On Existence of Nontrivial Fixed Points in Large $N$ Gauge Theory in More than Four Dimensions

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Abstract

Inspired by a possible relation between large $N$ gauge theory and string theory, we search for nontrivial fixed points in large $N$ gauge theory in more than four dimensions. We study large $N$ gauge theory through Monte Carlo simulation of the twisted Eguchi-Kawai model in six dimensions as well as in four dimensions. The phase diagram of the system with the two coupling constants which correspond to the standard plaquette action and the adjoint term has been explored.

PACS: 11.15.Ha; 11.10.Kk
Keywords: Large N, Monte Carlo simulation, string theory

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1 Introduction

String theory has been considered as a natural candidate for the unified theory including gravity. Intensive study revealed, however, that nonperturbative understanding of its dynamics is essential in making any predictions as to our present world. The double scaling limit of the large $N$ matrix model [1] is a rare example in which nonperturbative study of string theory has been successful. Although the development concerning matrix models so far is restricted to the space-time dimension less than 1, the above success shows that nonperturbative study of string theory in physical dimensions may be possible through the study of the double scaling limit of some kind of large $N$ field theory.

As such a field theory, we consider large $N$ gauge theory. It has been generally believed that large $N$ gauge theory is related to some kind of string theory, since 't Hooft showed within perturbation theory that the $1/N$ expansion of $U(N)$ gauge theory gives an expansion in terms of the topology of the Feynman diagrams [2]. If, in the strong coupling regime, dominant contributions come from such diagrams as those with very fine internal lines like fishnets, the theory can be viewed as a string theory by identifying the fishnet with a continuous worldsheet embedded in the space-time where the gauge field lives. Also in lattice gauge theory, the expectation value of a Wilson loop can be expressed, in the strong coupling expansion, as a sum over random surfaces spanning the loop. Although no rigorous connection between large $N$ gauge theory and string theory has not yet been established, there is a good reason to consider it seriously.

In four dimensions, since nonabelian gauge theory is asymptotically free, a possible relation to string theory can arise only as an effective theory in the infrared region, where the coupling constant becomes sufficiently large. This naturally leads us to consider the theory in more than four dimensions.

The $\epsilon$-expansion of $(4 + \epsilon)$-dimensional nonabelian gauge theory shows that there exists an ultraviolet fixed point between the confining phase and the deconfining phase when $\epsilon$ is positive. If this fixed point survives even in physical dimensions such as 5 or 6, there is a possibility of constructing a string theory by approaching the fixed point as one take the large $N$ limit. Considering that the critical dimension of bosonic string theory is 26, one may speculate that the fixed point might well exist in more than four dimensions.

Since gauge theory is perturbatively unrenormalizable in more than four dimensions, the existence of such a fixed point is quite nontrivial. An example is three-dimensional
nonlinear sigma model, which is perturbatively unrenormalizable and yet can be defined on the lattice by taking the continuum limit at the second order phase transition point. Although this example is rather trivial in the sense that the continuum theory belongs to the same universality class as three-dimensional $\phi^4$ theory which is superrenormalizable, it shows that constructive definition of perturbatively unrenormalizable theory is possible at least in principle. Constructing quantum gravity within ordinary field theory belongs to the same kind of problem and we expect that our study may also be suggestive to that direction.

Since we need a nonperturbative approach, we consider lattice gauge theory. Monte Carlo simulation of large $N$ gauge theory in a straightforward manner is, however, unfeasible due to the huge number of dynamical degrees of freedom, which is as large as $N^2 \times D \times L^D$, where $L$ is the size of the lattice, and $D$ is the dimension of the space-time. However, as Eguchi and Kawai showed in the early 80’s [3], the factorization valid in large $N$ gauge theory leads to the fact that the above system is equivalent to a one-site model ($L=1$) with $N^2 \times D$ dynamical degrees of freedom. This tremendous reduction of dynamical degrees of freedom makes the study of large $N$ gauge theory accessible by numerical simulation.

