Prepotential of $N = 2$ Supersymmetric Yang-Mills Theories in the Weak Coupling Region

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Abstract

We show how to obtain the explicit form of the low energy quantum effective action for $N = 2$ supersymmetric Yang-Mills theory in the weak coupling region from the underlying hyperelliptic Riemann surface. This is achieved by evaluating the integral representation of the fields explicitly. We calculate the leading instanton corrections for the group $SU(N_c), SO(N)$ and $SP(2N)$ and find that the one-instanton contribution of the prepotentials for these groups coincide with the one obtained recently by using the direct instanton calculation.

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There has been much progress in the study of N=2 four dimensional supersymmetric
gauge theories. Originated from the work of Seiberg and Witten for the analysis of
$SU(2)$ Yang-Mills theories[1], the framework has been extended to higher rank gauge
groups with or without matter hypermultiplets[2, 3, 4, 7, 5, 6, 8, 9]. Although the basic
framework has been studied extensively so far, the explicit evaluation of the prepotential
is a difficult subject. Up to now, determination of the prepotential has been achieved
mainly by solving Picard-Fuchs equations[10, 11]. Although more direct method is known
to be available for $SU(2)$ gauge groups[13], solving Picard-Fuchs equation seems to be
the best tool for the investigation when these equations turns out to be solved by the well
known functions. The $N = 2$ $SU(2)$ supersymmetric Yang-Mills theories with massless
hypermultiplets[12] and $SU(3)$ super Yang-Mills theory without matters[11] have been
analyzed elegantly by using the method, whereas the method given in Ref.[13] is useful
only for the group $SU(2)$. However, the situation changes drastically when we find that
Picard Fuchs equation cannot be solved by any special functions. This situation generally
occurs for the theories with massive hypermultiplets or higher rank gauge groups. In
these cases, we expect that the explicit evaluation of the integral representation of the
solutions is more powerful than finding the solutions of the differential equations. As a
matter of fact, the analytic properties of the functions are studied mostly by using the
integral representations of the functions rather than the differential equations themselves
even in the theory of special functions.

In our previous paper, we have evaluated the prepotential of $SU(2)$ Yang-Mills theories
with massive hypermultiplets by analyzing the integral representations explicitly[14], and
discussed how these are related to the results for the massless cases which have been
obtained by using Picard Fuchs equations[12]. Therefore, it is natural to expect that this
approach is also powerful even for the theories with higher rank gauge groups.

In this paper, we show how to evaluate the prepotential for the gauge groups $SU(N_c)$,
$SO(N)$ and $SP(2N)$ without matter hypermultiplets in weak coupling region. Since we
cannot rely on the analysis of any special functions, the corresponding method for the
analysis will be quite different from our previous analysis applied for the group $SU(2)$[14].
We just use the technique used in our previous paper[14] only for showing the equivalence of our expression and the results for $SU(3)$ Yang-Mills theory[11] where the solutions are written by Appell hypergeometric functions of type $F_4$. It turns out that the one instanton contribution of the prepotentials for the groups $G = SU(N), SO(N)$ and $SP(2N)$ agree completely with the one obtained recently by using the direct instanton calculation[15].

In writting this paper, we recieved a preprint[17] which contains some of our results for $SU(N_c)$.

We will first consider $N = 2$ supersymmetric $SU(N_c)$ gauge theories without matter hypermultiplets[11] in detail. The theory has an $N_c - 1$ complex dimensional moduli space of vacua which are parameterized by the expectation value of the higgs fields as

$$<\phi> = \sum_{i=1}^{N_c} e_i H_i = diag[a_1, \ldots, a_{N_c}], \quad (1)$$

where $H_i$ are the generators of the Cartan sub-algebra of $U(N_c)$ and

$$\sum_{i=1}^{N_c} a_i = 0. \quad (2)$$

The fields $a_D^i$ dual to $a_i$ can be defined as

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a_i}, \quad (3)$$

where $\mathcal{F}(a)$ is the prepotential. The curve describing the space of vacua can be identified as

$$y^2 = \prod_{k=1}^{N_c} (x - e_k)^2 - \Lambda^{2N_c}, \quad (4)$$

where $e_i$ is the value of the classical moduli space with a constraint;

$$\sum_{i=1}^{N_c} e_i = 0. \quad (5)$$

It should be noted that the classical root of $y, e_i$ splits into $e_i^+$ and $e_i^-$ in (4).

