depends only on the geodesic distance between the points, is equivalent to the requirement that under a shift in the origin or orientation of the coordinate system the Gaussianity is preserved and the correlation function is unaltered. We begin by showing that for a scalar field this is the case if and only if such a coordinate change corresponds to a unitary transformation of
which requires the matrix and the mode expansion in Eq. (9) becomes

\[ f(r, \theta, \phi) = \sum_{lm} f_{lm}(r, \theta, \phi) Z_{klm}(r, \theta, \phi) . \]  
(9)

The joint probability distribution for the coefficients is

\[ \text{probability} = N \exp \left( - \sum_{lm} |f_{lm}|^2 \frac{1}{2A_kP(k)} \right) , \]  
(10)

(where the normalization factor \( N \) is actually infinitesimal because the sum is infinite).

Under a change of coordinates, the mode functions undergo a linear transformation

\[ Z_{klm}(r, \theta, \phi) = \sum_{l'm'} U_{lm,l'm'}^k Z_{klm'}(r', \theta', \phi') , \]  
(11)

and the mode expansion in Eq. (9) becomes

\[ f(r, \theta, \phi) = \sum_{lm} f_{lm} U_{lm,l'm'}^k \sum_{l'm'} Z_{klm'}(r', \theta', \phi') , \]  
(12)

where

\[ f_{lm}^* = \sum_{l'm'} U_{lm,l'm'}^k f_{lm} . \]  
(13)

The form of the joint probability distribution, Eq. (10), with respect to the transformed coefficients is clearly unaltered if and only if

\[ \sum_{l'm'} |f_{lm'}^*|^2 = \sum_{lm} |f_{lm}|^2 , \]  
(14)

which requires the matrix \( U \) to be unitary:

\[ \sum_{lm} (U_{lm'l'm'})^* U_{lm'l'm'}^k = \delta_{ll'} \delta_{mm'} . \]  
(15)

Although we have shown this only for modes with a single value of \( k \), the corresponding expression for a continuous spectrum of modes will also be unaffected, since the transformation does not act on \( k \). One can show that the correlation function defined by Eq. (7), or the spectrum in Eq. (6), are also unaffected for a unitary transformation between modes. Thus the field is homogeneous if and only if \( U \) is unitary.

*The invariance of the correlation function was shown in Ref. [4]. As in that reference we treat infinite sums as finite, which should be valid if the infinite sum Eq. (7) is uniformly convergent. That has been demonstrated in Ref. [5] for \( k^2 > 0 \), but we have not investigated the question for \( k^2 \leq 0 \).

The unitarity requirement allows one to change the normalization of the modes by an arbitrary \( k \)-dependent real factor, and a completely arbitrary phase. What is important is the dependence of the magnitude of the normalization factor on \( l \) and \( m \). The normalization of the spherical harmonics \( Y_{lm}(\theta, \phi) \) ensures that they transform unitarily under an arbitrary rotation (and hence the distribution defined in Eq. (6) is isotropic), but we have to ensure that the complete basis functions \( Z_{klm} \) transform unitarily under both rotations and shifts of origin.

Note that a rotation about a fixed origin leaves \( k \) and \( l \) fixed but mixes the different \( m \)-multipoles, while a shift along the \( \theta = 0 \) axis leaves \( k \) and \( m \) fixed but mixes different \( l \) multipoles. Because an arbitrary shift and rotation of the origin can be decomposed into a rotation, followed by a shift along \( \theta = 0 \), followed by another rotation, the unitarity of the spherical harmonics under rotations fixes the \( m \)-dependence of the normalization. We now seek the correct \( l \)-dependence of the normalization of the radial functions \( \Pi_{kl}(r) \) which ensures homogeneity under a shift of the origin.

### III. HOMOGENEOUS SCALAR RANDOM FIELDS

#### Sub-curvature modes

For modes with \( k^2 > 1 \), we can split \( \Pi_{kl}(r) \) as

\[ \Pi_{kl}(r) = N_{kl} \tilde{\Pi}_{kl}(r) , \]  
(16)

where \( N_{kl} \) is a normalization factor, to be determined, and \( \Pi_{kl}(r) \) are the unnormalized functions [11]

\[ \tilde{\Pi}_{kl}(r) = q^{-2} \sinh q_r \left( \frac{1}{\sinh q_r} \right)^{l+1} \cos(qr) , \]  
(17)