This paper is organized as follows. In section 2, we explain the reduction of Wilson’s lattice gauge theory to the one-site model in the large $N$ limit. In section 3, we study the four-dimensional case. In addition to the standard plaquette action, we put the adjoint term in the action and study the system with the two coupling constants. In section 5, we show our results for the six-dimensional case. Section 6 is devoted to conclusion and discussion.

2 Reduction to one-site model

In Wilson’s lattice gauge theory, SU($N$) gauge theory can be defined with the action

$$S = -N\beta \sum_n \sum_{\mu \neq \nu} \text{tr}(U_{n,\mu} U_{n+\mu,\nu} U_{n+\nu,\mu} U_{n,\nu}^\dagger).$$

(1)

$U_{n,\mu}$’s are SU($N$) matrices and are called link variables. A gauge invariant observable can be given by the Wilson loop

$$W(C) = \frac{1}{N} \text{tr}(U_{n,\alpha} U_{n+\alpha,\beta} U_{n+\beta,\gamma} \cdots U_{n-\omega,\omega}),$$

(2)

which is the trace of the ordered product of link variables along the loop $C$.

The one-site model which is equivalent to the above Wilson’s theory can be defined with
the action
\[ S = -N\beta \sum_{\mu \neq \nu} Z_{\mu\nu} \text{tr}(U_\mu U_\nu U_\mu U_\nu^\dagger), \] (3)
where \( Z_{\mu\nu} \) is an element of the center group \( Z_N \) and is called ‘twist’. A successful choice of the twist is given by
\[ Z_{\mu\nu} = \exp(2\pi i/L) \quad \text{for} \quad \mu < \nu, \] (4)
\[ Z_{\nu\mu} = Z_{\mu\nu}^*, \] (5)
where \( L \) is an integer defined through
\[ N = L^{D/2} \] (6)
with the space-time dimension \( D \), which is taken to be even.\(^1\) The twist has been introduced by the authors of Ref. [4] in order to cure a problem in the original Eguchi-Kawai model [3], where \( Z_{\mu\nu} \)’s were taken to be unity. The model is thus called twisted Eguchi-Kawai model. For a general review on twisted Eguchi-Kawai models, we refer the reader to Ref. [5].

The observable corresponding to the Wilson loop (2) can be defined in the twisted Eguchi-Kawai model as
\[ w(C) = \frac{1}{N} \left( \prod_{\mu \neq \nu} Z_{\mu\nu}^{N_{P_{\mu\nu}}} \right) \text{tr}(U_\alpha U_\beta U_\gamma \cdots U_\omega), \] (7)
where \( N_{P_{\mu\nu}} \) is the number of plaquettes in the \((\mu, \nu)\) direction on the minimal surface spanning the loop \( C \).

Under the assumption of the factorization
\[ \langle W(C_1)W(C_2) \cdots W(C_k) \rangle = \langle W(C_1) \rangle \langle W(C_2) \rangle \cdots \langle W(C_k) \rangle + O\left( \frac{1}{N^2} \right), \] (8)
which is valid generically in large \( N \) gauge theory, one can show that \( \langle w(C) \rangle \) calculated in the twisted Eguchi-Kawai model is equal to \( \langle W(C) \rangle \) calculated in the original Wilson’s theory on the \( L^D \) lattice, where \( L \) is related to \( N \) through eq. (6).

In this sense, the twisted Eguchi-Kawai model is equivalent to the original Wilson’s theory on the infinite lattice, in the large \( N \) limit. As is indicated in the above statement, the finite \( N \) effects in the twisted Eguchi-Kawai model appear in two ways. One is the violation of the factorization which is the assumption of the equivalence, and the other is the finite \( L \) effects in the corresponding Wilson’s theory to which the one-site model is equivalent.

