The Two-Loop Master Diagram in the Causal Approach *

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Abstract

The scalar two-loop master diagram is revisited in the massive cases needed for the computation of boson and fermion propagators in QED and QCD. By means of the causal method it is possible in a straightforward manner to express the propagators as double integrals. In the case of vacuum polarization both integrations can be carried out in terms of polylogarithms, whereas the last integral in the fermion propagator cannot be expressed by known special functions. The advantage of the method in comparison with Feynman integral calculations is indicated.

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1 Introduction

It is the big advantage of Epstein and Glaser's causal approach to quantum field theory [1] that it explicitly uses the causal structure of the theory, which is invisible in the traditional Lagrangian approach using Feynman rules or in path-integral methods. As a consequence, it is possible to prove the general properties of the S-matrix in very direct and transparent manner [2]. However, the full strength of the causal method comes out in higher order calculations and can be summarized in the following points:

1.) There exists no ultraviolet problem in the causal theory, so that no regularization is necessary.

2.) All integrals are four-dimensional, so that no $\gamma^5$-problem exists. Furthermore, the number of non-trivial integrations is reduced to the minimum, essentially one one-dimensional integral for every additional loop.

3.) The method is inductive in the order $n$ of perturbation theory, so that the work done in lower orders is completely and automatically utilized in higher orders.

The main reason why the Feynman rules are not optimal for loop diagrams is the following. Suppose that time ordering can simply be done by multiplication with step functions $\Theta(x_0^k - x_0^j)$, as in the scalar theory we are going to consider. Then, in calculating a time-ordered product according to the rules, every pairing of field operators gets such a $\Theta$-function which leads to the Feynman propagators $D_F(x_k - x_j)$. However, the time ordering of the vertices is already specified by less $\Theta$-functions. Without the temporal $\Theta$-function the pairing function is $D^+$ instead of $D_F$. Consequently, some propagators $D_F$ can actually be simplified into $D^\pm$ which is simpler to integrate because it contains a $\delta$-distribution in $p$-space. For example, in the causal theory the third order scalar vertex function is given by

$$\Lambda(x_1, x_2, x_3) = D_F(x_1 - x_3)D^+(x_3 - x_2)D_F(x_1 - x_2) - D_F D^a v D^- + D^+ D^a v D^{ret}. \quad (1.1)$$

If one substitutes $D^+ = D_F - D^{av}$ in the first term, one arrives at the usual Feynman form

$$\Lambda = D_F D_F D_F - D_F D^{av}(D_F + D^-) + D^+ D^{av} D^{ret}, \quad (1.2)$$

because the last two terms are equal to $D^{av} D^{av} D^{ret}$ which vanishes by the support properties of the advanced and retarded distributions. But (1.1) is simpler to calculate than (1.2) because every member contains one $D^\pm$. This advantage is strongly increasing in higher loop diagrams.

It is the purpose of this paper to illustrate these features for the so-called two-loop master diagram of Fig.1. Interestingly enough, this diagram was already computed with the causal method by various authors without knowing it. The first were Källén and Sabry [3], their work was extended by Broadhurst [4] to other mass cases. The most extensive higher order calculations using the "dispersive method" were carried out over many years by the italian group Mignaco, Remiddi [5], Barbieri [6] and others (see [7] and references given therein). All these authors base their calculations on analytic properties of Feynman integrals which are often referred to as Cutkosky rules. The lack of understanding of these analytic properties has created the problem of anomalous thresholds. Fortunately, the latter do not appear in diagrams with two and three external legs, but they do appear in four- and more legs-diagrams [8].
Such basic subjects are discussed in the following section. As a first application, we then briefly describe the calculation of the scalar vertex function with arbitrary masses which is needed at various places in the later two-loop calculations. In Sect. 3 we turn to the master diagram for vacuum polarization. We show the calculation of the causal distribution in some detail to localize the infrared divergences which appear in the case of vanishing photon mass. The most difficult integration is the dispersion integral for the splitting. The necessary techniques to handle polylogarithms are shown in Sect. 4. Finally we consider the more complicated case of the fermion propagator. By ingenious tricks, Broadhurst [4] succeeded in expressing this propagator by a one-dimensional integral over the complete elliptic integral of the first kind \( K(k) \) together with elementary functions which can easily be computed numerically. In the causal theory one needs not be a genius. We get the result by a slight change of the standard procedure. As far as the spin structure is concerned, it is known that the general two-legs diagram can be expressed in terms of the scalar two-point functions [9]. Therefore, we restrict ourselves to the scalar case here.

2 The Causal Method and the Scalar Vertex

We first summarize the main ingredients of the causal method, for details we refer to [1,2]. In the causal theory the S-matrix is viewed as an operator-valued distribution of the following form

\[
S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n T_n(x_1, \ldots x_n) g(x_1) \ldots g(x_n),
\]

where \( g \in S(\mathbb{R}^4) \), the Schwartz space of functions of rapid decrease. The test function \( g \) plays the role of ”adiabatic switching” and provides a cutoff in the long-range part of the interaction, without destroying any symmetry. It can be considered as a natural infrared regulator. The adiabatic limit \( g \to 1 \) must be performed at the end of the calculation in the right quantities where this limit exists. The existence of the adiabatic limit becomes a problem if the theory contains massless fields.

