Dynamical Comptonization in spherical flows: black hole accretion and stellar winds

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ABSTRACT
The transport of photons in steady, spherical, scattering flows is investigated. The moment equations are solved analytically for accretion onto a Schwarzschild black hole, taking into full account relativistic effects. We show that the emergent radiation spectrum is a power law at high frequencies with a spectral index smaller (harder spectrum) than in the non–relativistic case. Radiative transfer in an expanding envelope is also analyzed. We find that adiabatic expansion produces a drift of injected monochromatic photons towards lower frequencies and the formation of a power–law, low–energy tail with spectral index $-3$.

Key words: accretion, accretion discs – radiation mechanisms – radiative transfer.

1 INTRODUCTION
It was realized long ago (Cowsik & Lee 1982 and references therein) that the divergence of the velocity field in astrophysical flows can provide a very efficient mechanism to transfer energy from the fluid to particles (photons, neutrinos, cosmic rays) diffusing through the medium, even in the absence of shocks. In the case of photons undergoing multiple scatterings off cold electrons, this effect is germane to thermal Comptonization, with the flow velocity $v$ playing the role of the thermal velocity, and is sometimes referred to as dynamical Comptonization. In a series of papers Blandford & Payne (1981a,b; Payne & Blandford 1981, PB in the following) were the first to emphasize the importance of repeated scatterings in a steady, spherical flow of depth $\tau \gg 1$. They have shown that monochromatic photons injected in a region where $\tau v/c \sim 1$ always gain energy, because of adiabatic compression, and emerge with a broad distribution which exhibits a distinctive power–law, high–energy tail. Under the assumption that $v \propto r^\beta$, the spectral index depends only on $\beta$.

Cowsik & Lee (1982), and later Schneider & Bogdan (1989), stressed that Blandford & Payne diffusion equation for the photon occupation number could be regarded as a particular case of the standard cosmic–ray transport equation in which the diffusion coefficient, $\kappa \propto r^{\alpha} v^{\gamma}$, does not depend on the photon energy ($\gamma = 0$) and $\alpha - \beta = 2$. Starting from this result, Schneider & Bogdan were able to generalize PB analysis to include the transition between Thomson ($\gamma = 0$) and Klein–Nishina ($\gamma = 1$) scattering cross–sections. The competitive role of dynamical and thermal Comptonization in accretion flows with a non–zero electron temperature was studied by Colpi (1988). More recently, Mastichiadis & Kylafis (1992) investigated the effects of dynamical Comptonization in near–critical accretion onto a neutron star. Their approach is very similar to PB, but the presence of a perfectly reflecting inner boundary (either the NS surface or the magnetosphere) was taken into account. They have shown that, in this case, the emergent spectrum is much harder than in PB and that the spectral index depends both on $\beta$ and on the depth at the inner boundary.

Even when dealing with black hole accretion, all previous analyses neglected both special and general relativity. The presence of an event horizon was not considered and only terms up to first order in $v/c$ were retained. In this paper we extend PB calculations to account for relativistic effects on radiative transfer which arise when the flow velocity approaches the speed of light in the vicinity of the hole horizon; as in PB, we assume that scattering is elastic in the electron rest frame. The motivation for this work is twofold, much like in Mastichiadis & Kylafis. First: to investigate the properties of the emergent spectrum in a more realistic accretion scenario, in which the optical depth near the horizon is not so large to prevent photons from escaping. Second: to check by means of an analytical calculation the results obtained with a numerical code recently developed for solving the complete transfer problem in spherical flows (Zane et al. 1996). Computed spectra show, in fact, a power–law, high–energy tail but the spectral index depends on the optical depth at the horizon, even for fixed $\beta = -1/2$, and it is always smaller than 2 (PB result). Here we show that advection/aberration effects in the high–speed flow near the
horizon, due to the finiteness of the depth there, produce a power–law tail flatter with respect to PB and enable photons to drift also towards energies lower than the injection energy.

In addition, we present an analysis of dynamical Comptonization in an expanding atmosphere, using the same assumptions of PB. We found that the solution for the emergent flux shows specular features with respect to PB. Adiabatic expansion produces a drift of injected monochromatic photons to lower energies and the formation of low–energy, power–law tail. In this case, however, the spectral index is independent on the velocity gradient and turns out to be always equal to $-3$.

## 2 Radiative Transfer in a Converging Flow

In this and in the following sections we deal with the transfer of radiation through a scattering, steady, spherical flow, characterized by a power–law velocity profile $v \propto r^\beta$. Under these assumptions the rest–mass conservation yields immediately a density profile $\rho \propto r^{-2-\beta}$ from which it follows that the electron–scattering optical depth is $\tau = \kappa_e \rho g/(1 + \beta) \propto r^{-1-\beta}$. The parameter $\beta$ is positive for outflows and it has to be $\beta > -1$ for the optical depth to decrease with increasing $r$. Units in which $c = G = 1$ are used unless explicitly stated.

