Calculation of the renormalized charmed-quark mass in Lattice QCD

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The correlator of heavy-quark currents is calculated in quenched Lattice QCD on a $16^3 \times 32$-lattice for $\beta = \{6,6.3\}$. The renormalized charmed quark mass is extracted from the short-distance part of that correlator: $m_{c}^{\overline{MS}}(m_{c}) = 1.22(5) \text{GeV}$. We study the sensitivity of our data to the strong coupling constant.

The point-to-point correlator of local heavy-quark currents:

$$\Pi(q^2)_{\mu\nu} = i \int dx e^{iqx} <0 | T \{j_{\mu}(x)j_{\nu}(0)\} | 0 >$$

where $j_{\mu} = \bar{c} \gamma_{\mu} c$, is reliably calculable in the loop expansion of perturbative QCD in the vicinity of zero momentum. If one knows the correlator from experiment or from lattice simulations one can then extract parameters of perturbative QCD such as the renormalized heavy-quark mass or strong coupling constant normalized on the heavy-quarks threshold [1]. In earlier work [2,3] we pointed out that applicability of the perturbative loop-expansion implies that the lattice artifacts of the short-distance part of the correlator $\Pi(q^2 \sim 0)$ may be studied analytically order-by-order in that expansion. Using this idea, our aim is to extract the renormalized charmed-quark mass $m_{c}$ and strong coupling constant $\alpha_s(m_{c})$.

Following [1] we consider the ratios $r_n = M_{n+1}/M_n$ of moments of the correlator (1):

$$M_n = \frac{1}{2^n n! (n+1)!} \int d^4x x^{2n} \Pi(x)$$

(2)

to separate the short and the long distances. The applicability of perturbative QCD near $q^2 = 0$ implies the following expansion for the lower ratios $r_{2,3,4}$, originating from short-distances:

$$r_n = \frac{a_n}{4m_c^2} \{ 1 + \omega_n \alpha_s(m_c) \}$$

(3)

where $\{a_n, \omega_n\}$ are known numbers [1]. The term $\propto a_n$ comes from one loop of free charmed quarks. The term $\propto \omega_n$ comes from the two-loop diagrams due to one-gluon exchange.

Figure 1. Charmonium spectrum obtained on a $16^3 \times 32$ lattice at $\beta = 6$ (squares) and $\beta = 6.3$ (diamonds) and on an $8^3 \times 16$ lattice at $\beta = 6$ (crosses) with the clover-and-tadpole-improved action. The pseudoscalar mass $m_{\eta_c} = 2.979 \text{GeV}$ is used for normalization. The wide horizontal lines are experimental data.

We have calculated the correlator (1) in quenched QCD on an $8^3 \times 16$ lattice for $\beta = 6$ and a $16^3 \times 32$ lattice for $\beta = \{6,6.3\}$ with the Wilson tadpole-and-clover-improved fermionic action [4], with 20 configurations in every case. We fixed the scale (lattice spacing $a$) from the mass of the low-lying resonance. To extract that mass in these relatively small volumes we incorporated the physical continuum spectrum at high energies in the way it is done in the QCD sum rules approach [1] to describe the behavior of the correlator at short distances. We used the following extension of the dispersion relation for the two-point corre-
latter $\Pi(q^2)$ to the lattice theory [3]:

$$
\Pi(q^2) = \int ds \frac{\rho(s)}{s + \frac{4}{\pi^2} \sum \sin^2(q_\mu a/2)}
$$

(4)

The higher ratios $r_8 - r_{11}$ originate from long distances. They are saturated by the single low-lying resonance contribution. Our results for the charmonium spectrum are shown in Fig. 1.

