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UNIVERSAL VERSUS DRIVE DEPENDENT EXPONENTS FOR
SANDPILE MODELS

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Universal versus Drive dependent exponents for Sandpile Models

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Abstract

We study the scaling relations of the Manna model (2-state sandpile model), and its connection to a recently proposed rice pile model. We found that the avalanche exponents depends crucially on whether one drives the system in the bulk or at the boundary while the cut off scaling exponent is invariant. The scaling relations relating these exponents are derived for various modes of driving, and they are examined numerically for one and two dimensional systems. It is shown that one dimensional Manna model and the rice pile model have the same exponents within the numerical error. Finally, a new class of non conserved SOC models is introduced, and a classification scheme for sand pile models is proposed.

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Following the introduction of the canonic sandpile model of Bak, Tang, and Wiesenfeld [1], there has been introduced a number of self-organized critical (SOC) models, which define a number of “universal classes” depending on their values of exponents. On the other hand, as for the 1-d systems, it seems that sandpile models show not simple finite-size scaling but multi-fractal scaling behaviors [2].

Recently, however, being inspired by the experiment on rice pile [3], a new model was proposed and demonstrated that it shows a simple finite-size scaling in 1-d for the avalanche size distribution,

$$P(s) \sim s^{-\tau} f_s(s/L^\sigma) \quad (1)$$

with $\tau = 1.53$ and $\sigma = 2.20$ [4–6].

This rice pile model has a stochastic dynamics in the redistribution process of slope during avalanche and is different from most of previous models, where the avalanche process is deterministic. It should be also noted that the system is driven only at the top, and the obtained exponent $\tau = 1.53$ is very large compared with other SOC models, where the exponent is usually less than the mean-field value $\tau = 1.5$.

In this paper, we study the sand pile model originally introduced by Manna, which has stochastic redistribution process as in the rice pile model [7]. Our numerical results show that the 1-d Manna model and the rice pile model are in the same universal class. It is also shown that the exponent τ crucially depend on the way how the system is driven, but these differences are irrelevant because they have the same exponent σ , from which τ can be derived through a simple scaling relation.

A version of Manna model we study here is defined as follows: Consider a lattice in d dimension with open boundaries. At each lattice point, the field variable n_i can take an integer value $n_i = \{0, 1, \dots\}$ counting the number of grains on that site. A grain is added to the n_i of a randomly selected site i iteratively, and an avalanche is initiated when one of the variable n_i exceeds 1. The avalanche propagates by redistributing all the grains on all the sites with $n_i > 1$ to their nearest neighbors *randomly and independently* until all the

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variable n_i 's become less than or equal to 1. We employ parallel updating scheme during the avalanches.

The model differs from the standard sandpile model in having randomness in the local redistribution rules. As already seen in extremum dynamics models [8–11], this immediately opens for critical behavior also in 1-d systems, which we make special focus on in the following.

Let us start by introducing some of exponents for scaling relations. Following the notation of Biham [12], the avalanche size s and its width w scale with the avalanche duration time t as

$$s \sim t^{\gamma_{st}}, \quad w \sim t^{\gamma_{wt}}. \quad (2)$$

This is illustrated in Fig.1 for our 1-d Manna model, in the case where the grain is always added at the boundary. Numerics show that

$$\gamma_{st} = 1.48 \pm 0.03, \quad \gamma_{wt} = 0.68 \pm 0.03. \quad (3)$$

Due to the scaling relations we discuss in the following, these two exponents are enough to characterize the critical behavior of the present models as well as many other sandpile models.

The distributions for the avalanche size s , the duration time t , and the width w are supposed to have the scaling forms

$$P(s) = \frac{1}{s^\tau} f_s \left(\frac{s}{L^\sigma} \right), \quad P(t) = \frac{1}{t^{\tau_t}} f_t \left(\frac{t}{L^{\sigma_t}} \right), \quad P(w) = \frac{1}{w^{\tau_w}} f_w \left(\frac{w}{L} \right), \quad (4)$$

where the τ 's are the exponents for the distributions and the σ 's the exponents for the cut off of the distributions with finite system sizes. These distribution functions are related to each other through the variable transformation (2), thus we can derive the scaling relations

$$\gamma_{st}(\tau - 1) = \tau_t - 1 = \gamma_{wt}(\tau_w - 1). \quad (5)$$

Notice that for all sandpile models, the cut off for the avalanche width must be given directly by the system size L .

As noted by Ben-Hur and Biham [12], there are some obvious relations among these exponents. For example, from (2) one obtain

$$s \sim w^D; \quad D = \gamma_{st}/\gamma_{wt}, \quad (6)$$

where the exponent D is often called the dimension of the avalanche (see the review of Paczuski *et al.* [11]) because it counts how the total mass of the avalanche scale with its spatial extent. The fact that the cut off for the width w is L gives us the cut off exponents as

$$\sigma = D = \gamma_{st}/\gamma_{wt}, \quad \sigma_t = 1/\gamma_{wt}. \quad (7)$$

From the above relations we have reduced the number of independent exponents to three: γ_{st} , γ_{wt} , and τ , for example.

