Critical exponents in abelian projected $SU(2)$ QCD

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The critical properties of the abelian Polyakov loop and the Polyakov loop in terms of Dirac string are studied in finite temperature abelian projected $SU(2)$ QCD. The critical point and the critical exponents are determined from each Polyakov loop in the maximally abelian gauge using the finite-size scaling analyses. Those critical points and exponents are in good agreement with those from non-abelian Polyakov loops.

1. Introduction

Abelian projected QCD is regarded as an abelian theory with electric charges and monopoles. Dual Meissner effect based on the condensation of the monopoles can be considered as the mechanism of confinement of quarks. This picture is likely to be realized at least in the maximally abelian gauge: the value of the string tension and the behavior of Polyakov loops are reproduced by the abelian link fields and by the monopoles (Dirac strings) \cite{1−3}; the effective monopole action is calculated and it indicates that QCD is always in the monopole condensed phase from the comparison of the energy and the entropy \cite{4}.

Figure 1\cite{3} shows the abelian and the monopole Polyakov loops defined later appear to be good order parameters. However, those curves seem to have different slopes. Their absolute values in the deconfinement phase are also different. The critical behavior of 4-dimensional $SU(2)$ lattice gauge theory is shown to be the same as that of 3-dimensional $Z_2$ theory. Since the abelian and the monopole Polyakov loops seem to be good order parameters, it is interesting to evaluate critical exponents and critical points exactly from these Polyakov loops. What kind of universality class will appear from these abelian quantities?

2. Definitions of Polyakov loops

After abelian projection is over, we can define abelian Polyakov loops\cite{1} written in terms of abelian link variables $u_\mu(x,t)$. We separate $u_\mu(x,t)$ from gauge-fixed link variables $\tilde{U}_\mu(x,t)$:

$\tilde{U}_\mu(x,t) = A_\mu(x,t) u_\mu(x,t)$

We define an abelian Polyakov loop

$P_{abel}(x_0) = \text{Re} \{ \text{exp}(i \sum \theta_4(x,t) J_4(x,t)) \}$. (1)

Here $J_\mu(x,t) = \delta_\mu, 4 \delta_x, x_0$ and $\theta_\mu(x,t)$ are the angle variables of $u_\mu(x,t) = \text{exp}(i \theta_\mu(x,t) \sigma_3)$, where $\sigma_3$ is a Pauli matrix.

The abelian Polyakov loop can be decomposed into two parts: a monopole part and a photon part\cite{3}. The abelian field strength $f_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu$ can be separated into two parts: $f_{\mu\nu} = \tilde{f}_{\mu\nu} + 2\pi n_{\mu\nu}$, where $n_{\mu\nu}$ is an integer and $\tilde{f}_{\mu\nu} \in [-\pi, \pi)$. Then, rewriting Eq.(1), we get

$P_{abel}(x_0) = \text{Re} \{ P_1(x_0) \cdot P_2(x_0) \}$,
\[ P_1(x_0) = \exp(-t \sum D(x - x', t - t') \times \partial_s \bar{f}_{s4}(x', t') J_4(x, t)), \]
\[ P_2(x_0) = \exp(-2\pi t \sum D(x - x', t - t') \times \partial_s \bar{n}_{s4}(x', t') J_4(x, t)). \]

Here \( D(x, t) \) is a lattice Coulomb propagator which satisfies \( \partial_{\mu} \partial_{\mu} D(x, t) = -\delta_{x,0} \delta_{t,0} \). The monopole Polyakov loop, \( P_{\text{mono}}(x_0) = \text{Re} \, P_1(x_0) \) is composed of Dirac strings of monopoles. Suzuki et al.[3] have indicated that \( P_{\text{abel}}(x_0) \) and \( P_{\text{mono}}(x_0) \) vanish in the confinement phase, whereas \( P_{\text{photon}} = \text{Re} \, P_2(x_0) \) is finite at the range from \( \beta = 2.1 \) to 2.5 and did not change drastically around the critical point.