Blandford & Payne (1981a, b) and PB restricted their analysis to converging flows with a non–relativistic bulk velocity and to conservative and isotropic scattering in the electron rest frame. Combining the first two moment equations, written in the source term. In the frame comoving with the fluid they found that, in diffusion approximation, the (angle–averaged) photon occupation number $n$ obeys a Fokker–Planck equation. Defining $\tau^* \equiv 3\tau v$ and looking for separable solutions of the form

$$n(\nu, \tau^*) = f(\tau^*) (\nu^3 + \beta) \nu^{-\lambda},$$

the resulting second order ordinary differential equation for $f$ reduces to a confluent hypergeometric equation (here the sign of $\beta$ is the opposite with respect to PB, e.g. $\beta = -1/2$ for free–fall, according to the assumptions at the beginning of this section). As discussed by PB, the solution corresponding to a constant radiative flux at infinity and to adiabatic compression of photons for $\tau \to \infty$ is expressed in terms of the Laguerre polynomial $L_n^{(3+\beta)}(\tau^*)$. The above two conditions give rise to a discrete set of eigenvalues for the photon index $\lambda$ which are given by

$$\lambda_n = \frac{3(n + 3 + \beta)}{2 + \beta} \quad (n = 0, 1, 2, \ldots)$$

(2)

The general solution is written as the superposition of different modes. Assuming that monochromatic photons with $\nu = \nu_0$ are injected at $\tau^* = \tau_0^*$, it is

$$n \propto \tau^{3+\beta} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+4+\beta)} L_n^{(3+\beta)}(\tau_0^*) L_n^{(3+\beta)}(\tau^*) \left( \frac{\nu}{\nu_0} \right)^{-\lambda_n}.$$  \hspace{1cm} (3)

In this case, bulk motion comptonization tends to create a power law, high energy tail. Defining the spectral index as

$$\alpha = -\frac{d\ln L(\nu, 0)}{d\ln \nu}$$

(4)

where $L$ is the luminosity, PB found

$$\lim_{\nu \to \infty} \alpha = \frac{3}{2 + \beta}$$

(5)

showing that the spectral slope at high frequencies is dominated by the fundamental mode $n = 0$. In particular, it is $\alpha = 2$ for a free–falling gas.

The same results can be recovered using the PSTF moment formalism introduced by Thorne (1981), as shown by Nobili, Turolla & Zampieri (1993). Here we outline the general method, mainly to introduce some basic concepts that will be used in the following sections. In particular, we consider the first two PSTF moment equations with only Thomson scattering included in the source term. In the frame comoving with the fluid they read

$$\frac{\partial w_1}{\partial \ln r} + 2w_1 + \frac{y}{y} (\nu_1 - \frac{\partial w_1}{\partial \ln \nu}) - v \left[ \frac{\partial w_0}{\partial \ln r} + (1 - \beta) \frac{\partial w_2}{\partial \ln \nu} - (2 + \beta) \left( \frac{1}{3} \frac{\partial w_0}{\partial \ln \nu} - w_0 \right) \right] = 0 \quad (6a)$$

$$\frac{1}{3} \frac{\partial w_0}{\partial \ln r} + \frac{\partial w_2}{\partial \ln r} + 3w_2 - \frac{y}{y} (\nu_2 - \frac{1}{3} \frac{\partial w_0}{\partial \ln \nu}) - v \left[ -\frac{3}{5} (4 + \beta) \nu_1 - \frac{1}{5} (2 + 3\beta) \frac{\partial w_1}{\partial \ln \nu} + \frac{\partial w_1}{\partial \ln r} \right] + (1 - \beta) \left( \frac{w_3 + \partial w_3}{\partial \ln \nu} \right) + \frac{(1 + \beta) \nu}{y} w_1 = 0 \quad (6b)$$

where a prime denotes the total derivative wrt $\ln r$, $y = \sqrt{1 - r_g/r}/\sqrt{1 - v^2}$, $r_g$ is the gravitational radius, and $v$ is taken positive for inward motion. As discussed by Turolla & Nobili (1988, see also Thorne, Flammang & Zytkow 1981), in diffusion...
approximation the hierarchy of the frequency–integrated PSTF moments \( W_1 \) is such that

\[
\begin{align*}
W_1 &\sim \frac{W_0}{\tau} \\
W_2 &\sim \frac{W_0}{\tau} \left( \frac{1}{\tau} - v \right)
\end{align*}
\]

(7)

The same hierarchy can be assumed to hold also for frequency–dependent moments in a scattering medium.

PB result can be reproduced in the limit of large \( \tau \) and small \( v \), i.e. retaining in equations (6) only terms of order \( w_0 \), \( w_0/\tau \) and \( w_0/\tau^2 \), and suppressing gravity, which is equivalent to set \( y = 1 \) and \( y' = 0 \). In particular, under such hypothesis, all terms containing both \( w_2 \) and \( w_3 \) can be neglected and the moment equations become

\[
\begin{align*}
\frac{\partial}{\partial \ln t} w_1 - v \frac{\partial w_0}{\partial \ln t} - 2w_1 + v(2 + \beta) \left( w_0 - \frac{1}{3} \frac{\partial w_0}{\partial \ln \nu} \right) &= 0 \\
\nu^\beta \frac{\partial w_0}{\partial \ln t} - tw_1 &= 0,
\end{align*}
\]

(8a)
\( (8b) \)

where \( t = (1 + \beta)^{-1} \). Equations (8a) and (8b) can be combined together to yield a second order, partial differential equation for the radiative flux

\[
t \frac{\partial^2 w_1}{\partial t^2} - (t + 1 - \beta) \frac{\partial w_1}{\partial t} + \left( 1 - \frac{2\beta}{t} \right) w_1 - \frac{2 + \beta}{3} \frac{\partial w_1}{\partial \ln \nu} = 0.
\]

(9)

Following PB, the solution of equation (9) can be found by separation of variables. Writing \( w_1 = t^\beta h_1(t)\nu^{-\alpha} \), it is easy to show that for \( p = 2 \) and for \( p = -\beta \), equation (9) becomes a confluent hypergeometric equation for \( h_1 \). Actually the requirement of constant radiative flux at infinity is met only for \( p = 2 \), and in this case we get

\[
t \frac{d^2 h_1}{dt^2} + (3 + \beta - t) \frac{dh_1}{dt} - \left( 1 - \frac{2 + \beta}{3\alpha} \right) h_1 = 0.
\]