The meromorphic one form $\lambda$ can be defined as

$$\lambda = \frac{dx}{2\pi i y} \frac{x}{\prod_{k=1}^{N_c} (x - e_k)}'. \quad (6)$$
The dual pair of fields $a$ and $a_D$ can be written as periods of a meromorphic one form $\lambda$ on the curve as

$$a_i = \oint_{\alpha_i} \lambda, \quad a_D^i = \oint_{\beta_i} \lambda,$$

(7)

where $\alpha^i, \beta_i$ form a basis of homology cycles on the curve. Therefore, we can compute the prepotential $\mathcal{F}(a)$ once we can evaluate $a_i$ and $a_D^i$ as functions of $e_k$ and $\Lambda$.

We are going to evaluate $a_i$ in the weak coupling region. To begin with, we expand the meromorphic differential with respect to $\Lambda^2$ by using an expansion

$$\frac{1}{y} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n \Lambda^{2n}}{n! (x-e_1)^{2n+1} \cdots (x-e_{N_c})^{2n+1}},$$

(8)

where $(a)_n$ is defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$  

(9)

After the use of a partial integration, we have

$$a_i = \sum_{k=1}^{N_c} \oint_{\alpha^i} \frac{dx}{2\pi i} \frac{x}{x-e_k} + \sum_{n=1}^{\infty} \oint_{\alpha^i} \frac{dx}{2\pi i n! 2n} \frac{(\frac{1}{2})_n \Lambda^{2n}}{(x-e_1)^{2n} \cdots (x-e_{N_c})^{2n}}.$$  

(10)

Originally, the $\alpha_i$ circle was chosen enclosing the roots of $y e^\pm_i, e^{-i}$. In our expression both of them shrink to the classical value $e_i$. Therefore, we can take the contour enclosing $e_i$ as $\alpha^i$ cycle to obtain

$$a_i = e_i + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n \Lambda^{2n} \Gamma(2n+m_k)}{n! 2n} \prod_{m_1 \cdots m_{N_c} = 2n-1 \ k \neq i} \frac{\Gamma(2n+m_k)}{\Gamma(2n) \Gamma(m_k+1)} (e_k - e_i)^{-2n-m_k}$$

$$= e_i + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n \Lambda^{2n} \Gamma(2n)}{n! (2n)!} \prod_{k, k \neq i} (e_k - e_i)^{-2n} \left(\frac{\partial}{\partial e_i}\right)^{2n-1}.$$  

(11)

In the case of the dual fields $a_D^i$, we have to use analytic continuation because of the logarithmic singularity. For such purpose, we write the meromorphic one form $\lambda$ by Barnes-type representation as

$$\lambda = \frac{dx}{2\pi i} \int_{-\infty}^{i\infty} ds \ \frac{\Gamma(-s) \Gamma(s+1/2)}{\Gamma(1/2) 2s} \prod_{k=1}^{N_c} \frac{(x-e_k)^{-2s} (-\Lambda^{2N_c})^s}{(x-e_k)^{-2s} (-\Lambda^{2N_c})^s},$$

(12)
where the path of integration is taken around the poles at \( s = 0, 1, 2, \ldots \). This expression can be obtained by considering Barnes-type representation for (8) and use a partial integration. Note also that we can obtain the strong coupling expansion of \( \lambda \) by taking the poles at \( s = -1/2, -3/2, \ldots \)

The \( \beta_{ij} = b_i - b_j \) cycle consists of the circle enclosing the root \( e_j^+ \) and \( e_i^+ \), which can be written as two times the line integral of \( \lambda \) from \( e_j^+ \) to \( e_i^+ \). When we use the expression (12), these roots shrink to the classical value. If we replace the contour integral to the line integral from \( e_j \) to \( e_i \), we have to subtract the contribution from the circle around \( e_i^− \) and \( e_j^− \), which can be evaluated as the half of the \( \alpha^i \) and \( \alpha^j \). We therefore obtain an expression of \( a_D \) as follows:

\[
a_{ij}^D \equiv a_D^i - a_D^j = 2 \int_{e_j}^{e_i} \lambda - \frac{1}{2}(a_i - a_j).
\]  

(13)

The \( a_D^i \) can be obtained from the procedure

\[
a_i^D = \frac{1}{N_c} \sum_{j=1}^{N_c} a_D^{ij}.
\]  

(14)