3. Finite-size scaling theory

We calculated the critical exponent of the non-abelian, the abelian and the monopole Polyakov loops from a finite-size scaling theory. The singular part of the free energy density on \( N_s^3 \times N_t \) lattice has the following form:

\[ f_s(x, h, N_s) = N_{s}^{-\beta/\nu} Q_s(x N_s^{1/\nu}, h N_s^{(\beta+\gamma)/\nu}, g_s N_s^{-\omega}), \]

where \( x = (T - T_c)/T_c \). Here the action contains the term \( h N_s^2 L \) (\( L \)) denotes the magnetization) and only the largest irrelevant exponent \( (\omega) \) is taken into account. By differentiating \( f_s \) with respect to \( h \) at \( h = 0 \), we get

\[ L(x, N_s) = N_s^{-\beta/\nu} Q_s(x N_s^{1/\nu}, g_s N_s^{-\omega}), \]
\[ \chi(x, N_s) = N_s^{\gamma/\nu} Q_s(x N_s^{1/\nu}, g_s N_s^{-\omega}), \]
\[ g_r(x, N_s) = Q_s(x N_s^{1/\nu}, g_s N_s^{-\omega}), \]

where \( \langle L \rangle, \chi \) and \( g_r \) are order parameter, susceptibility and 4-th cumulant, respectively. Expanding those equations with respect to \( x \), we have

\[ L(x = 0, N_s) = N_s^{-\beta/\nu} (c_0 + c_3 N_s^{-\omega}), \]
\[ \chi(x = 0, N_s) = N_s^{\gamma/\nu} (c_0 + c_3 N_s^{-\omega}), \]
\[ g_r(x = 0, N_s) = g_r^\infty + c_3 N_s^{-\omega}, \]

at \( x = 0 \). The critical point can be defined as that point where a fit to the leading \( N_s \)-behavior has the least minimal \( \chi^2 \)[5]. Actually, leading \( N_s \)-behavior of Eq.(2) and of Eq.(3) is given by

\[ \ln L(x = 0, N_s) = -\beta/\nu \ln N_s + \ln c_0, \]
\[ \ln \chi(x = 0, N_s) = \gamma/\nu \ln N_s + \ln c_0, \]
\[ \ln g_r(x = 0, N_s) = \ln (g_r^\infty) + \ln (c_3), \]

as in ref.[5]. From the fits to Eqs.(4), (5) and (6), we can find the position of critical point \( \beta_c \), and then obtain the values of \( \beta/\nu, \gamma/\nu \) and \( g_r^\infty \) at \( \beta_c \) simultaneously. We also considered the derivatives of the observables with respect to \( x \). The leading \( N_s \)-behavior of each derivatives at the critical point is obtained similarly:

\[ \ln \frac{\partial O}{\partial x} (x = 0, N_s) = \rho \ln N_s + \ln c_0. \]

Here \( O \) is \( L, \chi \) and \( g_r \) with \( \rho = (1 - \beta)/\nu, (1 + \gamma)/\nu \) and \( 1/\nu \).

4. Results and Discussions

We performed the numerical calculations on \( N_s^3 \times 4 \) lattices, where \( N_s = 8, 12, 16 \) and 24. The standard \( SU(2) \) Wilson action was adopted and abelian link valuables were defined in maximally abelian gauge. We calculated the following observables:

\[ L = \sum P(x)/N_s^3, \]
\[ \chi = N_s^2 \langle (L^2) - \langle L \rangle^2 \rangle, \]
\[ g_r = \langle L^4 \rangle/\langle L^2 \rangle^2 - 3, \]

where \( P(x) \) denotes \( P_{SU(2)}(x), P_{\text{abel}}(x) \) and \( P_{\text{mono}}(x) \). The values of observables at various \( \beta \) are needed in order to calculate the derivatives with respect to \( x \), where \( x = (\beta - \beta_c)/\beta_c \). Then, we used the density of state method(DSM)[6].
Table 1
The critical exponents calculated from various Polyakov loops.