The \( n \)-point operator-valued distributions \( T_n \) are the basic objects of the theory. They can be constructed inductively from \( T_1 \) through a number of physical requirements, the most essential one being causality. Unitarity plays no essential role. Let the operator-valued distributions \( \tilde{T}_n \) be defined by

\[
S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^3x_1 \ldots d^3x_n \tilde{T}_n(x_1, \ldots x_n) g(x_1) \ldots g(x_n).
\]

Then one defines, for arbitrary sets of points \( X, Y \) in Minkowski space, the following distributions

\[
A_n'(x_1, \ldots x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n)
\]

\[
R_n'(x_1, \ldots x_n) = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X),
\]
where the sums run over all partitions

\[ P_2 : \{x_1, \ldots, x_{n-1}\} = X \cup Y, \quad X \neq \emptyset \]

into disjoint subsets with \(|X| = n_1, |Y| \leq n - 2,\) and \(|X|\) means the number of points in the set \(X\). We also introduce

\[ D_n(x_1, \ldots, x_n) = R_n' - A_n'. \quad (2.5) \]

If the sums are extended over all partitions \(P_0^2\), including the empty set \(X = \emptyset\), we obtain the distributions

\[ A_n(x_1, \ldots, x_n) = \sum_{P_0^2} \tilde{T}_{n_1}(X)T_{n-n_1}(Y, x_n) = \]

\[ = A_n' + T_n(x_1, \ldots, x_n), \quad (2.6) \]

\[ R_n(x_1, \ldots, x_n) = \sum_{P_0^2} T_{n-n_1}(Y, x_n)\tilde{T}_{n_1}(X) = \]

\[ = R_n' + T_n(x_1, \ldots, x_n). \quad (2.7) \]

These two distributions are not known by the induction assumption because they contain the unknown \(T_n\). Only the difference

\[ D_n = R_n' - A_n' = R_n - A_n \]

is known.

One can determine \(R_n\) or \(A_n\) separately by investigating the support properties of the various distributions, this is the point where the causal structure becomes important. It turns out that \(R_n\) is a retarded and \(A_n\) an advanced distribution

\[ \text{supp} R_n \subseteq \tilde{\Gamma}_{n-1}^+(x_n), \quad \text{supp} A_n \subseteq \tilde{\Gamma}_{n-1}^-(x_n) \quad (2.10) \]

with

\[ \tilde{\Gamma}_{n-1}^\pm(x) \equiv \{(x_1, \ldots, x_{n-1}) \mid x_j \in \tilde{V}^\pm(x), \forall j = 1, \ldots, n - 1\} \quad (2.11) \]

\[ \tilde{V}^\pm(x) = \{y \mid (y - x)^2 \geq 0, \pm(y^0 - x^0) \geq 0\}. \quad (2.12) \]

Hence, by splitting of the causal distribution (2.8) one gets \(R_n\) (and \(A_n\)), and \(T_n\) then follows from (2.7) (or (2.6)). The \(T_n\)'s are well-defined time-ordered products. To carry out the splitting process, we write (2.8) in normally ordered form and then split the numerical distributions \(d^k_n(x)\), where \(x = (x_1 - x_n, \ldots, x_{n-1} - x_n)\). The causal splitting of \(d(x)\) can directly be done in momentum space by means of the following dispersion formula

\[ \hat{r}(p) = \pm \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp)}{(t \mp i0)^{\omega+1}(1-t \pm i0)}, \quad (2.13) \]

which holds true for \(p \in \Gamma^+\) (upper signs) or \(p \in \Gamma^-\) (lower signs). The result for arbitrary \(p\) is obtained by analytic continuation, based on the fact that the retarded distribution \(\hat{r}(p)\) is the boundary value of an analytic function, regular in \(\mathbb{R}^{4n-4} + i\Gamma^+\). This so-called central splitting solution is very convenient because it is obviously Lorentz covariant and
does not destroy any symmetry of the theory. \( \omega \) is the singular order of \( \hat{d} \) [2], in case of trivial splitting one has \( \omega = -1 \).

The formulae (2.3-4) contain what is usually called Cutkosky rules. But we would like to emphasize that the sums run over all non-trivial partitions \( X \cup Y \). If \( X \) and \( Y \) are connected the term corresponds to an ordinary cut through the diagram. Otherwise the diagram is decomposed into more than two pieces. In this situation anomalous thresholds appear. But this only occurs in diagrams with more than three external legs.

To calculate scalar diagrams with arbitrary masses, we start the inductive process from the following first order

\[
T_1(x) = i : \varphi_1^+(x)\varphi_2(x) : A(x) - \text{h.c.} = -\tilde{T}_1(x). \tag{2.14}
\]

Here \( \varphi_1, \varphi_2 \) are charged scalar fields with masses \( m_1 \) and \( m_2 \), respectively, and \( A(x) \) is a neutral (self-adjoint) scalar field of mass \( m_3 \). All fields are free fields satisfying the following commutation relations

\[
[\varphi_j(x), \varphi^+_j(y)] = -iD_m(x - y), \quad j = 1, 2 \tag{2.15}
\]

\[
[\varphi_j^-(x), \varphi^+_j(y)] = -iD^+_m(x - y), \tag{2.16}
\]

where \( D_m \) is the Jordan-Pauli distribution of mass \( m \) and \(( \pm )\) refers to the positive and negative frequency parts of the various quantities. \( A(x) \) fulfills the same commutation relations without the hermitian adjoint \(+\). The commutator (2.16) gives the contraction in Wick’s theorem, its Fourier transform is equal to

\[
\hat{D}^+_m(p) = \frac{i}{2\pi}\Theta(p^0)\delta(p^2 - m^2). \tag{2.17}
\]

The last vertex \( x_n \) plays a special role in the above equations (2.3-10). It is the splitting vertex, defining the edge of the causal cone. By translation invariance the numerical distributions only depend on the relative coordinates

\[
y_j = x_j - x_n, \quad j = 1, \ldots, n - 1. \tag{2.18}
\]

The Fourier transform is always understood with respect to these relative coordinates

\[
\hat{d}(p) = (2\pi)^{-2n+2}\int d(y)e^{ipy}d^4y_1 \ldots d^4y_{n-1}. \tag{2.19}
\]

Until now all \( n \)-th order distributions depend on \( n \) or \( n - 1 \) variables, the inner vertices are not integrated out. If the adiabatic limit exists, we can integrate the inner coordinates with \( g(x) = 1 \). In \( p \)-space this means that the inner momenta are put equal to 0. Then many terms in (2.3-4) vanish:

**Lemma 1.** In the adiabatic limit only those partitions \( X \cup Y \) contribute to \( A'_n \) (2.3) where \( X \) and \( \{ Y, x_n \} \) contain external vertices, and similarly for \( R'_n \) (2.4).