Now we will show that another scaling law can be derived, using the argument which is originally introduced heuristically by Kadanoff *et al.* [2] for sand pile models in general but is *exact* for the present model. We will extend the argument to derive different scaling relations for different ways of driving the system.

If one trace a particular grain, each grain propagates randomly. Each time it topple it does so to left and right with equal probability. Therefore, the distance that a particular grain travels is given as an ordinary random walk with time counted by the number of time it has toppled. Notice that this time counting is very different from the real time, where often a particular grain gets stuck for a long time. If one deposit grains randomly in the bulk, the distance that each grain has to travel before it falls out of the system at the boundary is order of L , thus number of topples s_0 each grain goes through is order of L^2 , which is a contribution of the grain to avalanches while it remains in the system. In the stationary state, every time a new grain is added to the system, one grain should go out of the system on average, therefore the equality

$$\langle s \rangle = \langle s_0 \rangle \quad (8)$$

holds, and thus we have $\langle s \rangle \approx L^2$.

This argument was first suggested by Kadanoff *et al.* [2] for deterministic versions of the sandpile models, and has later been verified analytically for the Abelian sandpile models by Dhar [15]. The implication is that the deterministic updating does not introduce long range correlation between the tumbling directions of the individual grains.

As for the present model, this picture is exact by the definition of the model, and it leads to the scaling law:

$$\sigma(2 - \tau) = 2. \quad (9)$$

which should be valid in all dimensions providing that the system is driven in the bulk.

On the other hand, if one deposit grains only at the boundary of the system as in the case of the rice pile model, the average number of steps s_0 that the grain moves before it falls out of the system should be estimated as the numbers of steps the grain moves before it returns the original place with the upper cut off L^2 :

$$\langle s_0 \rangle \approx \int_0^{L^2} \frac{s_0}{s_0^{3/2}} ds_0 \propto L. \quad (10)$$

The upper cut off represents the case where the grain falls off through the other end of the system. From (8) and (10), we obtain the scaling law in this case

$$\sigma(2 - \tau) = 1. \quad (11)$$

which is valid also in all dimensions when system is driven at the boundary. This scaling relation has been derived for the rice pile model using the transport properties of the directed motion of rice corns in the pile [5].

In Fig.2, we plot $s^\tau P(s)$ versus s/L^σ with $\sigma = 2.20$ and τ given by eqs.(9) and (11), or $\tau = 1.09$ and 1.55 , for the bulk and the boundary depositing cases, respectively. As for the bulk deposition case (Fig.2 a), the system size dependence in the scaling region persists even in a fairly large system as has been pointed out [12], but the convergence in the cut off region is quite convincing, thus we can determine the exponent τ through eq.(9) much better than direct observation of the scaling region. In the case of the boundary deposition case (Fig.2 b), overall convergence is very good.

It should be noted that in the both cases, the value of cut off exponent σ is the same although τ depends crucially on the way how the system is driven.

This is in fact understood intuitively; The stationary states for these system should be the same because the system configuration should be determined in the way that avalanche dynamics is critical. Thus, the way how a large avalanche develops deep inside the system should not depend on the position where the avalanche initiated, *i.e.* at the boundary or in the bulk. Thus, the exponent D of (6), which describes how avalanches spread in space, should be independent of the point of external driving. Therefore, we can regard the exponent D , which is equal to σ , defines the universality class whereas τ merely measures how fast excitations leave the system.

It is noted that eqs.(9) and (11) are valid also in higher dimensions, and in fact should be valid for all the undirected sandpile models that have discrete and strictly conserved dynamics. Furthermore, it is interesting that in higher dimensions one may deposits not only at boundaries, but also at corners of various co-dimensions. For example, the 2-d Manna model can be driven at a corner, then the average number of tumbles s_0 for a grain injected at the corner will be given by the conditional probability that it survives (does not return to 0) during s_0 steps of a random walk both along the x axis and along the y axis. Thus the probability that it survives more than s_0 steps in the lattices is $1/\sqrt{s_0} \cdot 1/\sqrt{s_0} = 1/s_0$. Accordingly, the chance that it survives exactly s_0 step is $1/s_0^2$, thus the mean life time for grains in the lattice is

$$\langle s_0 \rangle \approx \int_0^{L^2} \frac{s_0}{s_0^2} ds_0 \propto \log L \quad (12)$$

implying the scaling relation

$$\sigma(2 - \tau) = 0, \quad \text{or } \tau = 2 \quad (13)$$

for deposition at a corner in the open boundary 2-d lattice. Finite size scaling results of numerical simulation are given in Fig.3 for bulk (a), boundary (b), and corner (c) driving with $\sigma = 2.70$ and τ 's given by eqs.(9), (11), and (13), respectively: $\tau = 1.26$ (a), 1.63 (b),

and 2 (c). The value of $\tau = 2$ for the corner driving is, to our knowledge, the first time such a big exponent have been reported in any non deterministic SOC model.

