Periods and Prepotential of $N=2$ SU(2) Supersymmetric Yang-Mills Theory with Massive Hypermultiplets

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Abstract

We derive a simple formula for the periods associated with the low energy effective action of $N = 2$ supersymmetric SU(2) Yang-Mills theory with massive $N_f \leq 3$ hypermultiplets. This is given by evaluating explicitly the integral associated to the elliptic curve using various identities of hypergeometric functions. Following this formalism, we can calculate the prepotential with massive hypermultiplets both in the weak coupling region and in the strong coupling region. In particular, we show how the Higgs field and its dual field are expressed as generalized hypergeometric functions when the theory has a conformal point.
1 Introduction

Since Seiberg and Witten discovered how to determine exactly the low-energy effective theory of $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory by using the elliptic curve[1], many subsequent works have been made on the basis of their analysis by extending the gauge group and by introducing matter hypermultiplets[2, 3, 4, 6, 5, 7]. The exact solution for the prepotential which controls the low energy effective action, can be obtained from the period integrals on the elliptic curve. Associated with singularities of the theories coming from the massless states, these curves for various kinds of $N = 2$ supersymmetric Yang-Mills theories have been studied extensively[4]. Usual approach to obtain the period is to solve the differential equation which the periods obey, so called Pichard-Fuchs equation[8, 9]. When the theory is pure Yang-Mills with massless $N_f \leq 3$ hypermultipletsthe with gauge group $SU(2)$, this approach works successfully to solve the periods[10] because these theories have three singularity points if we use appropreate variables. Other more direct approach is known to be valid only in these cases[11]. However when hypermultiplets are massive, the situation changes drastically; additional massless states appear in the theory and the number of singularities becomes more than three. Therefore, the Picard-Fuchs equation can no longer be solved by any special function and the known solution is a perturbative solution in the weak coupling region[12].

In this article we derive a simple formula for the periods from which we can obtain the prepotential both in the weak coupling region and in the strong coupling region; we can evaluate the period integral of holomorphic one-form on the elliptic curve by using various identities of hypergeometric functions. As a result, the periods are represented as hypergeometric functions in terms of the function of $u, \Lambda$ and masses, which are known from the form of the elliptic curve. We show that the resulting expression agrees with the results for massless case[10] and also have a power to handle the theories with conformal points.
2 Period Integrals

We begin with reviewing some properties of the low-energy effective action of the \( N = 2 \) supersymmetric \( SU(2) \) QCD. In \( N = 1 \) superfields formulation\cite{1}, the theory contains chiral multiplets \( \Phi^a \) and chiral field strength \( W^a \) \((a = 1, 2, 3)\) both in the adjoint representation of \( SU(2) \), and chiral superfield \( Q^i \) in \( 2 \) and \( \tilde{Q}^i \) \((i = 1, \ldots, N_f)\) in \( \bar{2} \) representation of \( SU(2) \). In \( N = 2 \) formulation \( Q^i \) and \( \tilde{Q}^i \) are called hypermultiplets. Along the flat direction, the scalar field \( \phi \) of \( \Phi \) get vacuum expectation values which break \( SU(2) \) to \( U(1) \), so that the low-energy effective theory contains \( U(1) \) vector multiplets \((A, W_\alpha)\), where \( A \) are \( N = 1 \) chiral superfields and \( W_\alpha \) are \( N = 1 \) vector superfields. The quantum low-energy effective theory is characterized by effective Lagrangian \( \mathcal{L} \) with the holomorphic function \( \mathcal{F}(A) \) called prepotential,

\[
\mathcal{L} = \frac{1}{4\pi} \text{Im} \left( \int d^2\theta d^2\bar{\theta} \frac{\partial \mathcal{F}}{\partial A} \bar{A} + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial A^2} W_\alpha W^\alpha \right). \tag{2.1}
\]