\(^1\)For odd \( D \), one can apply the reduction scheme only to the \((D - 1)\) dimensions leaving one dimension unreduced. We prefer to restrict ourselves to even dimensions, i.e. \( D = 4 \) and \( D = 6 \) in order to avoid any technical complications.
Let us now explain how to perform Monte Carlo simulation of the twisted Eguchi-Kawai model. In contrast to the ordinary Wilson’s theory, the action (3) is not linear in terms of each link variable. In order to make the heat bath algorithm [6] applicable to the model, we use the technique proposed by Ref. [7]. The idea is to introduce an auxiliary field \( Q_{\mu\nu} (1 \leq \mu < \nu \leq D) \), which is a general complex \( N \times N \) matrix, with the following action.

\[
S = N\beta \sum_{\mu<\nu} \text{tr} Q^\dagger_{\mu\nu} Q_{\mu\nu} \\
- N\beta \sum_{\mu<\nu} \text{tr} Q^\dagger_{\mu\nu} \left( t_{\mu\nu} U_{\mu} U_{\nu} + t_{\nu\mu} U_{\nu} U_{\mu} \right) \\
- N\beta \sum_{\mu<\nu} \text{tr} Q_{\mu\nu} \left( t^*_{\mu\nu} U_{\mu}^\dagger U_{\nu}^\dagger + t^*_{\nu\mu} U_{\nu}^\dagger U_{\mu}^\dagger \right),
\]

where \( t_{\mu\nu} \) is the square root of \( Z_{\mu\nu} \). Since this action is linear in terms of \( U_\mu \), we can use the heat bath algorithm for the update of \( U_\mu \). We update \( U_\mu \) by successively multiplying it by matrices each belonging to the \( N(N-1)/2 \) SU(2) subgroups of the SU(\( N \)) [8]. After updating all the \( U_\mu \)'s in this way, we perform the update of \( Q_{\mu\nu} \), which can be done with little cost by generating Gaussian variables. This defines the ‘one sweep’ of our system. For further technical details of the algorithm, we refer the reader to Ref. [7].

3 Results for four-dimensional case

We first study the model in four dimensions. We take \( N = 16 \) which corresponds to \( L = 4 \) in the equivalent Wilson’s theory. In fig. 1, we plot the mean plaquette defined by

\[
P = \left\langle \frac{1}{N} \frac{1}{D(D-1)} \sum_{\mu \neq \nu} Z_{\mu\nu} \text{tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) \right\rangle.
\]

Each point is an average over 1000 sweeps. We also plot the leading terms of the strong- and weak-coupling expansion for the Wilson’s theory.

\[
P \sim \begin{cases} 
\beta & \text{for small } \beta \\
\frac{1}{2D^3} & \text{for large } \beta.
\end{cases}
\]

Our data are in good agreement with those of Ref. [4]. We observe a clear indication for a first order phase transition at \( \beta \approx 0.35 \). We should note that a first order phase transition is already present at finite \( N \), for \( N \geq 4 \), when one uses the standard plaquette action [9]. This first order phase transition is considered as a lattice artifact of the plaquette action,
and can be removed, for example, by using the modified action \(^2\)

\[
S = -N\beta \sum_P \text{tr}U(P) - \frac{\beta_A}{2} \sum_P \text{tr}_A U(P),
\]

(11)

where \(U(P)\) is the ordered product of link variables around the given oriented plaquette \(P\). The second term is the trace of \(U(P)\) taken in the adjoint representation, which can be rewritten as

\[
\text{tr}_A U(P) = |\text{tr}U(P)|^2 - 1.
\]

(12)

Indeed, in Ref. [11], Monte Carlo simulation of the above system in the ordinary Wilson’s theory has been done for \(4 \leq N \leq 8\) and it is shown that the line of first order phase transition terminates at a point with a sufficiently large negative \(\beta_A\).

We, therefore, study large \(N\) gauge theory with this modified \(\beta_A\). The adjoint term can be incorporated into the twisted Eguchi-Kawai model with the action

\[
S = -N\beta \sum_{\mu \neq \nu} Z_{\mu\nu} \text{tr}(U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}) - \frac{\beta_A}{2} \sum_{\mu \neq \nu} |Z_{\mu\nu} \text{tr}U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}|^2.
\]

(13)

The second term, which corresponds to the adjoint term, can be dealt with by Metropolis algorithm. In order to increase the acceptance rate, we rewrite the action as

\[
S = -N(\beta + \beta_A C) \sum_{\mu \neq \nu} Z_{\mu\nu} \text{tr}(U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}) - \frac{\beta_A}{2} \sum_{\mu \neq \nu} |Z_{\mu\nu} \text{tr}U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger} - NC|^2,
\]

(14)

taking the arbitrary real constant \(C\) to be approximately equal to the mean plaquette defined by eq. (9).