Although the method of the expansion for \( a \) in (11) is quite different from those of \( a_D \) (13), these will provide a consistent result. As a matter of fact, we can write \( a \) by using Barnes-type integral representation as

\[
a_{ij} \equiv a_i - a_j = \int_{e_j}^{e_i} \frac{dx}{2\pi i} \int_{-i\infty}^{i\infty} ds \left\{ \frac{\sin 2\pi s}{\pi} \frac{\Gamma(-s)\Gamma(s + 1/2)}{\Gamma(1/2)2s} \prod_{k=1}^{N_c} (x - e_k)^{-2s}(-\Lambda^{2N_c})^s \right\},
\]  

(15)

which we can derive by considering the integral enclosing \( e_i \) and \( e_j \) and by replacing it by line integral. The equivalence of (11) and (15) can also be shown by evaluating the poles of \( s \) in (15) explicitly.

As a check of the expression, let us consider the group \( SU(2)[1] \) and \( SU(3)[11] \) where the known expression are written in terms of the symmetric polynomial of roots rather than roots themselves[11]. In the case of \( SU(2) \), the expression (11) leads to

\[
a_1 = -a_2 = e_i + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n}{n!2n} \frac{\Gamma(4n - 1)}{\Gamma(2n)^2} (e_2 - e_1)^{-4n+1}.
\]  

(16)
By using Legendre’s duplication formula

\[ \Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2), \]  

we find that (16) can be written as

\[ a_1 = -a_2 = \sqrt{u} F(-1/4, 1/4; 1; \frac{\Lambda^4}{u^2}), \]

where we have set \( e_1 = -e_2 = \sqrt{u} \). This expression is identical to the result obtained in Refs.[1, 10, 11] Quite similarly, we can find that \( a_{12}^D \) coincides with the known expression.

Rather non-trivial check of our expression is for \( G = SU(3) \)[11] where the known expression is written in terms of Appell functions of type \( F_4 \) with respect to the variables \( u \) and \( v \), which are defined by

\[(x - e_1)(x - e_2)(x - e_3) = x^3 - ux - v. \]  

Let us first consider \( a_{23} \). From the equation (15), we have

\[ a_{23} = \int_{-i\infty}^{i\infty} ds \frac{\sin 2\pi s}{2\pi i} \frac{\Gamma(-s)\Gamma(s + 1/2)\Gamma(-2s + 1)^2}{\Gamma(-4s + 2)} (-\Lambda^6)^s \]

\[ \times (e_2 - e_3)^{-4s+1}(e_3 - e_1)^{-2s} F(2s, -2s + 1; -4s + 2; \frac{e_3 - e_2}{e_3 - e_1}). \]

These are written by root variables. We are going to re-write this expression in terms of symmetric polynomial which is \( u \) and \( v \) in (19). For this purpose, we are going to apply a method used in the case of \( SU(2) \) super-Yang Mills theories with massive hypermultiplets[14]. Before doing so, we have to choose the branch for the low energy expression. We will choose the branch that \( |\frac{e_3 - e_2}{e_3 - e_1}| \) is in the neighborhood of zero. By using the following quadratic transformation[16];

\[ F(a, b; 2b; z) = (1 - z)^{-1/2} a F\left(\frac{1}{2} a, b - \frac{1}{2} a; b + \frac{1}{2}; \frac{z^2}{4(z - 1)}\right), \]

for \( a = 2s, b = -2s + 1 \), we get

\[ a_{23} = \int \frac{ds}{2\pi i} \frac{\Gamma(-s)2^{2s-1} \pi^{1/2}}{\Gamma(s + 1)\Gamma(-2s + \frac{3}{2})} (e_2 - e_3)^{-4s+1}(e_3 - e_1)^{-s}(e_2 - e_1)^{-s} \]

\[ \times F(s, -3s + 1; -2s + \frac{3}{2}; \frac{(e_2 - e_3)^2}{4(e_3 - e_1)(e_1 - e_2)}), \]
where we have also used various identities of gamma functions.