**Proof.** Consider a partition where \( X \) contains no external vertex. Performing the contractions between \( X \) and \( \{ Y, x_n \} \) with \( D^+\)-distributions and transforming into \( p \)-space (2.17), we get a product of \( \Theta \)-functions

\[
\Theta(p^{0}_1)\Theta(p^{0}_2)\ldots \Theta(p^{0}_j),
\]
where all momenta add up to 0:

\[ p_1^0 + p_2^0 + \ldots + p_j^0 = 0. \]

Such a product is zero.

The exists one serious problem, however. The central splitting solution (2.13) is only true if all momenta \( p_j \) are inside the light cone. Therefore, strictly speaking, we cannot put the inner momenta equal to 0. But if the \( D_{m^+} \)-distribution is massive \( m > 0 \), the vanishing of the contribution of a wrong partition takes place for small enough inner momenta \( \tilde{p}_j \) in \( V^+ \), already. Then we can use (2.13) and take the limit \( \tilde{p}_j \to 0 \). For this reason we always calculate with massive fields first. If the limit \( m \to 0 \) is required, it must be carefully performed by taking cancellations of infrared divergences between different terms into account.

From lemma 1 it is clear that in diagrams with two and three external legs, the non-vanishing terms (in the adiabatic limit) correspond to ordinary cuts through the diagram. But in a four-legs diagram a pair of opposite legs can be in \( X \) and the other pair of opposite legs in \( \{ Y, x_n \} \). Then this decomposition is no longer a simple cut and an "anomalous threshold" appears. To our knowledge such a diagram was never computed by the naive "dispersive method", but in the causal theory this is no problem. The following second lemma further simplifies the later calculations. It is a consequence of parity- and time-reversal invariance.

**Lemma 2.** In a PT-invariant theory the numerical distributions \( d_n^k(x) \) in \( D_n \) (2.5) are essentially real (i.e. up to an overall factor \( i \)) in momentum space: \( \hat{d}_n^k(p)^* = \hat{d}_n^k(p) \).

**Proof.** We know that the causal \( D \)-distributions are PT-invariant ([2], p.281). This implies for the numerical distributions: \( d_n^k(-x) = d_n^k(x)^* \). The complex conjugate comes from the antiunitarity of time-reversal. After Fourier transformation this gives the desired result in momentum space. The overall factor \( i \) depends on whether the number of internal lines in the diagram is even or odd.

We now come to the calculation of the third order vertex diagram (Fig.2). To simplify the notation, we write the arguments in \( x \)-space without the dummy variable \( x \). From (2.4) we have

\[ R_3' = T_2(1,3)\tilde{T}_1(2) + T_2(2,3)\tilde{T}_1(1) + T_1(3)\tilde{T}_2(1,2). \]  

(2.20)

The first term herein contains a Compton subgraph

\[ R_{31}' = i : \varphi_2^+(1)D_{m_1}^F(1-3)\varphi_2(3) : A(1)A(3) : (-i)A(2) : \varphi_2^+ \varphi_1(2) :, \]  

(2.21)

where

\[ \hat{D}_m^F(p) = \frac{-(2\pi)^2}{p^2 - m^2 + i0} \]  

is the Feynman propagator. The product (2.21) is computed by Wicks theorem, restricting ourselves to those contractions which generate the vertex diagram:

\[ R_{32}' = - : \varphi_2^+(1)D_{m_1}^F(1-3)D_{m_2}^+(3-2)D_{m_3}^+(1-2)\varphi_1(2) : A(3). \]  

(2.23)

Similarly we get for the other two terms in (2.20)

\[ R_{32}' = - : \varphi_2^+(1)D_{m_1}^+(3-1)D_{m_2}^F(3-2)D_{m_3}^+(2-1)\varphi_1(2) : A(3), \]  

(2.24)
The anti-Feynman propagator $D^{AF}$ is the complex conjugate of $D^F$. It appears because we have used unitarity $T_2(1,2) = T_2(1,2)^+$. This is the only minor role which unitarity plays here.

The result for $A'_3$ (2.3) is obtained in the same way. Collecting the terms with field operators: $\varphi^+_2(1)\varphi_1(2) : A(3)$ in $D_3$ (2.5), the corresponding numerical distribution is given by

$$d_3(1,2,3) = -D_{m_1}^{-1}(1-3)D_{m_2}^{-1}(3-2)D_{m_3}^{-1}(1-2) - D_{m_1}^{-1}D_{m_2}^{-1}D_{m_3}^{-1}$$

$$+ D_{m_1}^-D_{m_2}^+D_{m_3}^+ + D_{m_1}^+D_{m_2}^-D_{m_3}^- + D_{m_1}^+D_{m_2}^+D_{m_3}^- - D_{m_1}^+D_{m_2}^-D_{m_3}^+, (2.26)$$

where we have used $D^-(x) = -D^+(-x)$ and the arguments in the 6 terms agree with the first term. The 6 terms come from three cuts through the vertex diagram, not only one.