(10)

As previously discussed, the physical solution for \( w_1 \) can be obtained as a superposition of the Kummer functions \( M(-n,3+\beta,t) \propto L_n^{(2+\beta)}(t) \), for \( n = 0,1,\ldots \), with corresponding eigenvalues

\[
\alpha_n = \frac{3(n+1)}{2+\beta}.
\]

(11)

In this case

\[
w_1 = t^2 \sum_{n=0}^{\infty} A_n L_n^{(2+\beta)}(t)\nu^{-\alpha_n},
\]

(12)

where the \( A_n \)'s are constants to be fixed by the boundary conditions. At sufficiently large frequencies the spectrum is dominated by the fundamental mode; in particular, for \( \beta = -1/2 \), it is again \( \alpha_0 = 2 \).

3 Importance of Relativistic Effects

Here we consider the effects of dynamical Comptonization in spherical accretion onto a non–rotating black hole, taking into full account both gravity and velocity terms in the moment equations. With reference to this particular problem, we can safely assume that matter is free–falling, \( v = (r/r_g)\beta \) with \( \beta = -1/2 \). Note that in spherical accretion onto black holes the radiative flux is never going to influence the flow dynamics close to the horizon (see e.g. Gilden & Wheeler 1980; Nobili, Turolla & Zampieri 1991; Zampieri, Miller & Turolla 1996). In this case dynamics cancels, locally, gravity, so that \( y = 1, y' = 0 \). The moment equations look then “non–relativistic” in form, although \( v \) can be arbitrarily close to unity. Corrections due to large values of the flow velocity were not considered in previous works despite the fact that they are bound to become important near the event horizon where \( v \sim 1 \). We note that the bulk of the emission in realistic accretion models is expected to come precisely from the region close to \( r_g \). As in the non–relativistic analysis presented in the last section, we consider the diffusion limit, truncating self–consistently both equations (6) to terms of order \( w_0/\tau \). The moments hierarchy, expressions (7), shows that all terms containing \( w_3 \) can be always neglected in equation (6b), since they are of order \( w_0/\tau^2 \). In the present case, however, all other terms must be retained. In fact, \( v w_1 \sim w_1 \sim w_0/\tau \) when \( v \sim 1 \) and \( w_2 \sim w_0/\tau \), at least for \( v \sim 1/\tau \) (see again expressions [7]). This implies that \( w_1 \) and \( w_2 \) contribute to the same extent to the anisotropy of the radiation field. Note that under such conditions it is \( W_2 = 4\pi(K - J/3) < 0 \), as already pointed out by Turolla & Nobili (1988), so that in high–speed, diffusive flows \( K \) may become less than \( J/3 \). Contrary to the case discussed in section 2 where \( w_2 \) is negligible,
now the system of the first two moment equations is not closed. However, up to terms of order \( w_0/\tau \), the second moment equation does not contain moments of order higher than \( w_2 \) and provides then the required closure equation

\[
\nu \frac{\partial w_2}{\partial \ln \tau} - \frac{4}{15} \frac{\partial w_1}{\partial \ln \nu} + \frac{15}{14} v w_2 - \frac{4}{15} w_1 + \frac{2}{5} v w_0 + \frac{3}{14} \nu \frac{\partial w_2}{\partial \ln \nu} - \frac{2}{15} \frac{\partial w_0}{\partial \ln \nu} + \frac{3}{10} w_2 = 0. \tag{13}
\]

The complete system (6a), (6b) and (13) is awkward and a solution can be obtained only numerically. It is possible, nevertheless, to find an analytical solution if we consider the closure condition for \( w_2 \) which follows from equation (13) with only terms of order \( w_0 \) retained

\[
w_2 = \frac{4}{9} \frac{v}{\tau} \left( \frac{\partial w_0}{\partial \ln \nu} - 3 w_0 \right). \tag{14}
\]

With this closure, \( w_2 \) is always negative provided that \( \partial w_0 / \partial \ln \nu < 0 \). This implies that equation (14) is strictly valid only for \( \tau \nu \gg 1 \) (see expression [7]), that is to say below the trapping radius. Introducing the new dependent variables \( f_0 = v w_0 \), \( f_1 = w_1 \) and \( f_2 = v w_2 \), the moment equations become

\[
\frac{t}{\gamma} \frac{\partial f_0}{\partial t} + \frac{1}{2} \frac{\partial f_0}{\partial \log \nu} - 2 f_0 - \frac{t}{\alpha} \frac{\partial f_1}{\partial t} + 2 f_1 = 0,
\]

\[
\frac{t}{\beta} \frac{\partial f_1}{\partial t} + \frac{1}{10} \frac{\partial f_1}{\partial \log \nu} + \left( \frac{t}{3} - \frac{9}{10} \right) f_1 + \frac{t}{9} \frac{\partial f_2}{\partial t} - \frac{7}{2} f_2 = 0
\]

\[
f_2 = \frac{4}{9 t h} \left( \frac{\partial f_0}{\partial \log \nu} - 3 f_0 \right) = 0,
\]

where \( t_h \) is the value of \( t \) at the radius where \( v = 1 \), i.e. at \( r = r_\gamma \) in the case under examination. We note that for \( t_h \rightarrow \infty \) equations (15a,b) give exactly the low–velocity limit of PB (equations [8a,b]), irrespective of the value of \( v \). This is because when \( t_h \rightarrow \infty \) the scattering depth itself near the horizon must be very large, so the radiation field there is very nearly isotropic. Departures from isotropy, due both to the radiative flux \( w_1 \sim w_0/\tau \) and to the radiative shear \( w_2 \sim (1/\nu - v) w_0/\tau \) become vanishingly small, no matter how large velocity is. Under such conditions PB approach is still valid just because both \( w_1 \) and \( w_2 \) become negligible in the moment equations, although they may be of the same order.