The scalar component of \( A \) is denoted by \( a \), and \( A_D = \frac{\partial \mathcal{F}}{\partial a} \) which is dual to \( A \) by \( a_D \). The pair \((a_D, a)\) is a section of \( SL(2, \mathbb{Z}) \) and is obtained as the period integrals of the elliptic curve parameterized by \( u, \Lambda \) and \( m_i \) \((0 \leq i \leq N_f)\), where \( u = \langle \text{tr} \phi^2 \rangle \) is a gauge invariant moduli parameter, \( \Lambda \) is a dynamical scale and \( m_i \) are bare masses of hypermultiplets. Once we know \( a \) and \( a_D \) as a holomorphic function of \( u \), we can calculate the prepotential \( \mathcal{F}(a) \) by using the relation

\[
a_D = \frac{\partial \mathcal{F}(a)}{\partial a}. \tag{2.2}
\]

General elliptic curves of \( SU(2) \) Yang-Mills theories with massive \( N_f \leq 3 \) hypermultiplets are\cite{4}

\[
y^2 = C^2(x) - G(x) \tag{2.3}
\]

\[
\begin{align*}
C(x) &= x^2 - u, & G(x) &= \Lambda^4, & (N_f = 0) \\
C(x) &= x^2 - u, & G(x) &= \Lambda^3(x + m_1), & (N_f = 1) \\
C(x) &= x^2 - u + \frac{\Lambda^2}{8}, & G(x) &= \Lambda^2(x + m_1)(x + m_2), & (N_f = 2) \\
C(x) &= x^2 - u + \frac{\Lambda}{4}(x + \frac{m_1 + m_2 + m_3}{2}), & G(x) &= \Lambda(x + m_1)(x + m_2)(x + m_3), & (N_f = 3)
\end{align*}
\]
These curves are formally denoted by

\[ y^2 = C^2(x) - G(x) = (x - e_1)(x - e_2)(x - e_3)(x - e_4), \quad (2.4) \]

where \( e_1 = e_4, \ e_2 = e_3 \) in the classical limit. In order to calculate the prepotential, we consider \( a \) and \( a_D \) as the integrals of the meromorphic differential \( \lambda \) over two independent cycles of these curves,

\[ a = \oint_{\alpha} \lambda, \quad a_D = \oint_{\beta} \lambda, \quad (2.5) \]

\[ \lambda = \frac{x}{2\pi i} d\ln \left( \frac{C(x) - y}{C(x) + y} \right). \quad (2.6) \]

where \( \alpha \) cycle encloses \( e_2 \) and \( e_3 \), \( \beta \) cycle encloses \( e_1 \) and \( e_3 \), \( \lambda \) is related to the holomorphic one-form as

\[ \frac{\partial \lambda}{\partial u} = \frac{1}{2\pi i} \frac{dx}{y} + d(*). \quad (2.7) \]

Since there are poles coming from mass parameters in the integrant of \( a \) and \( a_D \), we instead evaluate the period integrals of holomorphic one-form;

\[ \frac{\partial a}{\partial u} = \oint_{\alpha} \frac{dx}{y}, \quad \frac{\partial a_D}{\partial u} = \oint_{\beta} \frac{dx}{y}. \quad (2.8) \]

First of all, we consider \( \frac{\partial a}{\partial u} \);

\[ \frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2\pi} \int_{e_2}^{e_3} \frac{dx}{y} = \frac{\sqrt{2}}{2\pi} \int_{e_2}^{e_3} \frac{dx}{\sqrt{(x - e_1)(x - e_2)(x - e_3)(x - e_4)}}, \quad (2.9) \]

where the normalization is fixed so as to be compatible with the asymptotic behavior of \( a \) and \( a_D \) in the weak coupling region

\[ a = \frac{\sqrt{2u}}{2} + \cdots, \]

\[ a_D = i \frac{4 - N_f}{2\pi} a \ln a + \cdots. \quad (2.10) \]