To get contact with the convention in [2], we shall use the relative coordinates

$$y_1 = x_1 - x_3, \quad y_2 = x_3 - x_2, \quad (2.27)$$

and calculate the Fourier transform

$$\tilde{d}_3(p,q) = (2\pi)^{-4} \int d3(y_1, y_2)e^{ipy_1 + iqy_2}d^4y_1d^4y_2. \quad (2.28)$$

Then we arrive at

$$\tilde{d}_3(p,q) = (2\pi)^{-2} \int dk \left[D_{m_1}^- (p-k)D_{m_2}^+(q-k)D_{m_3}^+(k) - D_{m_1}^- (p-k)D_{m_2}^+(q-k)D_{m_3}^+(k) \right] [p \leftrightarrow q, m_1 \leftrightarrow m_2]. \quad (2.29)$$

Up to the arbitrary masses and the imaginary parts, this agrees precisely with the result in QED ([2], eq.(3.8.28)). By the same techniques as in the QED case, we then find

$$\tilde{d}_3(p,q) = \frac{\pi}{4(2\pi)^6} \left\{ \frac{\text{sgn} \, P_0}{\sqrt{N}} \Theta(p^2 - (m_1 + m_2)^2) \log_1$$

$$- \frac{\text{sgn} \, q_0}{\sqrt{N}} \Theta(q^2 - (m_2 + m_3)^2) \log_2 + \frac{\text{sgn} \, p_0}{\sqrt{N}} \Theta(p^2 - (m_1 + m_3)^2) \log_3 \right\}. \quad (2.30)$$

where $P = p - q, N = (pq)^2 - p^2q^2$ and

$$\log_1 = \log \frac{p^2 + m_1^2 - m_2^2 - pP(1 + \frac{m_1^2 - m_2^2}{p^2}) + \sqrt{N} \sqrt{1 - 2\frac{m_1^2 + m_2^2}{p^2} + \frac{(m_1^2 - m_2^2)^2}{p^2}}}{p^2 + m_1^2 - m_2^2 - pP(1 + \frac{m_1^2 - m_2^2}{p^2}) - \sqrt{N} \sqrt{1 - 2\frac{m_1^2 + m_2^2}{p^2} + \frac{(m_1^2 - m_2^2)^2}{p^2}}}, (2.31)$$

$$\log_2 = \log \frac{q^2 + m_2^2 - m_3^2 - qP(1 - \frac{m_2^2 - m_3^2}{q^2}) + \sqrt{N} \sqrt{(1 - \frac{m_2^2 + m_3^2}{q^2} - \frac{4m_2^2}{q^2})^2}}{q^2 + m_2^2 - m_3^2 - qP(1 - \frac{m_2^2 - m_3^2}{q^2}) - \sqrt{N} \sqrt{(1 - \frac{m_2^2 + m_3^2}{q^2} - \frac{4m_2^2}{q^2})^2}}, (2.32)$$

$$\log_3 = \log \frac{q^2 + m_3^2 - m_1^2 - qP(1 - \frac{m_3^2 - m_1^2}{q^2}) + \sqrt{N} \sqrt{(1 - \frac{m_3^2 + m_1^2}{q^2} - \frac{4m_3^2}{q^2})^2}}{q^2 + m_3^2 - m_1^2 - qP(1 - \frac{m_3^2 - m_1^2}{q^2}) - \sqrt{N} \sqrt{(1 - \frac{m_3^2 + m_1^2}{q^2} - \frac{4m_3^2}{q^2})^2}}, (2.33)$$

Because of lemma 2 we only need the real parts of the logarithms. The splitting of (2.30) by means of the central solution (2.13) is done later in (3.10).
3 Vacuum Polarization in Fourth Order

Now we envisage the calculation of the diagram shown in Fig. 1a, which contributes to vacuum polarization in fourth order. We restrict ourselves to the case with vanishing 'photon' mass \( m_3 \) and mass \( m \) of the 'electron'. From (2.4) we have

\[
R'_4 = T_3(1, 2, 4)\tilde{T}_1(3) + T_3(1, 3, 4)\tilde{T}_1(2) + T_3(2, 3, 4)\tilde{T}_1(1)
\]

\[+ T_2(1, 4)\tilde{T}_2(2, 3) + T_2(2, 4)\tilde{T}_2(1, 3) + T_2(3, 4)\tilde{T}_2(1, 2) + T_1(4)\tilde{T}_3(1, 2, 3). \tag{3.1}
\]

Note that we only consider terms with field operator : \( A(2)A(4) : \). According to lemma 1, the first, the third and the fifth term in (3.1) vanish in the adiabatic limit. Furthermore, the second term \( R'_{42} \) gives the same contribution as \( R'_{47} \), and the same holds true for \( R'_{44} \) and \( R'_{46} \).

In \( x \)-space, the three-particle contribution \( R'_{44} \) (Fig. 3a) is given by

\[
R'_{44}(1, 2, 3, 4) : A(2)A(4) : = 2iD_m^{AF}(2-3)D_m^F(1-4)D_m^+(1-2)D_m^+(4-3)D_m^+(1-3) \tag{3.2}
\]