<table>
<thead>
<tr>
<th></th>
<th>$SU(2)$</th>
<th>abel</th>
<th>mono</th>
<th>Engels et al.[5]</th>
<th>Ising[7]</th>
<th>$U(1)[8]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta/\nu$</td>
<td>0.504(18)</td>
<td>0.485(22)</td>
<td>0.528(64)</td>
<td>0.525(8)</td>
<td>0.518(7)</td>
<td></td>
</tr>
<tr>
<td>$(1 - \beta)/\nu$</td>
<td>1.117(27)</td>
<td>1.138(10)</td>
<td>1.091(84)</td>
<td>1.085(14)</td>
<td>1.072(7)</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.617(16)</td>
<td>0.616(12)</td>
<td>0.617(57)</td>
<td>0.621(6)</td>
<td>0.6289(8)</td>
<td>0.67</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.311(19)</td>
<td>0.299(19)</td>
<td>0.326(69)</td>
<td>0.326(8)</td>
<td>0.3258(44)</td>
<td>0.35</td>
</tr>
<tr>
<td>$\gamma/\nu$</td>
<td>1.977(29)</td>
<td>2.025(34)</td>
<td>1.991(88)</td>
<td>1.944(13)</td>
<td>1.970(11)</td>
<td>1.32</td>
</tr>
<tr>
<td>$(1 + \gamma)/\nu$</td>
<td>3.600(38)</td>
<td>3.646(44)</td>
<td>3.608(93)</td>
<td>3.555(15)</td>
<td>3.560(11)</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.616(25)</td>
<td>0.617(29)</td>
<td>0.618(68)</td>
<td>0.621(8)</td>
<td>0.6289(8)</td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.218(68)</td>
<td>1.249(81)</td>
<td>1.23(19)</td>
<td>1.207(24)</td>
<td>1.239(7)</td>
<td></td>
</tr>
<tr>
<td>$\gamma/\nu + 2\beta/\nu$</td>
<td>2.985(47)</td>
<td>2.995(56)</td>
<td>3.05(15)</td>
<td>2.994(21)</td>
<td>3.006(18)</td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.447(41)</td>
<td>1.438(42)</td>
<td>1.438(41)</td>
<td>1.403(16)</td>
<td>1.41</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.633(13)</td>
<td>0.621(14)</td>
<td>0.600(13)</td>
<td>0.630(11)</td>
<td>0.6289(8)</td>
<td></td>
</tr>
</tbody>
</table>

|        |         |      |      |      |      |         |

First we performed Monte-Carlo simulations at $\beta_0=2.2988$, and then the expectation values of the observables in the vicinity of $\beta_0$ were obtained using DSM. $L$, $\chi$ and $g_r$ at $\beta_0$ were calculated every 50 sweeps after 2000 thermalization sweeps. The number of samples was 100000, except on $24^3 \times 4$ lattice (47000 in the case). The errors were determined according to the Jackknife method dividing the entire sample into 10 blocks (4 blocks on $24^3 \times 4$ lattice).

We estimated the critical point $\beta_c$ from $\chi^2$ method[5]. The data of our DSM results are fitted to Eq.(4)-(6) and Eq.(7) at each $\beta$. The number of input data is 2 and that of fit parameters is 2 ($\omega$ in Eq.(4) is fixed to 1 in accordance with Engels et al.[5]). Figure 2 describes the typical curves of $\chi^2/N_f$ versus $\beta$.

Averaging the obtained minimal positions of $\chi^2/N_f$, we get

$$\beta^{SU(2)}_c = 2.29940(20),$$
$$\beta^{abel}_c = 2.99962(26),$$
$$\beta^{mono}_c = 2.29971(23).$$

Those critical points are very close to each other.

Table 1 lists the critical exponents on each critical point in the non-abelian, the abelian and the monopole case. We get the following results:

1. The critical exponents in the abelian and the monopole case are in agreement with non-abelian exponents within the statistical error.
2. Those critical exponents agree with those of $Z_2$ rather than those of $U(1)$.
3. Hyperscaling relations are well satisfied.
4. Non-abelian exponents obtained are consistent with those of Engels et al.[5].

The first and the second results indicate the abelian (monopole) dominance in quark confinement.

REFERENCES