In fig. 2, we show the mean plaquette as a function of \(\beta\) for \(\beta_A = -0.3, -0.7, -1.0\). Each point is an average over 1000 sweeps. We replot the data for \(\beta_A = 0\). As \(\beta_A\) goes to a larger negative value, the gap at the transition point decreases, and finally disappears. This implies that the first order phase transition can be removed by introducing a sufficiently large negative \(\beta_A\) even in the large \(N\) limit.

It is known that, in the large \(N\) limit, the theory with \((\beta, \beta_A)\) is equivalent to the theory with \((\beta', \beta'_A)\), if

\[
\beta' = \beta + (\beta_A - \beta'_A) P(\beta, \beta_A),
\]

(15)

where \(P(\beta, \beta_A)\) is the mean plaquette at the coupling constants \((\beta, \beta_A)\) [12, 13].

\(^2\)There may be various types of modifications that can be used instead of (11). In Ref. [10], it is reported that the first order phase transition is absent when one uses the renormalization-group improved action.
Using this equivalence theorem, we can predict the results for $\beta_A = 0$ with the input of the data for $\beta_A = -0.3, -0.7, -1.0$. In fig. 3 we show the prediction from the data for $\beta_A = -0.3, -0.7, -1.0$ together with the data for $\beta_A = 0$. Note that there are three predicted values for each $\beta$ in the critical region. The middle point is unstable and can never be seen in the simulation. The other two points are stable or metastable. The metastable vacuum is stabilized as $N$ increases due to the small tunneling probability $\sim \exp(-\text{const.}N^2)$.

Each of the three points corresponds to a solution to the Schwinger-Dyson equation for the Wilson loops [13]. The ‘equivalence’ of the systems with different $\beta_A$’s should, therefore, be understood in the sense that the two systems have a common solution to the Schwinger-Dyson equation. One can see that our data clearly satisfies the equivalence theorem.

4 Results for six-dimensional case

Let us turn to the six-dimensional case. In fig. 4, we show the mean plaquette as a function of $\beta$ for $N = 64$ ($L = 4$) with the standard plaquette action ($\beta_A = 0$). Each point is an average over 300 sweeps. We see a strong hysteresis indicating a first order phase transition. Note that the thermal cycle is substantially larger than that in the four-dimensional case. Fig. 5 shows the result for $N = 27$ ($L = 3$) with $\beta_A = -10.0$. Each point is an average over 300 sweeps. Here we see a striking difference from the four-dimensional case. We see that there are two different $\beta$’s that give the same value for the mean plaquette. It is obvious from the relation (15) that the phase transition will never become continuous, however large a negative value of $\beta_A$ we may take.

Using the equivalence theorem in the previous section, we can examine the asymptotic behavior of the system in the large negative $\beta_A$ region. From the relation (15), we have

$$P(\beta, \beta_A) = P(\beta + (\beta_A - \beta_A') P(\beta, \beta_A), \beta_A').$$

(16)

We fix $\beta$ and $\beta_A$ and consider the asymptotic behavior of the system with $\beta' = \beta + (\beta_A - \beta_A') P(\beta, \beta_A)$ and $\beta_A'$ for $\beta_A' \to -\infty$. Differentiating the above equation with respect to $\beta$, we obtain

$$\frac{\partial P}{\partial \beta} \bigg|_{\beta', \beta_A'} \sim \frac{1}{|\beta_A'|}.$$  

(17)

Therefore, denoting the horizontal distance of the two nearly parallel data lines in fig. 5 as
\[ \Delta P = \Delta \beta \cdot \left. \frac{\partial P}{\partial \beta} \right|_{\beta', \beta'_A} \sim \frac{\Delta \beta}{|\beta'_A|}. \]  

(18)