The Fourier transform of (3.3) is \((y_i = x_i - x_4)\)

\[
r'_{44}(p_1, p_2, p_3) = (2\pi)^{-6} \int dy_1 dy_2 dy_3 r'_4(x_1, x_2, x_3, x_4) e^{ip_1y_1+ip_2y_2+ip_3y_3}
\]

\[
= 2(2\pi)^{-11} \int dqdp' \frac{1}{(p_2 + q)^2 - m^2 - i\epsilon} \frac{1}{(p_1 - q - p')^2 - m^2 + i\epsilon} \Theta(q^0)\delta(q^2 - m^2) \Theta(-p_2^0 - p_3^0 - q^0 - p'^0)\delta((p_2 + p_3 + q + p')^2 - m^2) \Theta(p_0^0)\delta(p^2 - m_3^2) \tag{3.4}
\]

which becomes in the adiabatic limit \( p_1, p_3 \to 0 \), \( p_2 = p \)

\[
r'_{44}(p) = 2(2\pi)^{-11} \int dqdp' \frac{1}{(p + q)^2 - m^2 - i\epsilon} \frac{1}{(q - p - p')^2 - m^2 + i\epsilon} \Theta(q_0)\delta(q^2 - m^2) \Theta(p_0^0 - q_0)\delta((p - q)^2 - m^2) \Theta(-p_0^0 - p'^0)\delta((p + p')^2 - m_3^2). \tag{3.5}
\]

In fact, the calculation of (3.5) has already been performed in [2] by G.Källén and A.Sabry. Our calculations confirm their result exactly, namely:

\[
r'_{44}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0)\Theta(p^2 - 4m^2) B(z), \tag{3.6}
\]

\[
B(z) = 3\text{Li}_2(z) + 2\text{Li}_2(-z) + \log z \log(1-z) + \log z \log(1+z) + \frac{1}{4} \log^2 z - \frac{\pi^2}{3} + \log z \log \frac{m_3}{m}, \tag{3.7}
\]

up to terms that vanish for \( m_3 \to 0 \). Here we have introduced the well-known dilogarithm \( \text{Li}_2 \), and the variable

\[
z = \frac{1 - \sqrt{1 - 4m^2/p^2}}{1 + \sqrt{1 - 4m^2/p^2}}, \tag{3.8}
\]

which varies from 0 to 1 for \( p^2 \in [4m^2, \infty) \).
The two-particle contribution $r'_{42}$ (Fig.3b) is

$$r'_{42}(p) = 2i(2\pi)^{-2} \int dp' \, D^+_m(-p')\Lambda_3(p', p' - p)D^+_m(p' - p). \quad (3.9)$$

Here, $\Lambda_3$ is the vertex function. Since its first argument lies in the backward light-cone and the second one in the forward light-cone, we are in the case of lower signs in (2.13), because of our convention (2.27). Then the retarded vertex function $\Lambda^\text{ret}_3$ is given by the following dispersion integral:

$$\Lambda^\text{ret}_3(p, q) = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_3(tp, tq)}{1 - t - i0} \quad (3.10)$$

Again, the computation of (the real part of) $r'_{42}$ can be carried out in a straightforward manner. As we know from (2.30), the vertex function consists of three different parts. The first one leads to an infrared finite contribution to $r'_{42}$:

$$r'^1_{42}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0)\Theta(p^2 - 4m^2) C_1(z), \quad (3.11)$$

$$C_1(z) = 2\text{Li}_2(z) + 2 \log z \log(1 - z) - \frac{1}{2} \log^2 z + \frac{\pi^2}{6}, \quad (3.12)$$

whereas the second and third term are equal and contain an infrared divergent term $\sim \log(m_3/m)$, which cancels the infrared divergence in (3.7):

$$r'^2_{42}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0)\Theta(p^2 - 4m^2) C_2(z), \quad (3.11)$$

$$C_2(z) = -\text{Li}_2(z) - \log z \log(1 - z) - \frac{1}{4} \log^2 z + \frac{\pi^2}{6} - \log z \log \frac{m_3}{m}. \quad (3.12)$$

Now $d_4 = r'_4 - a'_4$ can immediately be written down if we note that

$$r'_{4i}(p) = a'_{4i}(-p), \quad i = 1, \ldots, 7. \quad (3.13)$$

4 The Splitting of $d_4(p)$

After having calculated the causal distribution $d_4$

$$d_4(p) = -\frac{1}{2(2\pi)^9} \frac{1}{p^2} \text{sgn} p_0 \Theta(p^2 - 4m^2) J(z),$$

$$J(z) = 4\text{Li}_2(z) + 2\text{Li}_2(-z) + 2 \log(z) \log(1 - z) + \log(z) \log(1 + z), \quad (4.1)$$
we must decompose it into retarded and advanced parts. The retarded distribution $r_4$ is given in the forward light-cone according to (2.13)

$$r_4(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_4(tp)}{1 - t - i0}.$$
\[
= \frac{i}{2\pi} \frac{1}{2(2\pi)^9 p^2} \int_{-\infty}^{+\infty} \text{sgn } t \Theta(t^2 p^2 - 4m^2) J \left( \frac{\sqrt{t^2 p^2 - \sqrt{t^2 p^2 - 4m^2}}}{\sqrt{t^2 p^2 + \sqrt{t^2 p^2 - 4m^2}}} \right) dt. \quad (4.2)
\]

It is very convenient to introduce the new integration variable

\[
x = \frac{\sqrt{t^2 p^2 - \sqrt{t^2 p^2 - 4m^2}}}{\sqrt{t^2 p^2 + \sqrt{t^2 p^2 - 4m^2}}}, \quad t = \frac{1}{\sqrt{b}} \frac{x + 1}{\sqrt{x}}, \quad b = \frac{p^2}{m^2}, \quad \frac{dt}{dx} = \frac{1}{\sqrt{b}} \frac{x - 1}{2x\sqrt{x}}, \quad (4.3)
\]

This leads to \((p \in V^+)\)

\[
r_4(p) = -\frac{i}{2(2\pi)^9 p^2} \int_{0}^{1} dx \left\{ -\frac{2}{x + 1} + \frac{1}{x - z} + \frac{1}{x - 1/z} \right\} J(x). \quad (4.4)
\]

The imaginary part \(i0\) in the denominator of (4.2) is included in \(p^2\) as discussed at the end of this section.