where \( q^2 = k^2 - 1 \). For these modes, \( q = \pm \sqrt{k^2 - 1} \) is real and the eigenfunctions are exponentially decreasing beyond the curvature scale [which equals unity for the line element in Eq. (2)]. It is possible to choose the normalization factor \( N_{kl} \) to give an orthogonality relation between the modes (not necessarily orthonormality) of the form

\[ \int dV Z_{klm}(x) Z_{kl'm'}(x) = B_k \delta(|q| - |q'|) \delta_{mm'} , \]  
(18)

where \( B_k \) is a finite real function of \( k \), independent of \( l \) and \( m \). This relation ensures unitarity of the matrix \( U \) for a coordinate shift [4], as we show in Appendix A. In particular, the usual normalization factor is taken to be [11]

\[ N_{kl} = \sqrt{\frac{2}{\pi}} q^2 \prod_{s=0}^{l} (s^2 + q^2)^{-1/2} , \]  
(19)
and the resulting correlation function falls away exponentially, which leaves these modes normable. These modes are not corresponding to $B_k = 1$ in Eq. (18).

Note that the radial functions $\tilde{R}_k(r)$ and the normalization factor given by Eq. (19) vanishes as $N_k \propto q$, for all multipoles. In the absence of any physical reason why the amplitude of these modes should be zero, one can construct a homogeneous field using an alternative normalization factor

$$\tilde{N}_k = \lim_{q \to 0} \frac{N_k}{q} = \sqrt{\frac{2}{\pi}} \frac{1}{k},$$

which leaves these modes finite. These modes are not normalizable, in the sense that the corresponding $B_k = 1/q^2$ in Eq. (18) diverges as $q \to 0$. However, orthogonality of the transformation matrix, Eq. (20), still holds in the analytic limit, which ensures the homogeneity of the field generated from the $k^2 = 1$ modes.

Super-curvature modes

For modes corresponding exactly to the curvature scale $q = 0$, the normalization factor given by Eq. (19) vanishes as $N_k \propto q$, for all multipoles. In the absence of any physical reason why the amplitude of these modes should be zero, one can construct a homogeneous field using an alternative normalization factor

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For super-curvature modes the wavenumber lies in the range $0 < k^2 < 1$, so $q = \pm i \sqrt{1 - k^2}$ is purely imaginary. The mode functions can be obtained from the super-curvature modes by analytic continuation, giving

$$\tilde{R}_k(r) = -k |q|^{-2} \left( \frac{d}{\sinh kr} \right)^{l+1} \cosh(|q|r).$$

These eigenfunctions extend beyond the curvature scale, and the resulting correlation function falls away exponentially only on scales greater than $1/k$.4

These mode functions are not normalizable, and neither are they linearly independent of the sub-curvature modes in any finite region (such as our observable universe). Nevertheless, they can be used to construct a more general random field than is possible from the sub-curvature modes alone [4]. Because there is no orthogonality relation as in Eq. (18), one cannot fix the normalization of these modes in the way we did for the sub-curvature modes. However, if we use the analytic continuation of the normalization factor for the sub-curvature modes

$$N_{kl} = \sqrt{\frac{2}{\pi}} |s|^l \prod_{i=s}^{l} (s^2 - |s|^2)^{-1/2},$$

which is purely imaginary for all $l$, the transformation matrix $U$ will also be an analytic continuation. We know that Eq. (15) holds in the sub-curvature regime, and in fact it also holds in the super-curvature regime. At first sight it appears that we cannot appeal to the uniqueness of analytic continuation to demonstrate this, because the left hand side of Eq. (15) is not holomorphic. However, if real mode functions are used the unitarity relation becomes the orthogonality relation Eq. (20), which is preserved under the analytic continuation into the super-curvature regime. The transformation matrices $U$ remain real in this regime because all the $N_{kl}$ in Eq. (23) are purely imaginary, and so Eq. (20) implies unitarity and homogeneity.

IV. POSITIVE-EIGENVALUE MODES

Modes with positive eigenvalues not only extend beyond the curvature scale, but actually diverge as $r \to \infty$. However we have seen that the super-curvature modes with $0 < k^2 < 1$ can form a homogeneous random scalar field despite being non-square integrable, so one might ask whether a similar analytic continuation from the normalized sub-curvature modes might also give a homogeneous random field for $k^2 < 0$.