The system (15) can be solved looking again for separable solutions of the type \( f_i = g_i(t) \nu^{-\alpha} \). After some manipulation, it can be transformed into a pair of decoupled, second order, ordinary differential equation for \( g_0(t) \) and \( g_1(t) \), having the same structure. In particular for \( g_1(t) \) it is

\[
t^2 (\beta + \gamma) \frac{d^2 g_1}{dt^2} + t (\delta + \epsilon) \frac{dg_1}{dt} - (\eta t + \lambda) g_1 = 0
\]

where

\[
\beta = 60 t_h, \quad \gamma = -(20/3) t_h \left[ 3 t_h - 4 (3 + \alpha) \right], \quad 
\delta = 20 t_h^2 - 18 (2 \alpha + 3) t_h - 40 \alpha (3 + \alpha), \quad 
\epsilon = 30 t_h \left[ t_h - 4 (3 + \alpha) \right], \quad 
\eta = 10 (\alpha + 2) t_h^2 + \left( 31 \alpha^2 / 3 + 7 \alpha - 54 \right) t_h - 4 \alpha (\alpha + 3) (\alpha + 9), \quad 
\lambda = (20/3) t_h \left[ 3 t_h - 28 (3 + \alpha) \right].
\]

Equation (16) can be reduced to a hypergeometric equation upon the change of variables \( g_1(t) = t^p h_1(z) \) and \( z = -(\beta/\gamma)t \), where \( p \) is the solution of the quadratic equation \( \gamma p^2 + (\epsilon - \gamma)p - \lambda = 0 \). A direct, but tedious, calculation shows that

\[
p_+ = 2, \quad p_- = \frac{7}{2} - \frac{9 t_h}{3 t_h - 4 (3 + \alpha)}, \tag{18}
\]

in the limit \( t_h \rightarrow \infty, p_- = 1/2 \) as in the case considered in the previous section, and \( p = p_+ \) will be used in the following to meet the requirement of constant radiative flux at infinity. Equation (16) can be now written in the form

\[
z (1 - z) \frac{d^2 h_1}{dz^2} + \left[ \frac{\epsilon + 2p}{\gamma} - \frac{\delta + \eta}{\beta + 2p} \right] z \frac{dh_1}{dz} - \left[ p (p - 1) + \frac{\delta}{\beta} - \frac{\eta}{\beta} \right] h_1 = 0 \tag{19}
\]

which is a hypergeometric equation. The general solution is expressed in terms of the hypergeometric function \( _2F_1(a, b; c; z) \).
and the three parameters \( a, b, c \) (see Abramowitz & Stegun 1972, AS in the following, for notation) are given by the relations

\[
c = \frac{\epsilon}{\gamma} + 2p
\]

\[
a + b + 1 = \frac{\delta}{\beta} + 2p
\]

\[
ab = p(p - 1) + \frac{\delta}{\beta} - \frac{\eta}{\beta}.
\]

Solving for \( a, b \) we obtain, after a considerable amount of algebra,

\[
a = \frac{2 - \alpha}{2} - \frac{2\alpha (\alpha + 3)}{3t_h}
\]

\[
b = \frac{t_h}{3} + \frac{11 - \alpha}{10}.
\]

It can be seen from equations (20) that, in the limit \( t_h \to \infty \), \( b \) diverges while \( a \) stays finite; in the same limit the hypergeometric equation reduces to the confluent hypergeometric equation (see e.g. Sneddon 1956). As discussed in section 2, the relevant solution in the non-relativistic case is given by Laguerre polynomials and is recovered imposing \( a = -n \), with \( n = 0, 1, \ldots \). The solution of equation (21) which reduces to PB for \( t_h \to \infty \) is found imposing again that \( a \) is either zero or a negative integer (although other classes of solutions that do not match PB may exist). In this case \( h_1 \) is still polynomial and takes the form

\[
h_1(z) = \frac{\Gamma(1)}{c_n} P^{(c-1,b-c-n)}_n (1 - 2z),
\]

where \( P^{(p,q)}(z) \) is the Jacobi polynomial and \( (c)_n = \Gamma(c + n)/\Gamma(c) \) is the Pochhammer’s symbol (see again AS). For \( t_h \to \infty \), it is \( c \sim 5/2, b \sim t_h/3, z \sim 3t/t_h = t/b \) and

\[
P^{(c-1,b-c-n)}_n (1 - 2\frac{t}{b}) \to L_n^{(3/2)}(t),
\]

so that, as expected, the solution of section 2 is recovered.

The discrete set of eigenvalues for the spectral index \( \alpha_n \) follows immediately from (20a) solving the quadratic equation \( a = -n \). For each \( n \) both a positive, \( \alpha_n^+ \), and a negative, \( \alpha_n^- \), mode is present

\[
\alpha_n^\pm = -\frac{(12 + 3t_h)}{8} \pm \sqrt{(12 + 3t_h)^2 + 96(n + 1)t_h}.
\]

We checked that the eigenvalues of the equation for \( g_0 \) are again given by equation (22), in agreement with the starting hypothesis that \( a \) is the same for all moments.