From now on, \(r_4(p)\) is considered as a function of \(z\) (3.8)

\[
z = \frac{1}{2} \left( b - 2 - b\sqrt{1 - 4/b} \right) = \frac{1 - \sqrt{1 - 4m^2/p^2}}{1 + \sqrt{1 - 4m^2/p^2}}. \quad (4.5)
\]

Then we note that \(z \in [0, 1]\) for \(p^2 \in [4m^2, \infty)\), and

\[
R(z) = \int_{0}^{1} dx \left\{ -\frac{2}{x + 1} + \frac{1}{x - z} + \frac{1}{x - 1/z} \right\} J(x) \quad (4.6)
\]

has the property \(R(z) = R(1/z)\). The integral

\[
R_1 = -2 \int_{0}^{1} \frac{dx}{x + 1} J(x) = -\frac{9}{4} \zeta(3) \quad (4.7)
\]

is just a constant and can be calculated with the formulae given in [10,11]. The calculation of

\[
R_2(z) = \int_{0}^{1} \frac{J(x)}{x - z} + \int_{0}^{1} \frac{J(x)}{x - 1/z} = R_3(z) + R_4(z) \quad (4.8)
\]

is most easily performed by first calculating the derivatives of \(R_3\) and \(R_4\):

\[
R_3'(z) = \partial_z \left\{ \frac{1}{z} \int_{0}^{1} dx \frac{J(x)}{x/z - 1} \right\} = \partial_z \left\{ \int_{0}^{1/z} dt \frac{J(tz)}{t - 1} \right\} = \frac{J(1)}{z(z - 1)} + \frac{1}{z} \int_{0}^{1} dx \frac{x}{x - z} J'(x),
\]

\[
R_4'(z) = \frac{J(1)}{z - 1} + \int_{0}^{1} dx \frac{J'(x)}{1/x - z}. \quad (4.9)
\]

This is a helpful trick, but all integrals coming up in (4.6) are also discussed in the literature mentioned above. We give the separate results for the two-particle and three-particle expressions:

\[
\int_{0}^{1} dx \left\{ -\frac{2}{x + 1} + \frac{1}{x - 1/z} + \frac{1}{x - z} \right\} B(x)
\]
\[
5 \text{Li}_3(z) + 3 \text{Li}_3(-z) - 3 \text{Li}(z) \log z - 2 \text{Li}(-z) \log z - \frac{1}{2} \log^2 z \log(1 - z) \\
-\frac{1}{2} \log^2 z \log(1 + z) - \frac{1}{12} \log^3 z + \frac{\pi^2}{2} \log(1 + z) + \frac{\pi^2}{2} \log z + \frac{3}{4} \zeta(3) + \frac{\pi^2}{2} \log 2,
\]
(4.10)
and
\[
\int_0^1 dx \left\{ -\frac{2}{x + 1} + \frac{1}{x - 1/z} + \frac{1}{x - z} \right\} C(x) \\
= \text{Li}_3(z) - \text{Li}(z) \log z - \frac{1}{2} \log^2 z \log(1 - z) + \frac{1}{12} \log^3 z \\
+ \pi^2 \log(1 - z) - \frac{\pi^2}{2} \log z + \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \log 2,
\]
(4.11)
where \( C(x) = C_1(x) + C_2(x) \). Here we have introduced the trilogarithm, which is defined by
\[
\text{Li}_3(z) = \int_0^z \frac{\text{Li}(x)}{x} dx.
\]
(4.12)
The infrared divergent terms in \( B \) and \( C \) which cancel mutually in \( r_4 \) have already been omitted. Finally \( r_4 \) is given by
\[
r_4(z) = -\frac{i}{2(2\pi)^4 p^2} R(z),
\]
(4.13)
\[
R(z) = 6 \text{Li}_3(z) + 3 \text{Li}_3(-z) - 4 \text{Li}_2(z) \log(-z) - 2 \text{Li}_2(-z) \log(-z) \\
- \log^2(-z) \log(1 - z) - \frac{1}{2} \log^2(-z) \log(1 + z) + \frac{3}{2} \zeta(3).
\]
(4.14)
If \( R(z) \) becomes complex the sign of the imaginary parts is determined by the \( i_0 \) in (4.2) as follows. For arbitrary time-like \( p \), the \( i_0 \) goes over into \( i_0 p_0 \) (see (2.13) and [2], Chap.3.6). To obtain the full time-ordered distribution \( t_4(p) \) we have to subtract \( r'_4(p) \). This changes \( i_0 p_0 \) into \( i_0 \) again. Consequently, \( t_4(p) \) is given by (4.13) with \( p^2 \) substituted by \( p^2 + i_0 \), i.e. \( z \rightarrow z + i_0 \). This fixes the signs of the imaginary parts at the logarithmic cuts in (4.14). For space-like \( p \), \( t_4(p) \) is simply given by (4.13) because \( r'_4(p) \) now vanishes and \( R(z) \) is real.

