The Gauss-Bonnet type identity is derived in a Weyl-Cartan space on the basis of the variational method.

1 Introduction

In the modern gravitational theory quadratic Lagrangians are used widely that is stimulated by the gauge treatment of gravitation and the renormalization problems in quantum gravity.\(^1\) In this connection the Gauss-Bonnet type identity becomes the object of a considerable amount of attention. The generalization of the Gauss-Bonnet formula to a 4-dimensional Riemann space \(V_4\) was performed by Bach\(^2\) and Lanczos\(^3\) and on the basis of the variational method by Ray.\(^4\) The Bach-Lanczos identity in Riemann spaces implies the one-loop renormalizability of pure gravitation.\(^5\) The generalization of the Bach-Lanczos identity to a Riemann-Cartan space \(U_4\) was performed in.\(^6\)–\(^8\) We shall obtain the Gauss-Bonnet type identity in a Weyl-Cartan space \(Y_4\) that can be essential for the dilatonic gravitational theory with quadratic Lagrangians. On the preliminary version of our results see Ref. 9.
Preliminaries to the variational procedure

We shall consider a Weyl-Cartan space \( Y_4 \) that is a connected 4-dimensional oriented differentiable manifold \( \mathcal{M} \) equipped with a linear connection \( \Gamma \) and a metric \( g \) (with the Lorenzian signature) which obey the constraints,

\[
Q^\alpha_{\lambda} = \frac{1}{4} Q_\lambda g^{\alpha\beta}, \quad Q_{\alpha\beta} := \nabla_\lambda g^{\alpha\beta}, \quad Q_\lambda := Q^\alpha_{\lambda} g_{\alpha\beta}.
\]

The tensor \( Q^\alpha_{\lambda} \) is a nonmetricity tensor.\(^{10,11}\)

We shall use a holonomic local vector frame \( \vec{e}_\mu = \vec{\partial}_\mu \) \((\mu = 1, 2, 3, 4)\) with \( \Gamma_{\sigma\rho}^\lambda \) as a connection coefficients. A space \( Y_4 \) contains a nonvanishing torsion tensor, \( T_{\sigma\rho}^\lambda := 2\Gamma_{[\sigma\rho]}^\lambda, \) in general. A curvature tensor of \( Y_4 \) and its various contractions read,

\[
R_{\mu\nu} = R_{\sigma\mu\nu\sigma}, \quad \tilde{R}_{\mu\nu} = R_{\mu\sigma\nu}, \quad R = R_{\sigma}\quad\text{and}
\]

\[
R_{\alpha\beta\sigma}^\lambda = 2\partial_\alpha \Gamma_{\beta\sigma}^\lambda + 2\Gamma_{[\alpha\rho]}^\lambda \Gamma_{\beta\sigma}\rho.
\]

Let us consider the Lagrangian density,

\[
L_0 = \sqrt{-g} L_0, \quad L_0 = R^2 - (R_{\alpha\beta} + \tilde{R}_{\alpha\beta})(R^\beta_{\alpha} + \tilde{R}_\beta^\alpha) + R_{\alpha\beta\mu\nu} R^\mu\nu_{\alpha\beta}.
\]

The variation of (3) with respect to the metric \( g^{\sigma\rho} \) and the connection \( \Gamma_{\lambda\nu}^\sigma \) reads,

\[
\delta L_0 = -\frac{1}{2} \sqrt{-g} H_{\sigma\rho} \delta g^{\sigma\rho} - \sqrt{-g} H^\nu_{\sigma} \delta \Gamma_{\lambda\nu}^\sigma + \text{total divergence},
\]

where

\[
\sqrt{-g} H_{\sigma\rho} := -2 \left[ \frac{\delta L_0}{\delta g^{\sigma\rho}} \right]_\Gamma=\text{const}, \quad \sqrt{-g} H^\nu_{\sigma} := - \left[ \frac{\delta L_0}{\delta \Gamma_{\lambda\nu}^\sigma} \right]_{g^{\sigma\rho}=\text{const}},
\]

\[
H_{\sigma\rho} = g_{\sigma\rho} L_0 - 4 R_{\alpha\beta(\sigma} R_{\rho)]^\alpha_{\beta} - 4 R_{\tau(\sigma\rho)}^\lambda (R^\lambda_{\tau\sigma} + \tilde{R}_\tau^\lambda)
\]