For $k^2 < 0$, the unnormalized radial functions given by Eq. (22) remain real; however the monopole normalization factor in Eq. (23) is purely imaginary, while the dipole acquires an extra factor of $i$ and becomes real. Thus the normalized mode functions no longer have a unique phase. This means that the transformation matrices $U$ can no longer be purely real, unless the modes of differing phase do not mix with one another under a change of coordinates. Thus Eq. (20), although it still holds, no longer guarantees unitarity, Eq. (15).4

For instance, for $-3 < k^2 < 0$, the $l = 1$ multipoles are all real but the monopole is purely imaginary. One might hope to build a homogeneous random field from only the higher multipoles, but the orthogonality relation, Eq. (20), involves a sum over all the multipoles. Moreover a distribution with no monopole in one coordinate system will in general acquire a monopole term after a shift of the origin, which implies inhomogeneity.

More generally, for $1 - (n+1)^2 > k^2 > 1 - n^2$ (where $n \geq 1$ is an integer), the normalization given by Eq. (23) has a phase $(l-1)\pi/2$ for $l < n$, while for all $l \geq n$ it has the same phase, $(n-1)\pi/2$. A shift of the origin mixes all the multipoles and any attempt to construct a random

4We have not been able to prove that there is no normalization for which the transformation between modes with different phases becomes unitary, only that the analytic continuation from sub-curvature modes does not give a unitary transformation.
Homogeneous tensor fields when $k^2 = 1 - n^2$

An interesting situation arises for the discrete set of modes when $k^2 = 1 - n^2$ with $n \geq 1$ an integer. These modes can be obtained from the discrete closed universe modes with $k^2 = n^2 - 1$ by analytically continuing both the radial coordinate $r \to ir$ and the wavenumber $k \to ik$ [11].

Note that Eqs. (22) and (23) give finite expressions for the first $n$ multipoles, while for the higher multipoles $N_{kl} \tilde{N}_{kl}(r) = O(\epsilon)$, where $\epsilon^2 = n^2 - 1 + k^2$. An alternative is to define

$$\tilde{N}_{kl} \equiv \lim_{\epsilon \to 0} \epsilon N_{kl},$$

$$\tilde{\Pi}_{kl} \equiv \lim_{\epsilon \to 0} \frac{\tilde{\Pi}_{kl}}{\epsilon^2},$$

which gives finite expressions for the $l \geq n$ multipoles, while with this normalization the lower multipoles diverge as $1/\epsilon$. From Eqs. (17) and (29), we have, for $l \geq n$,

$$\tilde{\Pi}_{kl} = \frac{1}{2n^2} (\sinh r)^l \left( \frac{-1}{\sinh r} \frac{d}{dr} \right)^{l+1} (r \sinh r).$$

Rescaling all the mode functions independently of $l$ and $m$ does not change the transformation matrix $U$.

We show in Appendix B that the matrix $U$ becomes block diagonal in the limit $\epsilon \to 0$, so that the orthogonality relation, Eq. (20), holds separately for transformations between the $l < n$ multipoles and for transformations between the $l \geq n$ multipoles.

In a closed universe it is well known that the $l < n$ multipoles form a closed unitary group under coordinate transformations. In an open universe, while the $l < n$ modes still form a closed group, the mode functions have alternating phases and so the transformation matrix is not purely real. Thus the orthogonality relation no longer implies unitary, and so one cannot form a homogeneous random scalar field.

By contrast, the $l \geq n$ multipoles all have the same phase in an open universe (whereas in a closed universe they have alternating phases). Hence the sub-matrix of $U$ connecting the higher multipoles is real, and the transformation is unitary. One might think that it is possible to construct a homogeneous scalar random field from the $l \geq n$ multipoles alone. However this is not possible, because the lower multipoles can be regenerated by a coordinate transformation. This is despite the fact that $U$ becomes block diagonal, as the contribution from the diverging low multipoles given in Eqs. (24) and (25) remains finite in the limit $k^2 \to 1 - n^2$ even though the matrix elements approach zero.

However if we act on the scalar field with an operator which kills the lower multipoles, we obtain a homogeneous tensor random field even though the underlying scalar field is inhomogeneous. We now discuss two physical cases in which this does in fact occur.