This expression is symmetric with respect to \(e_1\) and \(e_2\). In order to obtain fully symmetric form, we use the following cubic transformation [16];

\[
F\left(\frac{1}{3} - a, 3a; 2a + \frac{5}{6}; z\right) = (1 - 4z)^{-3a} F\left(a, a + \frac{1}{3}; 2a + \frac{5}{6}; \frac{27z}{(4z - 1)^3}\right),
\]

for \(a = -s + 1/3\). By this procedure, we obtain

\[
a_{23} = - \int_{-i\infty}^{i\infty} ds \frac{\Gamma(-s)2^{2s-1}\pi^{\frac{1}{2}}}{\Gamma(s+1)\Gamma(-2s + \frac{3}{2})} \Delta^{-2s + \frac{1}{2}} D^{3s-1}
\times F\left(\frac{1}{3} - s, \frac{2}{3} - s; -2s + \frac{3}{2}; \frac{27\Delta}{4D^3}\right),
\]

where

\[
\Delta = (e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = 4u^3 - 27v^2, \\
D = \frac{1}{2}\left[(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2\right] = 3u.
\]

It should be noted that these are written in the form which is totally symmetric with respect to roots variables so that these can be expressed in terms of \(u\) and \(v\).

In order to write this expression in terms of Appells functions, we make use of the formula [16];

\[
F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b; a + b - c + 1; 1 - z) + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - z)^{c-a-b} F(c - a, c - b, c - a - b + 1; 1 - z),
\]

apply a formula [16]

\[
F(a, b; c; z) = (1 - z)^{c-a-b} F(c - a, c - b; c; z),
\]

then we finally find that the function \(a_{23}\) can be written as

\[
a_{23} = u^{\frac{1}{2}}[-F_4\left(\frac{1}{6}, -\frac{1}{6}; \frac{1}{2}; 1; x; y\right) + (\frac{x}{3})^{\frac{1}{2}}F_4\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; 1; x; y\right)],
\]

where the function \(F_4\) is the Appel function [16] and \(x = \frac{27\Delta^6}{4v^3}, y = \frac{27\Delta^6}{4v^3}\). In the branch that \(|\frac{e_3 - e_2}{e_3 - e_1}|\) is in the neighborhood of zero, we should apply analytic continuation of the
variables in $a_{12}$ and $a_{31}$ before using the quartic and cubic transformations. The result $a_{12}$ is given by

$$a_{12} = 2u^\frac{1}{2} F_4\left(\frac{1}{6}, -\frac{1}{6}; \frac{1}{2}; 1; x; y\right).$$

(29)

You can find that the same procedure can be applied for $a^{ij}_D$. The resulting expressions completely agree with the ones obtained in ref. [11].

In the case of $N_c > 3$, we do not know how to express our result in terms of symmetric polynomials with respect to roots $e_k$ because not so many quartic transformations have been known for the hypergeometric functions of several variables. Although the corresponding transformation, if it exists, may be powerful when we consider the theories with matter multiplets as in the case of $SU(2)$ gauge theories[14], we here evaluate the integral for $a^i_D$ directly rather than trying to represent them in terms of symmetric polynomial. This approach seems to be convenient for obtaining the prepotential. Of course, the most difficult question is how to get an analytic continuation which is consistent with the integrability of the prepotential in (3). In this paper, we assume that the integration over $s$ in the expression (12) will regularize the function correctly. At present, we cannot justify the procedure, because many summations over some other integers appear in the course of the evaluation. However, we will show that we can obtain the result consistent with the integrability at least up to the leading order of the instanton corrections.

In the expression for $a^{ij}_D$, the singularities appear as double poles for the integral with respect to $s$, which consists of the contribution both from $a_i$ and $a_j$. Since $a_i$ and $a_j$ have different singularities, it seems not easy to extract these double poles in a concise manner. We therefore consider the path from zero to $e_i$ and define

$$\tilde{a}^i_D \equiv \frac{1}{\pi i} \int_0^{e_i} \lambda - \frac{1}{2} a_i,$$

(30)

where we have again subtracted the contribution caused by the degeneracy of the roots in the expression of $\lambda$ in (12). In the weak coupling region, $a^{ij}_D$ can be written as

$$a^{ij}_D = \tilde{a}^i_D - \tilde{a}^j_D.$$

(31)
From the relation (14), we have

\[ a^i_D = \bar{a}^i_D - \frac{1}{N_c} \sum_{k=1}^{N_c} \bar{a}^k_D. \]  

(32)

Let us evaluate \( \bar{a}^i_D \). By parameterizing \( x = e_i(1-t) \), we have

\[
\bar{a}^i_D = \frac{1}{\pi i} \frac{1}{\Gamma(1/2)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-s) \Gamma(s + 1/2)}{2s} \sum_{m_k, k \neq i} \int_0^1 t^{-2s+1+\sum m_k} dt \times \prod_{k, k \neq i} \frac{\Gamma(2s + m_k)}{\Gamma(2s) \Gamma(m_k + 1)} (e_i - e_k)^{-2s+m_k} e_i^{-2s+1+\sum m_k} (-\Lambda^2 N_c)^s - \frac{1}{2} a^i_i,
\]