The general solution for the spectral flux is obtained as a linear superposition of all modes

\[
w_1 = t^2 \left[ \sum_{n=0}^{\infty} A_n^+ (-1)^n \frac{(b)_n}{(c)_n} G_n(b - n, c, z) \left( \frac{\nu}{\nu_0} \right)^{-\alpha_n^+} + \sum_{n=0}^{\infty} A_n^- (-1)^n \frac{(b)_n}{(c)_n} G_n(b - n, c, z) \left( \frac{\nu}{\nu_0} \right)^{-\alpha_n^-} \right],
\]

where we have expressed \( h_1 \) in terms of the shifted Jacobi polynomials \( G_n \). We remind that \( b, c \) and \( z \) are all functions of \( \alpha_n^\pm \), although we dropped all indices to simplify the notation. The two sets of constants \( A_n^\pm \) are fixed imposing a boundary condition at the injection frequency \( \nu = \nu_0 \). The only boundary condition compatible with the assumption of a pure scattering flow for \( t < t_h \) is that all photons are created in an infinitely thin shell at \( t_* \). This is equivalent to ask that \( w_1(t, \nu_0) \propto \delta(t/t_* - 1) \), as in PB.

At variance with the results discussed in section 2, now the series in equation (23) can not be summed using the polynomial generating function because \( b, c \) and \( z \) depend on \( n \). The coefficients \( A_n^\pm \) are solution of an upper triangular, infinite system of linear algebraic equations (see Appendix A). It can be easily shown that the two series in equation (23) do not converge for any value of \( \nu/\nu_0 \). In fact, the general term of the first series, which is of the type \( f(n)/(\nu_0/\nu)^{\alpha_n^+} \), can not be infinitesimal for arbitrarily small frequencies unless the series truncates, which is not the case if it must reproduce the \( \delta \)-function at \( \nu = \nu_0 \). On the other hand, the series is absolutely convergent for \( \nu > \nu_0 \), provided that \( |f(n)| \) is bounded. For \( N \gg 1 \), the series has a majorant \( \sum_{n=N}^{\infty} (\nu_0/\nu)^{\sqrt{2}x} \) which is convergent because \( \int_{\nu_0}^{\infty} (\nu_0/\nu)^{\sqrt{2}x} dx \) is finite for \( \nu > \nu_0 \). The same argument applies to the second series for \( \nu < \nu_0 \), so that the solution satisfying our boundary condition is

\[
w_1(t, \nu) = \begin{cases} 
  t^2 \sum_{n=0}^{\infty} A_n^+ (-1)^n \frac{(b)_n}{(c)_n} G_n(b - n, c, z) \left( \frac{\nu}{\nu_0} \right)^{-\alpha_n^+} & \nu < \nu_0; \\
  t^2 \sum_{n=0}^{\infty} A_n^+ (-1)^n \frac{(b)_n}{(c)_n} G_n(b - n, c, z) \left( \frac{\nu}{\nu_0} \right)^{-\alpha_n^+} & \nu \geq \nu_0.
\end{cases}
\]
Equation (24) exhibits two striking features, not shared by its non-relativistic counterpart, which arise from the presence of advection/aberration terms in the moment equations. First of all, we note that according to equation (24) photons injected at $\nu = \nu_0$ can be shifted both to higher and lower energies by dynamical Comptonization. This is in apparent contrast with PB result that photons can only gain energy in scatterings with electrons in a converging flow (the adiabatic compression). PB statement is, however, correct up to $O(v)$ terms and their equation (8) is the low-velocity limit of the more general expression for the photon energy change along a geodesic (see e.g. Novikov & Thorne 1973)

$$\frac{1}{\nu} \frac{d\nu}{d\ell} = -\left( n^i a_i + \frac{1}{3} \theta + n^i n^j \sigma_{ij} \right),$$  \hspace{1cm} (25)

where $n^i$ is the unit vector along the photon trajectory and $a_i$, $\theta$ and $\sigma_{ij}$ are the flow 4-acceleration, expansion and shear, respectively. In free-fall $a^i$ vanishes while it can be safely neglected in PB approximation being $O(v^2)$. The remaining two terms are both of order $v$

$$\theta = -\frac{3}{2} \frac{v}{r}$$

$$n^i n^j \sigma_{ij} = \frac{1}{2} \frac{v}{r} (3\mu^2 - 1)$$  \hspace{1cm} (26)

where $\mu$ is the cosine of the angle between the photon and the radial directions. The mean photon energy change can be obtained angle-averaging equation (25) over the specific intensity

$$I_\nu(\mu) = w_0 + 3\mu w_1 + \frac{15}{4} (3\mu^2 - 1)w_2 + \ldots.$$  \hspace{1cm} (27)

Recalling the behaviour of the radiation moments in the diffusion limit, we get

$$\left\langle \frac{1}{\nu} \frac{d\nu}{d\ell} \right\rangle = \frac{v}{r} \left[ \frac{1}{2} + \frac{3}{v} \left( \frac{1}{\tau} - v \right) \right].$$  \hspace{1cm} (28)

The second term in square brackets arises because of shear and is negligible in PB approximation, being either $O(1/\tau^2)$ or $O(v/\tau)$. This implies that the mean photon energy change is always positive. However, when advection and aberration are taken into account (see equations [25], [26]) photons moving in a cone around the radial direction suffer an energy loss and the collective effect is stronger when the flow velocity approaches unity in regions of moderate optical depth.