$k^2 = 0$ modes

As $k^2 \to 0$, the normalization factor for the monopole in Eq. (23) gives $N_{00} = i \sqrt{2/\pi}$, while the higher multipoles diverge as $N_{kl} \propto i/k$. At the same time, the unnormalized monopole $\tilde{N}_{00} \to 1$, while the mode functions given by Eq. (22) vanish for $l \geq 1$ as $\tilde{\Pi}_{kl} \propto k^l$. One can construct a finite field from the multipoles $l \geq 1$ if we use the functions $[4,6]^\dagger$

$$\tilde{N}_{kl} \equiv \lim_{k \to 0} kN_{kl},$$

$$\tilde{\Pi}_{kl} \equiv \lim_{k \to 0} \frac{\tilde{\Pi}_{kl}}{k^2},$$

which leaves these modes finite (although the monopole becomes infinite with this normalization). Such a mode appears if one considers the quantum fluctuations of a massless scalar field in de Sitter space-time using an open universe coordinate system [5]. The anisotropy of the microwave background sky due to curvature perturbations with $k^2 = 0$ in an open universe has also been discussed recently [6].

The monopole $N_{00} \tilde{\Pi}_{00}(r) Y_{00}(\theta, \phi)$ is a constant when $k^2 = 0$. As a result any tensor field constructed by covariant differentiation of the scalar field will be homogeneous. The simplest example is the vector field

$$V^i \equiv \nabla^i f.$$

Note that, in the notation of Ref. [12], $V^i$ is indistinguishable from an intrinsically ‘vector’ quantity when $k^2 = 0$, as it is solenoidal, $\nabla_i V^i = 0$.

$k^2 = -3$ modes

Another interesting case of the $k^2 = 1 - n^2$ modes discussed above is $n = 2$. In the limit $k^2 \to -3$, the monopole normalization is $N_{10} = 2i \sqrt{2/3\pi}$ and that of the dipole is $N_{21} = 2\sqrt{2/3\pi}$, while the normalizations of the higher multipoles diverge as $N_{kl} \propto 1/\epsilon$, where $\epsilon^2 = 3 + k^2$. On the other hand, the radial functions behave as $\tilde{\Pi}_{00} \to \cosh r$, $\tilde{\Pi}_{11} \to -\sinh r$, and $\tilde{\Pi}_{kl} \propto \epsilon^2$ for $l \geq 2$. We can use the normalization given in Eqs. (24) and (25) to render the $l \geq 2$ multipoles finite.

\dagger Note there is a typographical error in Eq. (100) of Ref. [4].
To construct a homogeneous tensor random field in this l-dependent phase in the sub-curvature regime, they would still be suitably normalized there but the continuation of the phase factor to the super-curvature regime would in general spoil the normalization of the super-curvature modes. An example of this would be to replace the first factor in Eq. (33) by \( \Gamma(l + 1 + iq)/\Gamma(iq) \). Both normalizations are equivalent in the sub-curvature regime, but the latter is not suitable for analytic continuation to the super-curvature regime, where it gives an incorrect normalization and also fails to be symmetric under \( q \leftrightarrow -q \). Although Lyth and Woszczyna [4] and Hamazaki et al. [7] both quoted this latter form for the sub-curvature modes, they did not use it directly for analytic continuation and in fact both these papers obtained the satisfactory super-curvature mode normalization given by Eq. (33) above.

For \( k^2 \leq 0 \) there is no unique phase, and therefore one cannot use the analytically continued mode functions to construct a homogeneous scalar random field. However, in the specific case \( k^2 = 1 - n^2 \), for integer \( n \), the transformation between multipoles with \( l \geq n \) is unitary, which allows a homogeneous tensor random field to be constructed by acting on the scalar with a covariant differential operator, provided that the operator kills the lower multipoles. For \( k^2 = 0 \) we have noted that any differential operator does this, since the monopole is spatially constant. For \( k^2 = -3 \) we have seen that the traceless symmetric second-rank tensor does this. A physical example of the latter is the metric perturbation associated with quantum fluctuations of the bubble wall in open inflation models [7,8].