\[
-4 R_{\tau(\sigma} (R_{\rho)}^\tau + \tilde{R}_\tau^\rho) - 4 R_{\tau(\sigma\rho)},
\]

\[
\sqrt{-g} H^\nu_{\sigma} = 4 \nabla_\mu \{ \sqrt{-g} [R_{\sigma}^{\nu[\lambda\mu]} + (R_{\sigma}^{[\lambda} + \tilde{R}_{\sigma}^{[\lambda}) g_{\mu]\nu}
\]

\[
- (R_{\sigma}^{[\nu} + \tilde{R}_{\sigma}^{[\nu}) \delta_{\sigma}^{\mu]} - R\delta_{\sigma}^{[\lambda} g_{\mu]\nu]}
\]

\[
+ 2 \sqrt{-g} [R_{\sigma}^{\nu\alpha\beta} + (R_{\sigma}^{\alpha} + \tilde{R}_{\sigma}^{\alpha}) g_{\beta\nu} - (R_{\sigma}^{\nu\alpha} + \tilde{R}_{\sigma}^{\nu\alpha}) \delta^\beta_{\sigma} + R g^{\nu\alpha} \delta_{\sigma}^{\beta}] M_{\alpha\beta}^\lambda.
\]

\[
(7)
\]
Here the modified torsion tensor is introduced, $M_{\sigma\rho}^\lambda := T_{\sigma\rho}^\lambda + 2\delta_{[\sigma}^\lambda T_{\rho]}$, where $T_{\rho} := T_{\rho\tau}^\tau$ is the torsion trace.

3 The Gauss-Bonnet type Theorem

in Weyl-Cartan space

The following Gauss-Bonnet type Theorem generalized to $Y_4$ is valid.

**Theorem:** The integral quantity,

$$\int_M \sqrt{-g} [R^2 - (R_{\alpha\beta} + \tilde{R}_{\alpha\beta})(R_{\delta\alpha} + \tilde{R}_{\delta\alpha}) + R_{\alpha\beta\mu\nu} R^\mu_{\nu\alpha\beta}] \ d^4 x , \quad (8)$$

over the oriented 4-dimensional manifold $M$ without boundary equipped with the Weyl-Cartan differential-geometric structure does not depend on the choice of a metric and a connection of the manifold and is a topological invariant.

**Proof.** The main idea of the proof consists in the demonstration that the variation of the integrand of (8) with respect to a metric and a connection in a Weyl-Cartan space $Y_4$ is equal identically to a total divergence. In $Y_4$ the metric and the connection are not independent because of the constraints (1). Therefore one can vary the modified integrand expression,

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2} \sqrt{-g} H_{\alpha\beta}^\lambda \left( Q_{\lambda}^{\alpha\beta} - \frac{1}{4} Q_{\lambda} g^{\alpha\beta} \right) , \quad (9)$$

which in $Y_4$ coincides with the integrand of (8).

The variation of (9) has the form,

$$\delta \mathcal{L} = -\frac{1}{2} \left( \sqrt{-g} H_{\sigma}^{\nu} + \nabla_{\lambda} \left[ \sqrt{-g} (H_{\sigma}^{\nu\lambda} - \frac{1}{4} \delta_{\sigma}^{\nu} H_{\tau}^{\lambda}) \right] \right) g_{\nu\rho} \delta g^{\sigma\rho}$$

$$+ \frac{1}{2} \sqrt{-g} \left( H_{\sigma}^{\nu\lambda} (Q_{\lambda\nu\rho} - \frac{1}{4} Q_{\lambda} g_{\nu\rho}) - \frac{1}{2} g_{\sigma\rho} H_{\lambda\beta}^\lambda (Q_{\alpha}^{\beta\rho} - \frac{1}{4} Q_{\alpha} g^{\beta\rho}) \right) \delta g^{\sigma\rho}$$

$$- \sqrt{-g} \left( H_{[\sigma\rho]}^\lambda + \frac{1}{2} g_{\sigma\rho} H_{\tau}^{\lambda} \right) g^{\rho\nu} \delta \Gamma_{\lambda\nu}^\sigma$$

$$+ \frac{1}{2} \sqrt{-g} \left( Q_{\lambda}^{\alpha\beta} - \frac{1}{4} Q_{\lambda} g^{\alpha\beta} \right) \delta H_{\alpha\beta}^\lambda + \text{total divergence} , \quad (10)$$
where \( \hat{\nabla}_\lambda = \nabla_\lambda + T_\lambda \). If the constraints (1) are taken into account, then the hypothesis of the Theorem is the consequence of the Lemma.