Also the fluctuation of the mean plaquette \( P \) can be written as

\[ \delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \sqrt{\frac{1}{N^2} \left. \frac{\partial P}{\partial \beta} \right|_{\beta', \beta'_A}} \sim \frac{1}{\sqrt{|\beta'_A|}} \frac{1}{N}. \]  

(19)

Thus the fluctuation \( \delta P \) as well as the gap \( \Delta P \) vanishes in the \( \beta'_A \to -\infty \) limit. Note also that the gap \( \Delta P \) vanishes faster than the fluctuation \( \delta P \), so that the discontinuity, which is a clear signal of the first order phase transition, becomes difficult to observe in the actual simulation. These results naturally explain the observations of Ref. [14], where five-dimensional pure SU(2) lattice gauge theory has been studied.

In the large \( N \) limit, by virtue of the equivalence theorem, we can conclude that the line of first order phase transition continues to \( (\beta, \beta_A) = (\infty, -\infty) \). Considering that the data of Ref. [14] are qualitatively the same as ours, we may naturally expect that the same is true for \( 2 \leq N < \infty \).

One may think of enlarging the coupling-constant space where one searches for a second order phase transition. For this purpose, we consider those additional terms in the action which can be obtained from (11) by replacing plaquettes with three-dimensional loops which consist of six links. Such terms can be incorporated into the twisted Eguchi-Kawai model by the following additional terms.

\[ -\gamma_A \sum_{\mu < \nu < \lambda} \left| Z_{\mu \nu} Z_{\nu \lambda} Z_{\lambda \mu}^* \text{tr}(U_{\mu} U_{\nu} U_{\lambda} U_{\mu}^\dagger U_{\nu}^\dagger U_{\lambda}^\dagger) \right| - N|\alpha|^2 \]

\[ -\gamma_A \sum_{\mu < \nu < \lambda} \left| Z_{\nu \mu} Z_{\nu \lambda} Z_{\lambda \mu}^* \text{tr}(U_{\mu} U_{\nu} U_{\lambda} U_{\nu}^\dagger U_{\lambda}^\dagger) \right| - N|\alpha|^2. \]

Corresponding to the mean plaquette, we define the mean three-dimensional loop by

\[ Q = \frac{1}{N} \left[ \frac{2}{D(D-1)(D-2)} \sum_{\mu < \nu < \lambda} \sum_{\nu \neq \mu, \lambda} \text{Re}\{Z_{\mu \nu} Z_{\nu \lambda} Z_{\lambda \mu}^* \text{tr}(U_{\mu} U_{\nu} U_{\lambda} U_{\mu}^\dagger U_{\nu}^\dagger U_{\lambda}^\dagger)\} \right. \]

\[ + \left. \frac{6}{D(D-1)(D-2)} \sum_{\mu < \nu < \lambda} \text{Re}\{Z_{\nu \mu} Z_{\nu \lambda} Z_{\lambda \mu}^* \text{tr}(U_{\mu} U_{\nu} U_{\lambda} U_{\nu}^\dagger U_{\lambda}^\dagger)\} \right]. \]

A similar argument as with \( \beta, \beta_A \) and \( P \) applies to \( \gamma, \gamma_A \) and \( Q \), where \( \gamma = \gamma_A \alpha \).

In fig. 6, we show the data for \( (P,Q) \) in the ordered phase and in the disordered phase for four sets of \( (\beta, \gamma) \) with \( \beta_A = -1.0 \) and \( \gamma_A = -2.0 \). Each point is an average over 50
sweeps. Note that the quadrilaterals spanned by the data points in the two phases have an overlap. This means that the same \((P, Q)\) can be obtained for different \((\beta, \gamma)\)'s with fixed \(\beta_A\) and \(\gamma_A\). We can thus conclude that the phase transition will not become continuous even in the enlarged coupling-constant space.

5 Summary and Discussion

In this paper we study large \(N\) gauge theory in six dimensions as well as in four dimensions. The phase diagram in the large negative \(\beta_A\) region has been clarified by making use of a simple property of the system with the two coupling constants \(\beta\) and \(\beta_A\) due to the large \(N\).