![Figure 2. Monte-Carlo data for the vector currents on a 16$^3 \times 32$ lattice at $\beta = 6$ (diamonds), $\beta = 15$ (squares); $\kappa = 0.1100$. The lines show the ratios of moments for one loop of free Wilson fermions. The dashed line corresponds to $\kappa = 0.1100$; the solid lines fit the 3rd ratio of Monte-Carlo data by construction.](image)

We fit the lower ratios $r_{3,4}$ of Monte-Carlo data with the free-quarks approximation to the correlator (1):

$$
\text{fit to } r_{3,4}\text{(free Wilson quarks)} = \text{fit to } r_{3,4}\text{(Monte-Carlo)}
$$

(5)

The Wilson parameter $\kappa$ corresponding to those free quarks determines the renormalized quark mass. The relation between $\kappa$ in a finite volume and the corresponding quark mass of the continuum theory was clarified in [3].

To specify the subtraction scheme of the renormalized quark mass, one should be sensitive to the $\alpha_s$-correction in the ratios $r_\beta$. If we ignore lattice artifacts in the coefficients $\omega_n$ and take $\omega_n$ from the continuum theory [1], choosing the $\overline{MS}$ subtraction scheme and $\alpha_s(m_c) \approx 0.3$ [3], we get the results for the renormalized charmed-quark mass shown in Table 1. Since the two-loop correction is small in the lower moments, the uncertainty in the quark mass introduced by taking $\omega_n$ from the continuum theory rather than evaluating it on the lattice is small.

Table 1

<table>
<thead>
<tr>
<th>$\beta = 6.0$, $\kappa = 0.1060$, ($a \approx 1.9\text{GeV}$)</th>
<th>fit to $r_3$</th>
<th>$m_{res}$</th>
<th>$m_c$</th>
<th>$m_{c\overline{MS}}^{M}(\eta_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_c$</td>
<td>1.562(5)</td>
<td>.618(2)</td>
<td>1.208(5)</td>
<td></td>
</tr>
<tr>
<td>$J/\psi$</td>
<td>1.600(6)</td>
<td>.660(1)</td>
<td>1.246(6)</td>
<td></td>
</tr>
<tr>
<td>$\chi_c$</td>
<td>1.864(4)</td>
<td>.645(2)</td>
<td>1.230(8)</td>
<td></td>
</tr>
<tr>
<td>$\beta = 6.3$, $\kappa = 0.1150$, ($a \approx 3.7\text{GeV}$)</td>
<td>fit to $r_3$</td>
<td>$m_{res}$</td>
<td>$m_c$</td>
<td>$m_{c\overline{MS}}^{M}(\eta_c)$</td>
</tr>
<tr>
<td>$\eta_c$</td>
<td>0.805(6)</td>
<td>.307(2)</td>
<td>1.149(6)</td>
<td></td>
</tr>
<tr>
<td>$J/\psi$</td>
<td>0.836(7)</td>
<td>.336(2)</td>
<td>1.214(9)</td>
<td></td>
</tr>
<tr>
<td>$\chi_c$</td>
<td>0.89(6)</td>
<td>.338(5)</td>
<td>1.23(3)</td>
<td></td>
</tr>
</tbody>
</table>

One can see a rather stable value of the renormalized charmed-quark mass $m_{c\overline{MS}}^{M}(m_c) = 1.22(5)\text{GeV}$, which is in good agreement with estimates of the continuum theory [5]: $m_{c\overline{MS}}^{M}(m_c) \approx 1.23\text{GeV}$. The previously reported lattice result is [6]: $m_{c\overline{MS}}^{M}(m_c) = 1.5(3)\text{GeV}$.