(33)

Strictly speaking, in order to performing the \( t \)-integral, we have to make use of the analytic continuation from the strong coupling expression which takes the pole \( s = -1/2, -3/2, \ldots \), and after performing integral we have to use the analytic continuation to the weak coupling region. Let us evaluate the poles separately. At first, we are going to evaluate the pole at \( s = 0 \). The double pole arises in the case of \( m_k = 0 \) for all \( k \), and single poles appear when \( m_k \neq 0 \) for one of \( k \). We can evaluate the pole directly to find that the contribution from the zeroth order is given by

\[
\bar{a}^{i(0)}_D = \frac{1}{2\pi i} \left\{ -\sum_{k=1}^{N_c} (e_i - e_k) \ln \frac{(e_i - e_k)^2}{\Lambda^2} + [2 + (\psi(1/2) - \psi(1))/N_c] \sum_k (e_i - e_k)
\right. 
\]

\[
\left. - \sum_{k=1}^{N_c} e_k \ln \frac{e_k^2}{\Lambda^2} \right\}.
\]

(34)

From the relation (30) we have

\[
a^{i(0)}_D = \frac{1}{2\pi i} \left\{ -\sum_{k=1}^{N_c} (e_i - e_k) \ln \frac{(e_i - e_k)^2}{\Lambda^2}
\right. 
\]

\[
\left. + [2 + (\psi(1/2) - \psi(1))/N_c] \sum_k (e_i - e_k) \right\},
\]

(35)

which has exactly required form of \( a^i_D \) in the weak coupling region[2]. We next evaluate the contribution from \( s = 1 \). The double poles appear in the case \( \sum m_k = 1 \) in (33). Other terms have single poles so that we can evaluate them without any analytic continuation.
by going back to the original expression for $\lambda$ for $s = 1$. We can use the following decomposition

$$
\frac{1}{\prod_{k=1}^{N_c} (x - e_k)^2} = \sum_{k=1}^{N_c} \frac{1}{\prod_{l \neq k} (e_k - e_l)^2} \left( \frac{1}{(x - e_k)^2} + \frac{\partial}{\partial e_k} \left[ \frac{1}{\prod_{l \neq k} (e_k - e_l)^2} \right] \right). \tag{36}
$$

Of all the expansion of (30) by using the expansion (36), the term containing $1/(x - e_i)^2$ can be evaluated from the expression (33) as the term satisfying $\sum m_k = 0$ and the term of the form $1/(x - e_i)$ was evaluated as double pole in (33) so that we should subtract it. Other terms can be integrated explicitly. The result up to the leading order turns out to be

$$
a^i_D = \frac{i}{2\pi} \sum_k (a_i - a_k) \ln \left( \frac{e_i - e_k}{\Lambda^2} \right) - \frac{i}{\pi} \sum_k (e_i - e_k) - \frac{i}{2\pi} \left[ (\psi(1/2) - \psi(1))/N_c \right] \sum_k (a_i - a_k)$$

$$
\quad - \frac{i\Lambda^{2N_c}}{8\pi} \frac{\partial}{\partial e_i} \left[ \sum_{k \neq i} \frac{1}{\prod_{l \neq k} (e_k - e_l)^2} \right], \tag{37}
$$

which can be written by $a$ in the following form;

$$
a_D^i = \frac{i}{2\pi} \sum_k (a_i - a_k) \ln \left( \frac{a_i - a_k}{\Lambda^2} \right) - \frac{i}{\pi} \sum_k (a_i - a_k) - \frac{i}{2\pi} \left[ 2 + (\psi(1/2) - \psi(1))/N_c \right] \sum_k (a_i - a_k)$$

$$
\quad - \frac{i\Lambda^{2N_c}}{8\pi} \frac{\partial}{\partial a_i} \left[ \sum_{k \neq i} \frac{1}{\prod_{l \neq k} (a_k - a_l)^2} \right]. \tag{38}
$$