The second important feature concerns the slope of the power-law, high-energy tail of the spectrum. From equation (22) the fundamental mode is

$$\alpha_0^+ = \frac{- (12 + 3t_h) + \sqrt{(12 + 3t_h)^2 + 96t_h}}{8},$$  \hspace{1cm} (29)
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Figure 2. Emergent flux $F_{\nu}$ computed using our CRM code; the derived spectral index is 1.36. In this model $t_h \simeq 15$ and the corresponding value of $\alpha_0^+ = 1.43$ (dashed line).

and, for large values of $t_h$, equation (29) gives

$$\alpha_0^+ = 2 - \frac{40}{3t_h} + \frac{1120}{9t_h^2} + O \left(\frac{1}{t_h^3}\right).$$

(30)

At large enough frequencies the spectral index is dominated by the fundamental mode which, for $t_h \geq 1$, sensibly deviates from the value predicted by the non–relativistic calculation. Despite the fact that this effect is present below the trapping radius, we stress that, contrary to a widespread belief, the trapping radius does not act as a one–way membrane. Photons produced near or below the surface $\tau v = 1$ can still escape to infinity even if both the large optical depth and the strong advection caused by the inward flow dramatically reduce the emergent radiative flux. Moreover, these photons, although comparatively few, are the more strongly comptonized and will anyway dominate the high–energy tail of the spectral distribution. Equation (30) shows that the emergent spectrum turns out to be flatter with respect to PB case. The two main features of our solution, harder spectrum and drift of photons below $\nu_0$, can be clearly seen in figure 1, where the emergent spectrum is shown for $t_h = 20$.

It is interesting to compare the present, analytical solution with the numerical result obtained using the fully GR characteristic–ray code (CRM) described in Zane et al. (1996). In figure 2 we show the emergent spectrum relative to the “cold” solution for black hole accretion with $\varrho_h = 1.42 \times 10^{-5}$ g cm$^{-3}$. In this case both electron scattering and free–free emission/absorption are considered. At large enough frequencies scattering is the only source of opacity, so, in this limit, we expect our idealized analytical model to be representative of the realistic situation. The numerical model has $t_h \simeq 15$ which corresponds to $\alpha_0^+ = 1.43$. This value is in excellent agreement with the derived spectral index $\alpha = 1.36$. 

4 RADIATIVE TRANSFER IN AN EXPANDING ATMOSPHERE

In this section we discuss the case of a pure scattering, expanding atmosphere with a power–law velocity profile

$$v = v_\ast \left(\frac{r}{r_\ast}\right)^\beta$$

where the subscript “$\ast$” refers to the base of the envelope, now $v$ is taken positive outwards and PB approximation is used. The second order partial differential equation for the radiation flux is given by equation (9), upon the substitution of $t$ with $-t$

$$t \frac{\partial^2 w_1}{\partial t^2} + (t - 1 + \beta) \frac{\partial w_1}{\partial t} - \left(1 + \frac{2\beta}{t}\right) w_1 + \frac{2 + \beta}{3} \frac{\partial w_1}{\partial \ln \nu} = 0.$$  

(32)

This equation could be integrated using the same technique discussed in section 2, looking for separable solutions $w_1 = t^2 h_1(t) \nu^{-\alpha}$. It can be easily checked that equation (32) yields again, upon factorization, a Kummer equation for $h_1(t)$, as
in the converging flow case. A problem arises, however, as far as boundary conditions are concerned: in section 2 the only physically meaningful solution was selected asking that the flux become a constant for \( t \to 0 \) and that adiabatic compression of photons hold for \( t \to \infty \). In that case the existence of these physical constraints was sufficient to fix univocally the mathematical solution. However, this particular issue turns out to be much more delicate in the wind problem. We preferred to look for an alternative method of solution which allows for an easier handling of boundary conditions. Equation (32), describing diffusion of photons through a moving medium, is a Fokker–Planck equation and can be brought into the standard Fokker–Planck form

\[
\frac{\partial^2 (u_1)}{\partial t^2} - \frac{\partial}{\partial t}[(1 - \beta - t)u_1] = \frac{\partial u_1}{\partial x},
\]

(33)

where we have defined \( w_1 = t^2 u_1 \) and \( x = -3/(2 + \beta) \ln \nu \). The solution can be found by Fourier transforming equation (33) with respect to \( t \), solving the equation for the Fourier transform \( \hat{u}_1 \), and then transforming back (see e.g. Risken 1989). The equation for the Fourier transform is obtained from equation (33) replacing \( \partial/\partial t \) by \( ik \) and \( t \) by \( i\partial/\partial k \),

\[
ik(1 + ik) \frac{\partial \ln \hat{u}_1}{\partial k} + \frac{\partial \ln \hat{u}_1}{\partial x} = (1 + \beta)ik.
\]

(34)

This is a first order PDE which can be solved by standard methods (see e.g. Sneddon 1957) once a boundary condition is specified. If we assume that monochromatic photons of frequency \( \nu_0 \) are injected at \( t = t_* \), the boundary condition for equation (34) is just \( \hat{u}_1 = u_0^\nu \exp(-ikt_\nu) \), here \( u_0^\nu \) is the luminosity emitted at \( \nu_0 \), and the corresponding solution is

\[
\hat{u}_1 = u_0^\nu \left[ (1 - \exp(x - x_0)) ik + 1 \right]^{1+\beta} \exp \left\{ -\frac{\exp(x - x_0)ikt_\nu}{[1 - \exp(x - x_0)] ik + 1} \right\}.
\]

(35)

The solution to equation (33) is given by the Fourier integral

\[
u_0 \int_{-\infty}^{+\infty} \hat{u}_1 \exp(ikt) dk
\]

which can be evaluated analytically in terms of the modified Bessel function \( I_q \) (see Appendix B)

\[
u_0 \left( \frac{\nu}{\nu_0} \right)^{3/2} \left( \frac{t_*}{t} \right)^{(2+\beta)/2} \left[ 1 - \left( \frac{\nu}{\nu_0} \right)^{3/(2+\beta)} \right]^{-1} \exp \left\{ -\frac{t + (\nu/\nu_0)^{3/(2+\beta)} t_\nu}{1 - (\nu/\nu_0)^{3/(2+\beta)}} \right\} \times
\]

\[
I_{2+\beta} \left[ 2\sqrt{(\nu/\nu_0)^{2/(3+\beta)} t_*} \left( 1 - (\nu/\nu_0)^{2/(3+\beta)} \right) \right].
\]

(36)

The main advantage of solving equation (32) following the method outlined here is that the Fourier transform is automatically selecting the regular solution, because it can be computed only for functions that are \( L_2 \) in \( [ -\infty, \infty ] \). In other words, it is the method of solution itself which is suited for finding only regular solutions and in doing so no extra constraint is required.