After changing the variable and using the integral representation of hypergeometric function;

\[ F(a, b; c; x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \int_{0}^{1} ds s^{b-1} (1 - s)^{c-b-1} (1 - sx)^{-a} \quad (2.11) \]
where
\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}, \]
(2.12)
we obtain \( \frac{\partial a}{\partial u} \) as
\[ \frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} (e_2 - e_1)^{-1/2} (e_4 - e_3)^{-1/2} F \left( \frac{1}{2}, \frac{1}{2}; 1; z \right), \]
(2.13)
where
\[ z = \frac{(e_1 - e_4)(e_3 - e_2)}{(e_2 - e_1)(e_4 - e_3)}. \]
(2.14)
Similarly we get the following expression for \( \frac{\partial a_D}{\partial u} \);
\[ \frac{\partial a_D}{\partial u} = \frac{\sqrt{2}}{2\pi} \int_{e_1}^{e_3} \frac{dy}{y} = \frac{\sqrt{2}}{2} \left[ (e_1 - e_2)(e_4 - e_3) \right]^{-1/2} F \left( \frac{1}{2}, \frac{1}{2}, 1; 1 - z \right). \]
(2.15)
In this case \( a_D \) is obtained as a hypergeometric function around \( z = 1 \), so we have to do the analytic continuation which gives the logarithmic asymptotic in the weak coupling region.

Since elliptic curves are not factorized in general, it is difficult to obtain their roots in a simple form. Even if we know the form of roots, the variable \( z \) in (2.13) and (2.15) is very complicated in terms of \( u \) in these representations. So we will transform the variable to the symmetric form with respect to roots, by using the identity of the hypergeometric functions, so that the new variable is given easily from the curve directly without knowing the form of roots.

### 3 Quadratic and cubic transformation
3.1 Quadratic transformation

Before we treat a variety of SU(2) Yang-Mills theory with hypermultiplets, we consider the case where the elliptic curve is of the form

\[ y^2 = (x^2 + a_1 x + b_1)(x^2 + a_2 x + b_2). \] (3.1)

There are two possibilities that \( e_1 \) and \( e_2 \) are roots of the first quadratic polynomial or \( e_1 \) and \( e_4 \) are. First of all, we consider the former case. If the variable of the hypergeometric function become symmetric about \( e_1, e_2, e_3, e_4 \), it is quite easy to read the variable from the form of this curve. To this end, we use the quadratic transformation[13] for the hypergeometric functions to (2.13)

\[ F(2a, 2b; a + b + 1/2; z) = F(a, b; a + b + 1/2; 4z(1 - z)), \] (3.2)

where \( a = b = 1/4 \), so that the new variable \( z' = 4z(1 - z) \) of hypergeometric function is symmetric with respect to \( e_1, e_2, e_3, e_4 \):

\[ z' = 4z(1 - z) = \frac{(e_1 - e_3)(e_2 - e_4)(e_1 - e_4)(e_3 - e_2)}{(e_2 - e_1)^2(e_4 - e_3)^2}, \] (3.3)

and \( z' \) can be easily expressed by \( a_1, b_1, a_2, b_2 \) as

\[ z' = -4(ab - b_2)^2 - (ab + b_2)a_1a_2 + a_2^2a_1 + a_1^2b_2 \] (3.4)

Therefore, \( \frac{\partial a}{\partial u} \) can be written as

\[ \frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} [(e_2 - e_1)(e_4 - e_3)]^{-1/2} F \left( \frac{1}{4}, \frac{1}{4}, 1; z' \right). \] (3.5)

Similarly for \( \frac{\partial a_D}{\partial u} \), after using the analytic continuation and the quadratic transformation, we get

\[ \frac{\partial a_D}{\partial u} = \frac{\sqrt{2}}{2} [(e_2 - e_1)(e_4 - e_3)]^{-1/2} \left[ \frac{6 \ln 4}{2\pi} F \left( \frac{1}{4}, \frac{1}{4}, 1; z' \right) - \frac{1}{\pi} F^* \left( \frac{1}{4}, \frac{1}{4}, 1; z' \right) \right], \] (3.6)

where \( F^*(\alpha, \beta; 1; z) \) is another independent solution around \( z = 0 \) of the differential equation which \( F(\alpha, \beta; 1; z) \) obeys, which is expressed as