The prepotential at this order can be obtained as

$$
\mathcal{F}(a) = \frac{i}{4\pi} \sum_{i < j} (a_i - a_j)^2 \ln \left( \frac{a_i - a_j}{\Lambda^2} \right) + \frac{\tau_0}{2N_c} \sum_{i < j} (a_i - a_j)^2 + \mathcal{F}_1(a), \tag{39}
$$

where $\tau_0$ is the bare coupling;

$$
\tau_0 = \frac{i}{2\pi} (2 \ln 2 - 3N_c), \tag{40}
$$

and the one-instanton contribution $\mathcal{F}_1(a)$ is given by

$$
\mathcal{F}_1(a) = -\frac{i\Lambda^{2N_c}}{8\pi} \sum_{k=1}^{N_c} \frac{1}{\prod_{l \neq k} (a_k - a_l)^2}. \tag{41}
$$

Therefore, our method for analytic continuation is consistent with the integrability of the prepotential at least up to the leading order of the instanton corrections. It should be
noted that the expression (39) agrees completely with the known results for $G = SU(2)$ and $SU(3)$\textsuperscript{1}. Moreover, it coincides with the result obtained by the direct instanton method for $SU(N_c)$\textsuperscript{15}.

Let us consider the theories with other gauge groups. It is straightforward to apply the method to other groups. We are now going to list the curve and the one instanton contribution of other classical groups.

For $SO(2N + 1)$, the curve is identified as\textsuperscript{5}

$$y^2 = \prod_{k=1}^{N} (x^2 - e_k^2)^2 - \Lambda^{2(2N-1)}x^2.$$  
(42)

From this curve, we can calculate the one instanton contribution as

$$\mathcal{F}_1(a) = -\frac{i\Lambda^{2(2N-1)}}{32\pi} \sum_{k=1}^{N} \frac{1}{\prod_{l\neq k}(a_k^2 - a_l^2)^2}.$$  
(43)

For $SO(2N)$, the curve is given by\textsuperscript{6}

$$y^2 = \prod_{k=1}^{N} (x^2 - e_k^2)^2 - \Lambda^{2(2N-1)}x^4;$$  
(44)

from which the one instanton contribution is obtained as

$$\mathcal{F}_1(a) = -\frac{i\Lambda^{4(N-1)}}{32\pi} \sum_{k=1}^{N} \frac{a_k^2}{\prod_{l\neq k}(a_k^2 - a_l^2)^2}.$$  
(45)

In the case of $SO(4)$, you can find the decomposition to $SU(2) \times SU(2)$.

The Weierstrass form of the curve for the groups $SP(2N)$ is given by

$$y^2 = P^2(x) - \Lambda^{2(N+1)}P(x).$$  
(46)

where

$$P(x) = x^2 \prod_{k=1}^{N} (x^2 - e_k^2).$$  
(47)

The equivalent Riemann surface is\textsuperscript{9}

$$f = (z + \frac{\Lambda^{2(N+1)}}{z})^2 - 4P(x) = 0,$$  
(48)

\textsuperscript{1}The bare coupling agrees when we correct a misprint in ref.\textsuperscript{11}. 
whose equivalence can be checked by evaluating the periods. The meromorphic one form of the curve is obtained as
\[
\lambda = \frac{dx}{2\pi i} \left( \frac{\Gamma(-s)\Gamma(s + 1/2)}{\Gamma(1/2)s} \right) x^{-2s} \prod_{k=1}^{N} (x^2 - e_k^2)^{-s} (-\Lambda^{2(N+1)})^s,
\]
where the integration over \( s \) takes the poles at \( s = 0, 1, \ldots \). It can be shown that the classical part of the prepotential agrees with the general form and we find that the one instanton contribution is given by
\[
\mathcal{F}_1(a) = -\frac{i\Lambda^{2(N+1)}}{4\pi} \left( \frac{(-1)^N}{\prod_{k=1}^{N} a_k^2} \right). \tag{50}
\]
Note that all these results agree completely with the one obtained by the direct instanton method[15].

We have shown how to calculate the effective action of \( N = 2 \) supersymmetric Yang-Mills theories without matter hypermultiplets. At this moment, it is not clear whether our method of analytic continuation is consistent with the integrability of the prepotential at all orders. Although we could obtain rather compact expression for \( a \), we have not been able to obtain the general form of the dual fields \( a_D \). Actually the calculation of the next leading order seems very complicated and requires more simplification of our method.

When we consider the \( SU(2) \) theories with matter hypermultiplets, the use of symmetric polynomial has been shown to be useful[12, 14]. Therefore, it is natural that we can obtain a compact expression by using the symmetric polynomial of the root variables even for the theories having higher rank gauge groups. It seems interesting to analyze the quartic and quadratic transformations in these cases.

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References