Thus, in all cases, we have reduced the number of independent exponents to 2, for example γ_{st} and γ_{wt} , for which we obtained $\gamma_{st} = 1.48 \pm 0.03$ and $\gamma_{wt} = 0.68 \pm 0.03$ in the 1-d Manna model. In higher dimensions Ben-Hur and Biham [12] reported that $\gamma_{st} = 1.70$ and $\gamma_{wt} = 0.67$ for 2-d, and $\gamma_{st} = 1.80$ and that $\gamma_{wt} = 0.54$ for 3-d, respectively. In all cases the simulated γ 's agrees well with reported values of τ : 1.09 ± 0.03 for 1-d (present work), 1.26 ± 0.03 for 2-d (present work),

We would now like to discuss the connection between the Manna model studied here and other SOC models. We will see that the Manna model in fact cover a fairly large universality class, consistent of a variety of models suggested for very different systems.

First, we would like to point out that eq.(11) was derived for the Oslo rice pile model by of Paczhuski and Boetcher [5] using a slightly different picture. In fact the connection between the Manna model and the rice pile model is very close. The rice pile model may be seen as an integrated version of the Manna model, in the sense that the n_i should be identified as the slopes in the rice pile model. The only difference is then that in the rice pile model the random updates always involves a simultaneous redistribution of n_i to the left and right neighbor, whereas it is random in our formulation of the Manna model. This difference should be insignificant on large scales. In fact, the obtained exponents for the rice pile model ($\sigma = 2.20$, $\tau = 1.53$) are the same with the 1-d Manna model, thus we conclude that the Manna model and the Oslo rice pile model are in the same universality class.

Paczhuski and Boetcher [5] has further demonstrated that the 1-d rice pile model can be mapped to the linear interface model $dH_i/dt = d^2H/dx^2 + \eta(x, H)$ when this is driven critically (as done by extremum dynamics) [13] Thus, as a result, the cut off exponent σ in the 1-d Manna model equals the $1 + \chi$, where χ is the roughness exponent of the linear interface model in 1-d. The exponent is measured as $\chi = 1.25 \pm 0.05$ by Leschhorn [13]. It should be noted that the exponents of 2-d Manna model obtained here are consistent with those for the 2-d linear interface model [11].

Finally, as already observed by Maslov [14], then the linear interface model is presumably in the same universality class as the Zaitsev model [8] because the conservation law imposed by the Zaitsev model corresponds to a Laplacian redistribution of stresses in the linear interface model.

Thus, the 1-d Manna model can indeed be mapped to a large class of SOC models: the rice pile model, the linear interface model, and Zaitsev.

More interestingly, one may also consider a modified version of the Manna model, where one grain is dissipated from or added to a trophing site with equal probability. Such a model has the conservation law of grain only on average and it may be violated locally. This relaxation of conservation actually changes the system behavior drastically; Preliminary results of simulations show $\tau = 1.5 \pm 0.1$ and $\tau = 1.8 \pm 0.1$ when the system is driven at the bulk at the boundary, respectively. Notice that the measured value of τ can become very large in this 1-d model, maybe making it particularly relevant for understanding of the 1-d rice pile experiment by Frette *et al.* [3], where one observe $\tau = 2.1 \pm 0.1$.

In summary, we have studied scaling relations and universality of the model introduced by Manna, and demonstrated that it is related to a variety of other SOC models: the rice pile model, the linear interface model, and the Zaitsev model. We have presented evidence for σ as the exponent characterizing the universal class, for sandpile models in much the same way as D does it for extremum dynamics models [11]. Our study have shown that the exponent for avalanche size distribution is crucially dependent on driving, but the cut off exponent is invariant, from which universality class should be defined. Finally, we propose to subdivide sand pile models in classes after degree of randomness in local redistribution rules; Bak-Tang-Wiesenfeld type sand pile model with deterministic avalanche dynamics, the Manna and rice pile type model with stochastic avalanche dynamics but with the strict conservation law, and a new type of model with the conservation law which holds only on average.

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FIGURES

FIG. 1. Simulation results for spreading and the mass of avalanches initiated on the boundary of a open 1-d system of size $L = 4096$. The lines are to indicate the slope of 1.48 and 0.68.

FIG. 2. Finite-size scaling plot for 1-d Manna model. We employ $\sigma = 2.20$ and $\tau = 2 - 2/\sigma = 1.09$ for the bulk driven system (a), and $\sigma = 2.20$ and $\tau = 2 - 1/\sigma = 1.55$ for the boundary driven system (b). The system sizes are $L = 128, 256, 512, 1024$, and 2048 for (a), and $L = 256, 1024$, and 4096 for (b).

FIG. 3. Finite-size scaling plot for 2-d Manna model. We employ $\sigma = 2.70$ and $\tau = 2 - 2/\sigma = 1.26$ for the bulk driven system (a), and $\tau = 2 - 1/\sigma = 1.63$ for the boundary driven system (b), and $\tau = 2$ for the corner driven system (c). The system sizes are $L = 32, 64, 128$, and 256.

Fig.1 Nakanishi and Sneppen

Fig.2 Nakanishi and Sneppen

Fig.3 Nakanishi and Sneppen