\[ F^*(\alpha, \beta, 1, z) = F(\alpha, \beta, 1, z) \ln z + \sum_{n=1}^{\infty} \frac{(\alpha)^n (\beta)^n}{(n!)^2} z^n \sum_{r=1}^{n-1} \left[ \frac{1}{\alpha + n} + \frac{1}{\beta + n} - \frac{2}{n + 1} \right]. \] (3.7)
Therefore, we obtain the general expression for $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$ in the weak coupling region valid in the case of the elliptic curve (3.1). Notice that the quadratic transformation (3.2) is valid if $|z'| \leq 1$. The region of $z$-plane which satisfies this condition consists of two parts; one is around $z = 0$, one is around $z = 1$. The region around $z = 0$ corresponds to the weak coupling region, and the regions around $z = 1$ corresponds to the strong coupling region where monopole condensates. So we can construct the formula valid in the strong coupling region by continuing the expression (2.13) and (2.15) analytically to around $z = 1$ and by applying the quadratic transformaion (3.2).

Similarly if we consider the latter case where $e_1$ and $e_3$ are roots of first quadratic polynomial of the curve (3.1), we have to do the transformation which make the variable symmetric about $e_1, e_4$ and $e_2, e_3$. Thus we use another quadratic transformaion[13]

$$F(a, b; 2b; z) = (1 - z)^{-a/2} F\left(\frac{a}{2}, b - \frac{a}{2}; b + 1, \frac{z^2}{4(1 - z)}\right),$$

(3.8)

where $a = 1/2$. The new variable $\tilde{z}' = z^2/4(1 - z)$ is symmetric about $e_1, e_4$ and $e_2, e_3$ as follows;

$$\tilde{z}' = \frac{z^2}{4(1 - z)} = \frac{(e_1 - e_4)^2(e_3 - e_2)^2}{4(e_2 - e_1)(e_4 - e_3)(e_1 - e_3)(e_4 - e_2)}$$

$$= -\frac{1}{4((b_1 - b_2)^2 - (b_1 + b_2)a_1 a_2 + a_1^2 b_2 + a_2^2 b_1)}$$

(3.9)

By applying this transformation to (2.13) and (2.15), we get $\frac{\partial a}{\partial u}$, $\frac{\partial a_D}{\partial u}$ as

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} \left[ (e_3 - e_1)(e_4 - e_2)(e_2 - e_1)(e_4 - e_3) \right]^{-1/4} F\left(\frac{1}{4}, \frac{1}{4}, 1; \tilde{z}'\right),$$

(3.10)

$$\frac{\partial a_D}{\partial u} = i\frac{\sqrt{2}}{2} \left[ (e_1 - e_3)(e_4 - e_2)(e_2 - e_1)(e_4 - e_3) \right]^{-1/4}$$

$$\times \left[ 3\ln 4 - \frac{i\pi}{2\pi} F\left(\frac{1}{4}, \frac{1}{4}, 1; \tilde{z}'\right) - \frac{1}{2\pi} F^*\left(\frac{1}{4}, \frac{1}{4}, 1; \tilde{z}'\right) \right].$$

(3.11)

In both cases we can read the variable directly from the coefficients of the curve.

In the next subsection, we generalize the formalism of this subsection to all kinds of $SU(2)$ Yang-Mills theory with massive $N_f \leq 3$ hypermultiplets.
3.2 Cubic transformation

We denote the curve as

\[ y^2 = x^4 + ax^3 + bx^2 + cx + d. \]  \hspace{1cm} (3.12)

In general, the variable of the hypergeometric function is still very complicated even after the quadratic transformation. So in addition to the quadratic transformation, we must use the following cubic transformation[13] subsequently

\[ F \left( 3a, a + \frac{1}{6}; 4a + \frac{2}{3}; z' \right) = \left( 1 - \frac{z'}{4} \right)^{-3a} F \left( a, a + \frac{1}{3}; 2a + \frac{5}{6}; -27 \frac{z'^2}{(z' - 4)^3} \right), \]  \hspace{1cm} (3.13)

or

\[ F \left( 3a; a + \frac{1}{3} - a; 2a + \frac{5}{6}, \bar{z}' \right) = (1 - 4\bar{z}')^{-3a} F \left( a, a + \frac{1}{3}; 2a + \frac{5}{6}, \frac{27\bar{z}'}{(4\bar{z}' - 1)^3} \right), \]  \hspace{1cm} (3.14)