The spectrum is shifted towards lower frequencies and it broadens at the same time, developing a power law, low–energy tail. The overall behaviour is similar to that of the converging flow but somehow reversed, since now photons can drift only to frequencies lower than \( \nu_0 \). There is, however, a major difference in the power–law index \( \alpha \) between the two cases since \( \alpha \) does not depend on \( \beta \) for the wind solutions, as can be seen examining the spectral behaviour of equation (36) at low frequencies. Since \( I_q(z) \sim (z/2)^q/\Gamma(q + 1) \) when the argument is small, we have for the emergent luminosity

\[
L_\nu \propto \left( \frac{\nu}{\nu_0} \right)^3 \exp \left\{ -\frac{(\nu/\nu_0)^{2/(3+\beta)} t_*}{1 - (\nu/\nu_0)^{2/(3+\beta)}} \right\} \left[ 1 - \left( \frac{\nu}{\nu_0} \right)^{(2+\beta)/3} t_* \right]^{-3-\beta} \sim \left( \frac{\nu}{\nu_0} \right)^3
\]

if \( \nu \ll \nu_0 \), which shows that \( \alpha = -3 \) irrespective of the value of \( \beta \). The monochromatic flux at \( t = 0 \) is shown in figure 3 for \( \beta = 1 \) and \( t_* = 1 \); the power–law tail at low energies is clearly visible.

5 DISCUSSION AND CONCLUSIONS

In this paper we have reconsidered the transfer of radiation in a scattering, spherically–symmetric medium, extending PB analysis of converging flows to the relativistic case and investigating the effects of bulk motion Comptonization in expanding atmospheres. In the low–velocity limit and assuming diffusion approximation, PB found that monochromatic photons injected at the base of the atmosphere always gain energy as they propagate outwards. The emergent spectrum exhibits an overall shift to higher frequencies and a power–law, high–energy tail with a spectral index depending on the velocity gradient. Under the same assumptions, the wind solution shows similar, although reversed, features. Adiabatic expansion now produces an overall drift toward lower energies and the formation of a power–law tail at low frequencies. In this case, however, the spectral index is independent on the velocity gradient and turns out to be always equal to \( -3 \).

Both these analyses are correct to first order in \( \nu/c \) and can be thought to adequately describe situations in which bulk motion is non–relativistic in regions of moderate scattering depth. Relativistic corrections are, in fact, related to the
anisotropy of the radiation field and are washed out if $\tau \gg 1$. Obviously, if the flow is optically thin, repeated scatterings are ineffective no matter how large the velocity is. In outflowing atmospheres high velocities are expected at large radii, where the optical depth has dropped below unity, so that our assumption can be reasonable. On the other hand, in accretion flows onto compact objects the condition $\tau \gg 1$ where $v \sim 1$ is likely to be met only when the accretion rate becomes hypercritical. This shows that a relativistic treatment of dynamical comptonization is indeed required in investigating the emission properties of accretion flows. For $v \sim 1$ the diffusion limit is not recovered simply asking that the radiative flux is proportional to the gradient of the energy density, since the radiative shear is as important as the flux. Relativistic corrections produce two main effects: first, photons are shifted toward both higher and lower frequencies by dynamical comptonization and, second, the spectrum at large frequencies is sensibly flatter than in the non-relativistic case. The spectral index now depends not only on the velocity gradient, but also on the value of the scattering depth at the horizon and goes to its non–relativistic limit when $\tau_h$ tends to infinity. Despite the fact that relativistic effects are important only where $\tau v > 1$, that is to say below the trapping radius, their signature is still present in the emergent spectrum. In particular, the high energy tail is populated by the strongly comptonized photons coming just from this region. A similar effect was found by Mastichiadis & Kylafis (1992, see also Zampieri, Turolla, & Treves 1993) in an accretion flow onto a neutron star. In their case the formation of an essentially flat ($\alpha \approx 0$) spectrum is due to the fact that photons experience a very large number of energetic scatterings before emerging to infinity, since no advection is present being the star surface a perfect reflector. Our spectrum is softer with respect to Mastichiadis & Kylafis just because a sizable fraction of the more boosted photons are dragged into the hole, but, at the same time, it is harder than PB since in the relativistic regime the mean energy gain per scattering is higher. From the mathematical point of view it is noteworthy that the assumption of a finite optical depth at the inner boundary (i.e. at the horizon in our model or at the reflecting surface in Mastichiadis & Kylafis) produces a fundamental mode which is flatter with respect to PB; in both cases PB result is recovered in the limit $\tau_h \to \infty$. The possibility that scattering of photons in an accretion flow onto a black hole produces a power–law tail with spectral index flatter than 2 was also suggested in a very recent paper by Ebisawa, Titarchuk, & Chakrabarti (1996). Using a semi–qualitative analysis they found that the spectral index is close to 3/2 for large values of the optical depth at the horizon and discussed the possible relevance of this result in connection with the observed hard X–ray emission from black hole candidates in the high state.