The lowest non-vanishing modes are the quadrupole and octopole whose radial dependence is given from Eq. (26) as

\[
\hat{\Pi}_{l2} = -\frac{1}{8} \left( \frac{2 \cosh r - 3 \cosh r}{\sinh^2 r} - \frac{3r}{\sinh^2 r} \right),
\]

\[
\hat{\Pi}_{l3} = -\frac{1}{8} \left( \frac{2 \sinh r}{\sinh r} - \frac{5}{\sinh^2 r} - \frac{15r \cosh r}{\sinh^2 r} + \frac{15r \cosh r}{\sinh^4 r} \right).
\]

Note that these higher multipoles of the scalar field diverge as \( \hat{\Pi}_{l1} \sim e^r \) as \( r \to \infty \). The action of the tensor operator renders some components of \( T_{ij} \) finite at infinity, such as \( T_{rr}, T_{r\theta} \) and \( T_{\theta \phi} \) which are of order \( e^{-r} \), but the remaining components of \( T_{ij} \) still diverge as \( e^r \). Nonetheless, due to the form of the metric inverse \( [\gamma^{ij}] = \text{diag}(1, 1/\sinh^2 r, 1/\sinh^2 \theta \sinh^2 r) \) this is sufficient to leave scalar invariants finite at infinity, e.g., \( T_{ij} T^{ij} \sim e^{-2r} \).

**V. CONCLUSIONS**

To summarize, the normalization defined by Eq. (19) for sub-curvature modes can be used to generate a homogeneous field because of the orthogonality relation in Eq. (18). We have shown that it remains valid in the super-curvature regime, by virtue of the fact that the radial mode functions for a given eigenvalue \( k^2 \) have the same phase. These normalized eigenfunctions can be written, for both sub- and super-curvature modes, as

\[
\Pi_{nl}(r) = \left[ \frac{\Gamma(l + 1 + iq)\Gamma(l + 1 - iq)}{\Gamma(iq)\Gamma(-iq)} \right]^{1/2} \times \frac{P_{n-1/2}(-i/\sqrt{\sinh r})}{\sqrt{\sinh r}}.
\]

where \( q^2 = k^2 - 1 \). For sub-curvature modes, these functions are real, while for super-curvature modes they are purely imaginary.
APPENDIX A: MODE ORTHOGONALITY AND TRANSFORMATION UNITARITY

Here we prove that the orthogonality of the sub-curvature modes implies that the transformation induced by shifts in origin and orientation of coordinates is unitary. An alternative proof was given in Ref. [4]. This means that a random field generated from them is homogeneous.

Substituting the transformation

$$Z_{klm}(r,\theta,\phi) = \sum_{l'm'} U_{lml'l'm'}^k Z_{k'l'm'}(r',\theta',\phi'),$$  \hspace{1cm} (A1)

into the orthogonality relation

$$\int dV Z_{klm}^*(x) Z_{k'l'm'}(x) = B_k \delta(|q| - |q'|) \delta_{ll'} \delta_{mm'},$$  \hspace{1cm} (A2)

gives

$$\int dV Z_{klm}^*(x) Z_{k'l'm'}(x) = \int dV \sum_{l'm'} (U_{lml'l'm'}^k)^* Z_{k'l'm'}^*(x') \times \sum_{l'm'} U_{lml'l'm'}^k Z_{k'l'm'}(x')$$
$$= \sum_{l'm'} (U_{lml'l'm'}^k)^* U_{lml'l'm'}^k B_k \delta(|q| - |q'|),$$  \hspace{1cm} (A3)

where the last equality uses the orthogonality relation, Eq. (A2), for the $x'$ coordinates. Thus we have

$$\sum_{l'm'} (U_{lml'l'm'}^k)^* U_{lml'l'm'}^k = \delta_{ll'} \delta_{mm'},$$  \hspace{1cm} (A4)

and hence the transformation is unitary.

Note that orthogonality also gives us an expression for the matrix $U_{lml'm'}^k$, namely

$$\int dV Z_{k'l'm'}^*(x') Z_{klm}(x)$$
$$= \int dV Z_{k'l'm'}^*(x') \sum_{l'm'} U_{lml'l'm'}^k Z_{k'l'm'}(x')$$
$$= U_{lml'm'}^k B_k \delta(|q| - |q'|).$$  \hspace{1cm} (A5)

APPENDIX B: BLOCK DIAGONALITY OF $U$ FOR $k^2 = 1 - n^2$

In this Appendix we shall demonstrate that the matrix $U$ becomes block diagonal in the limit $k^2 \to 1 - n^2$ for integer $n \geq 1$. We split the transformation matrix $U$ into block matrix form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$  \hspace{1cm} (B1)

where $A_{ll'm'm'} = U_{ll'm'm'}^l$ for $l,l' < n$ and so on. Then the orthogonality condition $UU^T = I$ gives

$$AA^T + BB^T = I, \hspace{1cm} (B2)$$
$$AC^T + BD^T = 0, \hspace{1cm} (B3)$$
$$CA^T + DB^T = 0, \hspace{1cm} (B4)$$
$$CC^T + DD^T = I, \hspace{1cm} (B5)$$

where $I$ is the identity matrix.