where \( a = \frac{1}{12} \), so that the new variable \( z'' = -27z'^2/(z' - 4)^3 = 27\bar{z}'/(4\bar{z}' - 1)^3 \) become completely symmetric in \( e_i \). Notice that \( z'' \) is represented by coefficients of the elliptic curve

\[ z'' = -27 \frac{z'^2}{(z' - 4)^3} = 27 \frac{\bar{z}'}{(4\bar{z}' - 1)^3} = \frac{27z^2(1 - z)^2}{4(z^2 - z + 1)^3} = \frac{-27\Delta}{4D^3}, \]  \hspace{1cm} (3.15)

where \( \Delta \) is the discriminant of the elliptic curve

\[
\Delta = \prod_{i<j} (e_i - e_j)^2
= -[27a^4d^2 + a^3c(4c^2 - 18bd) + ac(-18bc^2 + 80b^2d + 192d^2)] + a^2(-b^2c^2 + 4b^3d + 6c^2d - 144bd^2) + 4b^3c^2 + 27c^4
- 16b^4d - 144bc^2d + 128b^2d^2 - 256d^3],
\]  \hspace{1cm} (3.16)

and \( D \) is given by

\[ D = \sum_{i<j} \frac{1}{2} (e_i - e_j)^2 = -b^2 + 3ac - 12d. \]  \hspace{1cm} (3.17)

Applying (3.13) to (3.5) or (3.14) to (3.10), without knowing precise forms of \( e_i \) we obtain a general expression for \( \frac{\partial a}{\partial u} \) in the weak coupling region valid even in the theory.
with massive $N_f \leq 3$ hypermultiplets,
\[
\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} (-D)^{-1/4} F\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3}\right). \tag{3.18}
\]

Similarly, after the analytic continuation and quadratic and cubic transformations, we obtain an expression for $\frac{\partial a_D}{\partial u}$ as
\[
\frac{\partial a_D}{\partial u} = i \frac{\sqrt{2}}{2} (-D)^{-1/4} \left[ \frac{3}{2\pi} \ln 12 \ F\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3}\right) \right.
\]
\[
- \frac{1}{2\pi} \ F^*\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3}\right) \right]. \tag{3.19}
\]

For the consistency check, we consider the asymptotic behavior in the weak coupling region $u \to \infty$,
\[
\Delta = (-1)^{N_f} 256u^{N_f + 2} \Lambda^{2(4-N_f)} + \cdots,
\]
\[
D = -16u^2 + \cdots, \tag{3.20}
\]
\[
-\frac{27}{4} \ \frac{\Delta}{D^3} = \frac{27(-1)^{N_f}}{64} \left(\frac{\Lambda^2}{u}\right)^{4-N_f} + \cdots.
\]

Thus we have
\[
\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{4\sqrt{u}} + \cdots,
\]
\[
\frac{\partial a_D}{\partial u} = i \frac{\sqrt{2}}{4\sqrt{u}} \frac{4 - N_f}{2\pi} \ln \left(\frac{\Lambda^2}{u}\right) + \cdots, \tag{3.21}
\]

which is compatible with (2.10).

The formula (3.18) and (3.19) are useful in the case where we cannot obtain any simple expression of roots, whereas (3.5) and (3.6) or (3.10) and (3.11) can be used when we have a factorized form for $y^2$ as (3.1).

Next we consider the periods in the strong coupling region. The quadratic and cubic transformation are valid if $|z''| \leq 1$. The region of $z$-plane which satisfies this condition consists of three parts; one is around $z = 0$, one is around $z = 1$ and the last is around $z = \infty$. The region around $z = 0$ corresponds to the weak coupling region, and the region around $z = 1$ corresponds to the strong coupling region where the monopoles condensate and $z = \infty$ is the dyonic point. So we can construct the formula valid in the strong...
coupling region by analytic continuation to around \( z = 1 \) or \( z = \infty \) and by using the quadratic and cubic transformation subsequently. For example, the formula around the strong coupling region \( z = 1 \) is given by

\[
\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} (-D)^{-1/4} \left[ \frac{3}{2\pi} \ln 12 \ F \left( \frac{1}{12}, \frac{5}{12}, 1, -\frac{27\Delta}{4D^3} \right) \right.
\]

\[
- \frac{1}{2\pi} F^* \left( \frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3} \right),
\]

(3.22)

\[
\frac{\partial a_D}{\partial u} = i\frac{\sqrt{2}}{2} (-D)^{-1/4} F \left( \frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3} \right).
\]

(3.23)

The expression (3.18), (3.19) and (3.23), (3.23) show a manifest duality of the periods.