We note that for $1 < \tau_h < 32/9$ the predicted spectral index (see equation [29]) is smaller than 1, implying a divergent frequency–integrated luminosity; this behaviour is not new and was already found by Schinder & Bogdan (1989) and Mastichiadis & Kylafis (1992). It simply reflects the fact that photons can gain an arbitrarily large amount of energy in collisions with the free–falling electrons. It should be taken into account, however, that, when $h\nu \approx m_e c^2$ the electron recoil in the particle rest frame can not be neglected anymore, so for large enough energies our treatment is not valid, as discussed in more detail in Zampieri (1995). The decrease of the cross–section in the quantum limit makes the scattering process less efficient, producing a sharp cut–off in the spectral distribution. Finally, as already stressed by Blandford & Payne (1981a)
and Colpi (1988), thermal comptonization dominates over dynamical comptonization when $v^2 \lesssim 12kT/m_e$. The spectral distribution depends then on the relative strength of competitive processes such as heating/cooling by thermal comptonization and compressional heating and must be derived solving the radiative transfer equation in its complete form.

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Appendix A

Since the polynomials $G_n(b - n, c, z)$ appearing in equation (24) are not an orthogonal system, is not possible to derive an explicit expression for the coefficients $A^\pm_n$. Here we show that these constants can be, in principle, obtained as the solution of an infinite system of linear algebraic equations. We note that the two sets $A^\pm_n$ are not independent, because the two expressions in (24) must match at $\nu = \nu_0$, where

$$w_1(t, \nu_0) = A6(t/t_\ast - 1)$$  \hspace{1cm} (A1)

(A) is a constant related to the monochromatic flux injected at the inner boundary. Since $z \propto t/t_b$ and $x = t/t_\ast$, the polynomials $G_n(b - n, c, z)$ can be expressed in terms of $G_n(3, 3, x)$, which form an orthogonal system, as

$$G_n(b - n, c, z) = \sum_{m=0}^{\infty} C_{nm} G_m(3, 3, x).$$  \hspace{1cm} (A2)

The coefficients $C_{nm}$ are solution of the upper triangular system of linear algebraic equations

$$\sum_{m=k}^{n} (-1)^{m-n} \binom{m}{k} \frac{(m + 2)!(m + 2 + k)!}{(2m + 2)!(k + 2)!} C_{nm} = \binom{n}{k} \frac{\Gamma(c + n)\Gamma(b + k)}{\Gamma(b + n)\Gamma(c + k)} \left(\frac{\beta}{\gamma t_\ast}\right)^k$$  \hspace{1cm} $k = 0, \ldots, n.$  \hspace{1cm} (A3)

Recalling the standard expansion of the $\delta$–function over an orthogonal set of eigenfunctions and using again $G_n(3, 3, x)$ as a basis, it is

$$\delta(x - 1) = x^2 \sum_{m=0}^{\infty} \frac{(2m + 3)!}{m!(m + 2)!} G_m(3, 3, x).$$  \hspace{1cm} (A4)

Inserting (A2) and (A4) into (A1) and equating the coefficients of the polynomials of the same degree, we obtain

$$\sum_{m=m}^{\infty} (-1)^n \binom{n}{m} A_n^+ C_{nm} A_m^+ = \frac{A}{t^2} \frac{(2m + 3)!}{m!(m + 2)!} \quad m \geq 0.$$

The numerical evaluation of $A_n^+$ has been carried out truncating the series appearing in (A5) to a maximum order $N \sim 60$ and solving the system by backsubstitution.
APPENDIX B

In this Appendix we derive the expression for $u_1$, equation (36), starting from the Fourier integral. By defining, for the sake of conciseness, $a = (\nu/\nu_0)^{3/(2+\beta)}$, the Fourier integral can be written as

$$u_1 = \frac{u_0}{2\pi} \int_{-\infty}^{+\infty} [(1-a)ik+1]^{1+\beta} \exp \left[ ik t - \frac{ait_\ast}{(1-a)ik+1} \right] dk,$$

which can be transformed into an integral in the complex plane by introducing the new, complex, integration variable $z = (1-a)ik+1$:

$$u_1 = \frac{u_0}{2\pi i} \frac{1}{1-a} \exp \left[ \frac{-t+at_\ast}{1-a} \right] \int_{-i\infty+1}^{i\infty+1} z^{1+\beta} \exp \left[ \frac{t}{1-a} z + \frac{at_\ast}{1-a} z^{-1} \right] dz.$$

The integral appearing in (B2) defines the Bessel function of imaginary argument (see Prudnikov, Brychkov, & Marichev 1986),

$$\int_{-i\infty+1}^{i\infty+1} z^{1+\beta} \exp \left[ \frac{t}{1-a} z + \frac{at_\ast}{1-a} z^{-1} \right] dz = 2\pi i \left( \frac{at_\ast}{t} \right)^{(2+\beta)/2} J_{2+\beta} \left[ 2(\nu t_\ast)^{1/2}/(1-a) \right].$$

so that finally we have

$$u_1 = \frac{u_0}{1-a} \left( \frac{at_\ast}{t} \right)^{(2+\beta)/2} \exp \left[ -\frac{t+at_\ast}{1-a} \right] I_{2+\beta} \left[ 2(\nu t_\ast)^{1/2}/(1-a) \right],$$

which is exactly equation (36). We note that, although the absolute convergence in the complex plane of the integral representation (B3) is proved only for $at_\ast/(a-1) > 0$, $2+\beta < 1$, direct substitution of (B4) into the Fokker–Planck equation (33) shows that (B4) is a solution with the only restriction $a < 1.$

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