5 The Electron Propagator in Fourth Order

After having demonstrated the instructive example of vacuum polarization, we proceed now with the discussion of the electron propagator in fourth order (Fig.1b). Again, we start from the general expression (3.1) and consider only terms with field operator \( \varphi(2)\varphi^+(4) \). Just as in the case of vacuum polarization, the first, the third and the fifth term in (3.1) vanish in the adiabatic limit, and the second and the seventh term
\[
R'_{42} = T_3(1, 3, 4) \tilde{T}_1(2), \quad R'_{47} = T_1(4) \tilde{T}_3(1, 2, 3)
\]
(5.1)
give the same contribution to the propagator. But we have to distinguish the different cuts in \( R'_{44} \) and \( R'_{46} \).
In x-space, the three-particle contribution in \( R'_{46} \) with two photons and one electron as intermediate state (Fig. 4) is given by

\[
R'_{46}^1 = r'_{46}^1(1, 2, 3, 4) : \varphi(2)\varphi^+(4) :, \quad (5.2)
\]

\[
r'_{46}^1 = iD_m^F(3 - 4)D_m^{AF}(1 - 2)D_m^+(3 - 1)D_{m_3}^+(4 - 1)D_{m_3}^+(3 - 2) \quad (5.3)
\]

The Fourier transform is after the adiabatic limit

\[
r'_{46}^1(p) = (2\pi)^{-11} \int dqdp' \frac{1}{(p + q)^2 - m^2 - i0} \frac{1}{(q - p - p')^2 - m^2 + i0} \Theta(q_0)\delta(q^2 - m_3^2)\Theta(p_0' - q_0)\delta((p' - q)^2 - m_3^2)\Theta(-p_0 - p_0')\delta((p + p')^2 - m^2). \quad (5.4)
\]

Straightforward calculation of (5.4) leads to the simple result

\[
r'_{46}^1 = \frac{1}{8(2\pi)^9} \frac{1}{p^2} \Theta(-p_0)\Theta(p^2 - m^2)B(x^2), \quad (5.5)
\]

\[
B(x^2) = \frac{1}{2} \text{Li}_2(x^2) + \frac{1}{2} \log(x^2) \log(x^2 - 1) - \frac{\pi^2}{12}, \quad x^2 = p^2/m^2. \quad (5.6)
\]

The two-particle contribution can be evaluated without any problem with the aid of (2.30). We therefore quote only the result:

\[
R'_{42} = r'_{42}(1, 2, 3, 4) : \varphi(2)\varphi^+(4) :, \quad (5.7)
\]

\[
r'_{42}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0)\Theta(p^2 - m^2)C(x^2), \quad (5.8)
\]

\[
C(x^2) = \frac{1}{2} \text{Li}_2(x^2) - \frac{\pi^2}{12}. \quad (5.9)
\]

In (5.6) and (5.9) only the real parts of the dilogarithms contribute.

Since we have to treat the three-particle contribution with three electrons as intermediate state \((r'_{44}^2 \) and \( r'_{46}^2 \)) separately, we already give the splitting results for the expressions obtained so far. The splitting procedure leads to the same type of integrals as in the case of vacuum polarization in Sect. 4, hence we refer again to [10,11]. We have for \( p \) in the forward light-cone and \( p^2 > m^2 \)

\[
d_1(p) = -\frac{1}{8(2\pi)^9} \frac{1}{p^2} \text{sgn}p_0 \Theta(p^2 - m^2) \left[ 3\text{Li}_2(x^2) + \log x^2 \log(x^2 - 1) - \frac{\pi^2}{2} \right], \quad (5.10)
\]

\[
r_1(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_1(tp)}{1 - t + i0} \quad (5.11)
\]

\[
= -\frac{i}{8(2\pi)^9} \frac{1}{p^2} \left[ 4\text{Li}_3(1 - x^2) - 3\log(1 - x^2)\text{Li}_2(1 - x^2) - \log x^2 \log^2(1 - x^2) - 4\zeta(3) \right]. \quad (5.12)
\]

The calculation of the second diagram in Fig. 4

\[
r'_{44}^2(p) = (2\pi)^{-11} \int dqdp' \frac{1}{(p + q)^2 - m_3^2 - i0} \frac{1}{(q - p - p')^2 - m_3^2 + i0}
\]
\[ \Theta(q_0)\delta(q^2 - m^2)\Theta(p'_0 - q_0)\delta((p' - q)^2 - m^2)\Theta(-p_0 - p'_0)\delta((p + p')^2 - m^2). \]  

is difficult and requires extra consideration. \( r_{44}^2(p) \) turns out to be infrared finite, so we may drop the small photon mass in the following. Making use of all \( \delta \)-distributions, the integral

\[ I = \int dq \Theta(q_0)\delta(q^2 - m^2)\Theta(p'_0 - q_0)\delta((p' - q)^2 - m^2) \frac{1}{(q + p)^2 - i0} \frac{1}{(q - p' - p)^2 + i0} \]  

can be transformed into

\[ I = \frac{\pi}{2} \frac{\Theta(p'_0)\Theta(p'^2 - 4m^2)}{\sqrt{(p'p)^2 - p'^2p^2}} \int_{x_1}^{x_2} dx \frac{1}{4(x + m^2)(x + pp')}, \]

where

\[ \tilde{p} = -p - p', \quad x_{1,2} = \frac{1}{2}p'\tilde{p} \mp \frac{1}{2} \sqrt{(p'\tilde{p})^2 - p'^2\tilde{p}^2} \sqrt{1 - \frac{4m^2}{p'^2}}. \]

Then we obtain

\[ r_{44}^2(p) = -\frac{1}{4(2\pi)^9 p^2} \text{sgn} p_0 \Theta(p^2 - 9m^2) \int \frac{1}{\sqrt{p'^2m^2}} dy \int_{x_1}^{x_2} dx \frac{1}{4(x + m^2)(x - y)}. \]

Introducing \( x = (y + z - m^2)/2 \), \( d_2(p) = 2(r_{44}^2(p) - a_{44}^2(p)) \) becomes

\[ d_2(p) = -\frac{1}{4(2\pi)^9 p^2} \text{sgn} p_0 \Theta(p^2 - 9m^2) \int \frac{1}{\sqrt{p'^2m^2}} dy \int_{-\xi}^{+\xi} dy \int dz (y + z + m^2)(y - z + m^2). \]