Notice that the ratio of two period integrals is the coupling constant of the theory,

\[
\tau = \frac{\partial^2 F}{\partial a^2} = \frac{\partial a_D}{\partial a} = \frac{\partial a_D}{\partial u} / \frac{\partial a}{\partial u} = \frac{i F(1/2, 1/2, 1, 1 - z)}{F(1/2, 1/2, 1, z)}.
\]

(3.24)

Though \( z \) is not invariant under the modular transformation of \( \tau \), the argument \( z'' = -27\Delta/4D^3 = 27z^2(1 - z)^2/(z^2 - z + 1)^3 \) is invariant completely. As a matter of fact, this variable can be written by the absolute invariant form \( j(\tau) \) as \( z'' = 1/j(\tau) \). Therefore it is quite natural to represent the period in terms of \( z'' \).

4 Examples

In this section we calculate the period, \( a, a_D \) and the prepotential of a variety of supersymmetric \( SU(2) \) Yang-Mills theory with massive hypermultiplets as examples of our formula. For consistency check we also consider massless case. Moreover we consider the cases where the theory has conformal points.

4.1 \( N_f = 1 \) theory

We consider the theory with a matter hypermultiplet whose curve is given by

\[
y^2 = (x^2 - u)^2 - \Lambda^3(x + m),
\]

(4.1)
from which $\Delta$ and $D$ is obtained as

$$\Delta = -\Lambda^6(256u^3 - 256u^2m^2 - 288um\Lambda^3 + 256m^3\Lambda^3 + 27\Lambda^6), \quad (4.2)$$

$$D = -16u^2 + 12m\Lambda^3. \quad (4.3)$$

Substituting these to (3.18) and (3.19), we can obtain $\alpha$, $\alpha_D$, by expanding (3.18) and (3.19) at $u = \infty$ and integrating with respect to $u$. Representing $u$ in terms of $\alpha$ inversely, and substituting $u$ to $\alpha_D$, and finally integrating $\alpha_D$ with respect to $\alpha$, we can get the prepotencial in the weak coupling region as

$$\mathcal{F}(\tilde{\alpha}) = i\tilde{\alpha}^2 \pi \left[ \frac{3}{4} \ln \left( \frac{\tilde{\alpha}^2}{\Lambda^2} \right) + \frac{3}{4} (-3 + 4\ln 2 - i\pi) - \frac{\sqrt{2\pi}}{2i\tilde{\alpha}} (n'm) \right.\left. - \ln \left( \frac{\tilde{\alpha}}{\Lambda} \right) \frac{m^2}{4\tilde{\alpha}^2} + \sum_{i=2}^{\infty} \mathcal{F}_i \tilde{\alpha}^{-2i} \right]. \quad (4.4)$$

where we introduce $\tilde{\alpha}$ subtracted mass residues from $\alpha$. These $\mathcal{F}_i$ agree with the perturbative result up to the orders cited in [12]. In principle we can calculate $\mathcal{F}_i$ to arbitrary order in our formalism. Quite similarly, we can obtain the prepotential in the strong coupling region.

To compare to the periods for massless case where the explicit form is known by solving the Picard-Fuchs equation[10], we start with our expression for the massless theory,

$$\Delta = -\Lambda^6(256u^3 + 27\Lambda^6), \quad D = -16u^2, \quad z'' = -\frac{27\Lambda^6(256u^3 + 27\Lambda^6)}{16^3u^6}. \quad (4.5)$$

If we set $w = -27\Lambda^6/256u^3$ then $z'' = 4w(1-w)$, thus using the quadratic transformations (3.2), we get the expression for the massless case,

$$\frac{\partial \alpha}{\partial u} = \frac{\sqrt{2}}{2} \frac{1}{2\sqrt{u}} F\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{6}; w\right),$$

$$\frac{\partial \alpha_D}{\partial u} = i\frac{\sqrt{2}}{2} \frac{1}{2\sqrt{u}} \left[ 3\ln 3 + 2\ln 4 \right] F\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{6}; w\right) - \frac{1}{2\pi} F^\ast\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{6}; 1, w\right). \quad (4.6)$$