**Lemma:** In a Weyl-Cartan space \( Y_4 \) the following identities are valid,

\[
(a) \quad \sqrt{-g} H_{\sigma}^\rho + \hat{\nabla}_\lambda (\sqrt{-g} H_{\sigma}^{\rho \lambda}) = 0, \quad (b) \quad H_{[\sigma \rho]}^\lambda = 0, \quad (c) \quad H_{\tau}^{\tau \lambda} = 0. \tag{11}
\]

**Proof.** The statement (c) follows immediately from (7). Using (7) and the Bianchi identities in \( Y_4 \) one gets,

\[
H_{[\sigma \nu]}^\lambda = [8 R_{\tau}^{\alpha \beta \gamma} \delta_\nu^\beta + 4 R_{\tau}^{\alpha \beta \gamma} \delta_\nu^\beta \lambda + 4 (R_{\tau}^{\alpha \beta \gamma} + \tilde{R}_{\tau}^{\alpha \beta \gamma}) \delta_\nu^\beta \delta_\nu^\gamma + 2 (R_{\tau}^{\alpha \beta \gamma} + \tilde{R}_{\tau}^{\alpha \beta \gamma}) \delta_\nu^\beta \delta_\nu^\gamma T_{\alpha \beta} \delta] T_{\alpha \beta} \delta \tag{12}
\]

Let us consider the expression,

\[
B_{\sigma \nu}^\lambda = \frac{1}{2} \eta_{\alpha \beta \gamma \delta} \eta_{\mu \nu \tau \rho} \eta_{\phi \omega \Sigma} R_{\phi \omega \Sigma} T_{\mu \nu \tau \rho} \delta, \tag{13}
\]

where \( \eta_{\alpha \beta \gamma \delta} = \sqrt{-g} e_{[\alpha \beta \gamma \delta]} \) is the Levi-Civita tensor \( (e_{[1234]} = -1) \), and calculate this expression in two ways: the first way consists in combining the first factor with the second one and the third factor with the forth one, while the second way consists in combining the first factor with the third one and the second factor with the forth one. By equating the two results one gets that the statement (b) of the Lemma is valid.

With the help of the Bianchi identities one can find,

\[
\sqrt{-g} H_{\sigma}^\nu + \hat{\nabla}_\lambda (\sqrt{-g} H_{\sigma}^{\nu \lambda}) = \sqrt{-g} \delta_\nu^\rho L_0 + 4 R_{\sigma}^{\alpha \beta \gamma} R_{\alpha \beta}^{[\tau \nu]} + 4 R_{\sigma}^{\alpha \beta \gamma} \delta_\nu^\rho L_0 + 2 (R_{\tau}^{\alpha \beta \gamma} + \tilde{R}_{\tau}^{\alpha \beta \gamma}) \delta_\nu^\rho M_{\alpha \beta} \delta \tag{14}
\]

Let us consider the expression,

\[
B_{\sigma \rho} = \frac{1}{4} \eta_{\lambda \mu \nu \tau} \eta_{\phi \omega \Sigma} \delta_{\rho \lambda} \delta_{\mu \alpha} \delta_{\nu \beta} \delta_{\tau \gamma} R_{\sigma \rho} \delta \tag{15}
\]
and calculate it in two ways, as before. After equating the two results one can be convinced that the statement (a) of the Lemma is valid. The proof of the Lemma is finished.

Using the Lemma one can see that in $Y_4$ the variation of (10) and therefore the variation of (8) is equal to a total divergence. Q.E.D.

References


2. R. Bach, Math. Z. 9, 110 (1921).


6. V.N. Tunjak, Izvestija vyssh. uch. zaved. (Fizika) N9, 74 (1979) [in Russ.].