At this stage, we will not proceed the same way as in the case of vacuum polarization. We first apply the splitting formula to (5.19)

\[ r_2(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d^2_4(p)}{1 - t + i0} \]

\[ = -\frac{i}{4(2\pi)^{10} p^2} \int_{\frac{m^2}{p^2}}^{+\infty} ds \frac{1}{s(1 - s + i0)} \frac{1}{\sqrt{p'^2m^2}} \int dy \int_{-\xi(s,y)}^{+\xi(s,y)} dy \int dz (y + z + m^2)(y - z + m^2). \]

\[ \xi(s, y) = \frac{1}{\sqrt{y^2 - p'^2m^2} s} \sqrt{1 + \frac{4m^2}{2y - m^2 - p'^2s}}, \quad p \in V^+ \]
and perform a partial integration with respect to $s$. This leads to the following integral:

$$r_2(p) = \frac{i}{2(2\pi)^{10}p^2} \int_{g/m^2}^{+\infty} ds \log \left( \frac{s}{1 - s - i0} \right) \int_{\sqrt{p^2m^2}}^\infty dy \frac{d}{ds} \xi(s, y) \left( y^2 - 2 \right)$$

(5.22)

\[ r_2(p) = \frac{i}{2(2\pi)^{10}p^2} \int_{g/x^2}^{+\infty} ds \log \left( \frac{s}{1 - s - i0} \right) \int_{\sqrt{x^2s}}^\infty dy \frac{\xi'}{(y + 1)^2 - \xi'^2}, \]

(5.23)

\[ x^2 = p^2/m^2, \quad y_1 = \frac{1}{2}(x^2s + 1), \quad y_2 = \frac{1}{2}(x^2s - 3), \]

\[ \xi = \sqrt{y^2 - x^2s} \sqrt{y_2 - y \over y_1 - y}. \]

(5.24)

We obtain after some simple manipulations

$$r_2 = \frac{i}{2(2\pi)^{10}m^2} \int_{g/x^2}^{+\infty} ds \log \left( \frac{s}{1 - s - i0} \right)$$

$$\int_{\sqrt{x^2s}}^\infty dy \frac{y_2}{\sqrt{(y^2 - x^2s)(y - y_1)(y - y_2)}} \left[ (x^2s - 1)y - \frac{1}{4} (x^2s + 1)^2 + 1 \right].$$

(5.25)

The last integral can be expressed by complete elliptic integrals of the first and third kind. It is

$$\int_{\sqrt{x^2s}}^{y_2} \frac{dy}{\sqrt{(y^2 - x^2s)(y - y_1)(y - y_2)}} = \frac{2}{\sqrt{(y_1 - \sqrt{x^2s})(y_2 + \sqrt{x^2s})}} K(k),$$

(5.26)

where

$$k^2 = \frac{(y_2 - \sqrt{x^2s})(y_1 + \sqrt{x^2s})}{(y_1 - \sqrt{x^2s})(y_2 + \sqrt{x^2s})},$$

(5.27)

and

$$\int_{\sqrt{x^2s}}^{y_2} \frac{ydy}{\sqrt{(y^2 - x^2s)(y - y_1)(y - y_2)}} =$$

$$\frac{2}{\sqrt{(y_1 - \sqrt{x^2s})(y_2 + \sqrt{x^2s})}} \left[ 2\sqrt{x^2s} \Pi(\alpha^2, k) - \sqrt{x^2s} K(k) \right],$$

(5.28)

$$\alpha^2 = \frac{y_2 - \sqrt{x^2s}}{y_2 + \sqrt{x^2s}}.$$

(5.29)

Introducing $\lambda = \sqrt{x^2s} \in [3, \infty)$, we find that the modulus $k$ and the parameter $\alpha^2$ are related by the following identities:

$$\alpha^2 = \frac{(\lambda - 3)(\lambda + 1)}{(\lambda + 3)(\lambda - 1)}, \quad k^2 = \frac{(\lambda - 3)(\lambda + 1)^3}{(\lambda + 3)(\lambda - 1)^3}.$$  

(5.30)
Then it is in fact possible to express $\Pi(\alpha^2, k)$ by $K(k)$:

$$\Pi(\alpha^2, k) = \frac{\lambda + 3}{6} K(k).$$  \hfill (5.31)

This has already been observed by A. Sabry [12]. Our relation follows from his equation (85) by the substitution $\lambda \rightarrow 1/\lambda$. Finally we arrive at a remarkably simple expression for $r_2$:

$$r_2(x) = \frac{i}{6(2\pi)^{10}m^2} \int_{9/x^2}^{\infty} ds \frac{\lambda - 3}{(\lambda - 1)^2(\lambda + 1)} \sqrt{\frac{\lambda + 3}{\lambda - 1}} \log\left(\frac{s}{1 - s - i0}\right) K(k).$$  \hfill (5.32)

So far we have calculated the retarded distribution for time-like $x^2 = p^2/m^2$. By the same argument as given at the end of the last section, this also gives the time-ordered distribution $t_2(p)$ for arbitrary $p^2$. To get the full fermion propagator, one must add $t_1(p)$ given by (5.12) with $x^2 \rightarrow x^2 + i0$. The integral in (5.32), which can easily be computed numerically, is in agreement with eq. (30) of Broadhurst [4]. Our funny normalization factors are the correct ones for the calculation of S-matrix elements according to (2.1).

References

Figure Captions

Fig.1 The two-loop master diagrams:
   (a) the 'photon' propagator;
   (b) the 'electron' propagator
Fig.2 The vertex diagram with arbitrary masses
Fig.3 (a) The three-particle cut, (b) the two-particle cut for fourth order
       vacuum polarization
Fig.4 The two different three-particle cuts in the electron propagator