We adopt the normalization of the mode functions given in Eqs. (24) and (25), which leaves the $l \geq n$ multipoles finite but leads to the $l < n$ multipoles diverging as $1/\epsilon$ where $k^2 = 1 - n^2 + \epsilon^2$. Consider a function composed solely of modes with $k^2 = 1 - n^2 + \epsilon^2$

$$f_k(x) = \sum_{lm} f_{lm} \tilde{N}_{kl}(r) Y_{lm}(\theta, \phi).$$  \hspace{1cm} (B6)

If this is to remain bounded at finite $r$ as $\epsilon \to 0$, we must have $f_{lm} = O(\epsilon)$ for $l < n$ and $f_{lm}$ finite for $l \geq n$. Under a coordinate transformation

$$f_k(x) = \sum_{l'm'} f'_{l'm'} \tilde{N}_{kl}(r') Y_{l'm'}(\theta', \phi'),$$  \hspace{1cm} (B7)

where $f'_{l'm'}$ is given by Eq. (13), and we must likewise have $f'_{l'm'} = O(\epsilon)$ for $l' < n$.

Thus the sub-matrix $C = O(\epsilon)$. In the limit $\epsilon \to 0$, Eq. (B5) becomes $DD^T = I$ and Eq. (B4) becomes $DB^T = 0$. Because $D$ is orthogonal this implies $B = 0$, and hence the matrix $U$ is block diagonal. Finally, we note from Eq. (B2) that $AA^T = I$.

APPENDIX C: $k^2 = -3$ TENSOR MODES

In this Appendix we give the actual components of the (symmetric) operator $T_{ij}$ given in Eq. (30) for the metric of Eq. (2),

$$T_{rr} = \frac{\partial^2}{\partial r^2} - \frac{1}{3} \nabla^2,$$  \hspace{1cm} (C1)
$$T_{\theta\theta} = \frac{\partial^2}{\partial \theta^2} + \sin \theta \left( \cosh \theta \frac{\partial}{\partial r} - \frac{1}{3} \sinh \theta \nabla^2 \right),$$  \hspace{1cm} (C2)
$$T_{\phi\phi} = \frac{\partial^2}{\partial \phi^2} + \sin \theta \cos \theta \frac{\partial}{\partial \theta}.$$
\[ \frac{1}{2}\sin^2 \theta \sinh r \left( \cosh r \frac{\partial}{\partial r} - \frac{1}{3} \sinh r \nabla^2 \right), \tag{C3} \]

\[ T_{r \theta} = \left( \frac{\partial}{\partial r} - \frac{\cosh r}{\sinh r} \right) \frac{\partial}{\partial \theta}, \tag{C4} \]

\[ T_{r \phi} = \left( \frac{\partial}{\partial r} - \frac{\cosh r}{\sinh r} \right) \frac{\partial}{\partial \phi}, \tag{C5} \]

\[ T_{\theta \phi} = \left( \frac{\partial}{\partial \theta} - \frac{\cos \theta}{\sin \theta} \right) \frac{\partial}{\partial \phi}. \tag{C6} \]

The unnormalized \( k^2 = -3 \) monopole and dipole modes are given, using Eq. (22), by

\[ \hat{\Pi}_{00}(r)Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \cosh r, \tag{C7} \]

\[ \hat{\Pi}_{10}(r)Y_{10}(\theta, \phi) = -\sqrt{\frac{3}{4\pi}} \sinh r \cos \theta, \tag{C8} \]

\[ \hat{\Pi}_{11}(r)Y_{11}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sinh r \sin \theta \cos \phi, \tag{C9} \]

\[ \hat{\Pi}_{1(-1)}(r)Y_{1(-1)}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sinh r \sin \theta \sin \phi. \tag{C10} \]

It is then straightforward to verify that each component of \( T_{ij} \) vanishes everywhere when applied to the monopole and dipole modes. Thus the action of the operator in Eq. (30) on the \( k^2 = -3 \) modes can indeed form a homogeneous tensor field from the \( l \geq 2 \) multipoles of the scalar field whose normalized radial functions are given by Eqs. (24) and (25).