In four dimensions the line of first order phase transition terminates at a sufficiently large negative \(\beta_A\) and the two phases are actually connected analytically. At the end point of the line, the first order phase transition becomes second order. What is the continuum theory that can be defined at this end point? Since the phase transition is not a deconfining one, the string tension does not scale to zero at the end point. This means that minimum surfaces dominate the summation over surfaces spanning the Wilson loops. Considering, for example, the two-point function of plaquettes, dominant contributions come from tube-like surfaces connecting the two plaquettes. This means that the continuum theory looks more like particle theory rather than string theory. This is consistent with the claim for SU(3) gauge theory that the continuum theory defined at this end point is (trivial) \(\phi^4\) theory [15].

In six dimensions, the line of first order phase transition continues to infinity. Moreover, we see that this is the case even if we enlarge the coupling-constant space to include three-dimensional loops in addition to plaquettes. It is natural to expect that this pattern is repeated for other types of loops. Thus we consider that there is no nontrivial fixed point in large \(N\) gauge theory in six dimensions.

In the context of string theory, it is tempting to interpret our conclusion as a result of tachyon instability of bosonic strings, which may be cured by introducing space-time supersymmetry. It is, therefore, probable that we may find a nontrivial fixed point in six dimensional supersymmetric large \(N\) gauge theory, though it will require much more effort since we have to deal with dynamical fermions.
Acknowledgment

I would like to thank H. Kawai for stimulating discussion. I am also grateful to S.R. Das and Y. Iwasaki for helpful communications during this work. This work was supported by National Laboratory for High Energy Physics (KEK) as KEK Supercomputer Project, and the calculations were carried out on Fujitsu VPP500 at KEK.
References


Figure captions

Fig. 1 The mean plaquette $P$ as a function of the coupling constant $\beta$ in four-dimensional SU(16) gauge theory with the standard plaquette action ($\beta_A = 0$). The dashed line represents the strong coupling expansion $P = \beta$, whereas the dash-dotted line represents the weak-coupling expansion $P = 1/(8\beta)$.

Fig. 2 The mean plaquette $P$ as a function of the coupling constant $\beta$ in four-dimensional SU(16) gauge theory with $\beta_A = -0.3$ (triangles), $-0.7$ (squares), $-1.0$ (diamonds). We replot the data for $\beta_A = 0$ (crosses) shown in fig. 1.

Fig. 3 The predicted values for the mean plaquette $P$ as a function of the coupling constant $\beta$ for $\beta_A = 0$ with the input of the data for $\beta_A = -0.3$ (triangles), $-0.7$ (squares), $-1.0$ (diamonds) shown in fig. 2. We also plot the original data for $\beta_A = 0$ (crosses).

Fig. 4 The mean plaquette $P$ as a function of the coupling constant $\beta$ in six-dimensional SU(64) gauge theory with the standard plaquette action ($\beta_A = 0$). The dashed line represents the strong coupling expansion $P = \beta$, whereas the dash-dotted line represents the weak-coupling expansion $P = 1/(12\beta)$.

Fig. 5 The mean plaquette $P$ as a function of the coupling constant $\beta$ in six-dimensional SU(27) gauge theory with $\beta_A = -10.0$. The triangles are the data in the disordered phase, whereas the squares are the data in the ordered phase.

Fig. 6 The mean plaquette $P$ and the mean three-dimensional loop $Q$ in six-dimensional SU(27) gauge theory are plotted for four sets of $(\beta, \gamma)$ with $\beta_A = -1.0$ and $\gamma_A = -2.0$. The triangles are the data in the disordered phase, whereas the squares are the data in the ordered phase.
fig. 1
$\beta_A = 0$ $\beta_A = -0.3$ $\beta_A = -0.7$ $\beta_A = -1.0$

fig. 2
fig. 3
fig. 4
fig. 5
(β,γ) = (0.59, 0.24)
(β,γ) = (0.62, 0.25)
(β,γ) = (0.66, 0.22)
(β,γ) = (0.63, 0.21)

fig. 6