Table 2

<table>
<thead>
<tr>
<th>$\beta = 6.0$, $\kappa = 0.1060$, ($a \approx 1.9\text{GeV}$)</th>
<th>fit to $r_4$</th>
<th>$m_{res}$</th>
<th>$m_c$</th>
<th>$m_{c\overline{MS}}^{M}(\eta_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_c$</td>
<td>1.562(5)</td>
<td>.645(1)</td>
<td>1.224(6)</td>
<td></td>
</tr>
<tr>
<td>$J/\psi$</td>
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<td>.677(1)</td>
<td>1.258(6)</td>
<td></td>
</tr>
<tr>
<td>$\chi_c$</td>
<td>1.864(4)</td>
<td>.686(3)</td>
<td>1.27(1)</td>
<td></td>
</tr>
<tr>
<td>$\beta = 6.3$, $\kappa = 0.1150$, ($a \approx 3.7\text{GeV}$)</td>
<td>fit to $r_4$</td>
<td>$m_{res}$</td>
<td>$m_c$</td>
<td>$m_{c\overline{MS}}^{M}(\eta_c)$</td>
</tr>
<tr>
<td>$\eta_c$</td>
<td>0.805(6)</td>
<td>.321(2)</td>
<td>1.165(6)</td>
<td></td>
</tr>
<tr>
<td>$J/\psi$</td>
<td>0.836(7)</td>
<td>.344(2)</td>
<td>1.222(6)</td>
<td></td>
</tr>
<tr>
<td>$\chi_c$</td>
<td>0.89(6)</td>
<td>.359(8)</td>
<td>1.27(4)</td>
<td></td>
</tr>
</tbody>
</table>

We have found previously [3] that the lattice version (4) of the dispersion relation works extremely well (i) to reproduce the lattice correlator on the one-loop level in perturbation theory, and (ii) to fit Monte-Carlo data in small volumes with the phenomenological form of spec-
The coefficient $\omega_n$ obtained via the lattice dispersion relation with the Schwinger spectral density on different lattices for $am_c = 0.6$ vs. the continuum-theory result (dashed line).

trum incorporating both the low-lying resonance and the smooth continuum spectrum at high energies. Here we use the dispersion relation (4) to estimate lattice artifacts in the coefficients $\omega_n$. We use in eqn.(4) the Schwinger expression for the spectral density, which determines the correlator of vector currents on the two-loop level. The quark mass is fixed in the $\overline{MS}$-scheme in the continuum theory (see [1], where it was fixed in the MOM-scheme). One can see from Fig.3 that the coefficient $\omega_n$ obtained in this way remarkably coincides with the continuum-theory results for the ratios of interest $n = 3, 4$ already on a $32^4$ lattice. Since the use of the lattice dispersion relation (4) helps to approach the continuum limit faster [3], Fig.3 does suggest that one can have very small lattice artifacts in the coefficients $\omega_n$ on a $32^4$-lattice for reasonable values of the quark mass $am_c \sim 0.6$. Finite-size effects are seen to be significant on smaller lattices.

Keeping $\kappa$ the same as at $\beta = 6$, we have generated Monte-Carlo data on a $16^3 \times 32$ lattice for $\beta = \{10, 15\}$. Moment ratios are shown on Fig.2 for $\beta = 15$. The corresponding fermion mass $\tilde{m}_c$ is very heavy, since $\tilde{m}_c \cdot a[\beta = 15] = m_c \cdot a[\beta = 6]$. Nonperturbative effects are then strongly suppressed in the range of distances probed by the given lattices, hence only perturbative terms survive. Indeed one can see the Monte-Carlo data much better fitted with one loop of free Wilson fermions in a wide range of ratios at $\beta = 15$ than at $\beta = 6$, which is in the confining phase. Nonetheless the deviation from the one-loop approximation is clearly visible. Therefore, we fit our data with the two-loop expression (3), taking the coefficients $\omega_n$ from continuum theory. Two adjacent ratios $r_n, r_{n+1}$ can be used to fix the quark mass and the effective coupling constant $\alpha'_s$. Fig.4 shows our results for $\alpha'_s$. Since $\alpha'_s = \alpha_s(\tilde{m}_c) \sim \alpha_s(a)$, one can check that the relative height of the plateaus in $\alpha'_s$ for $n = 5 - 8$ is consistent with asymptotic scaling, as expected at those high $\beta$s.

Figure 3. The coefficient $\omega_n$ obtained via the lattice dispersion relation with the Schwinger spectral density on different lattices for $am_c = 0.6$ vs. the continuum-theory result (dashed line).

Figure 4. Strong coupling constant for very heavy fermions obtained on a $16^3 \times 32$ lattice at $\beta = 15$ (squares) and $\beta = 10$ (diamonds) for $\kappa = 0.1100$.

REFERENCES