Integrating with respect to $u$, we can get the expression given by Ito and Yang[10]. The expression around the strong coupling region can be obtained from (3.22) and (3.22) by using the identity (3.2) for $w = 1 - \frac{27\Lambda^6}{256u^3}$. In general when $m \neq 0$, because of the singularity coming from additional massless states[1], we cannot represent $\alpha$ and $\alpha_D$ as
any special functions by integrating the expression for $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$. However if masses take critical values with which the number of the singularity goes down to the same number as in the massless case, that is three, $a$ and $a_D$ seem to be expressed by special functions. The number of the singularity is the number of the root of the equation $\Delta = 0$ plus one, which is the singularity at $u = \infty$. Since $\Delta$ is third order polynomial in terms of $u$ in $N_f = 1$ theory, $\Delta = 0$ must have one double root when the mass takes critical value. This condition is satisfied if $m = 3/4\Lambda$ where the parameters of the periods are given by

$$\Delta = -\Lambda^6(16u + 15\Lambda^2)(4u - 3\Lambda^2)^2, \quad D = -(4u + 3\Lambda)(4u - 3\Lambda), \quad (4.7)$$

$$z'' = -\frac{27}{4} \frac{\Lambda^6(16u + 15\Lambda^2)}{(4u + 3\Lambda^2)(4u - 3\Lambda^2)}. \quad (4.8)$$

Such factorization of $\Delta$ means that theory has a conformal point $u = 3\Lambda^2/4$ where the curve become [14, 15]

$$y^2 = \left(x + \frac{\Lambda}{2}\right)^3 \left(x - \frac{3\Lambda}{2}\right). \quad (4.9)$$

If we set

$$w = \frac{27\Lambda^2}{16u + 15\Lambda^2}, \quad (4.10)$$

then $z'' = -64w^3/(w - 9)^3(w - 1)$. In order to obtain $a$ and $a_D$ we need the quartic transformation which makes the variable simple enough. We can prove the following transformation of fourth order;

$$F \left( \frac{1}{12}, \frac{5}{12}, 1, -\frac{64w^3}{(w - 9)^3(w - 1)} \right) = \left(1 - \frac{w}{9}\right)^{1/4} \left(1 - w\right)^{1/12} F \left(\frac{1}{3}, \frac{1}{3}, 1, w\right). \quad (4.11)$$

Using this identity and the identity

$$F(a, b; c; z) = (1 - z)^{-a} F(a, c - b; c; z/(z - 1)), \quad (4.12)$$

we get $\frac{\partial a}{\partial u}$, $\frac{\partial a_D}{\partial u}$

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{8} \left(-27\Lambda^2\right)^{1/2} y^{1/2} F \left(\frac{1}{3}, \frac{2}{3}, 1, y\right), \quad (4.13)$$

$$\frac{\partial a_D}{\partial u} = i \frac{\sqrt{2}}{8} \left(-27\Lambda^2\right)^{1/2} y^{1/2} \left[\frac{(3\ln 3 - i\pi)}{2\pi} F \left(\frac{1}{3}, \frac{2}{3}, 1, y\right) - \frac{3}{2\pi} F^* \left(\frac{1}{3}, \frac{2}{3}, 1, y\right)\right], \quad (4.14)$$
where

\[ y = \frac{27\Lambda^2}{-16u + 12\Lambda^2}. \]  

(4.15)

Integrating with respect to \( u \), we get \( a \) and \( a_D \) in the weak coupling region as

\[
a = -\frac{i\sqrt{2}}{8} 3\sqrt{3}\Lambda y^{-\frac{1}{2}} 3F_2 \left( \frac{1}{3}, \frac{2}{3}, -\frac{1}{2}; 1, \frac{1}{2}; y \right),
\]

(4.16)

\[
a_D = +\frac{\sqrt{2}}{8} 3\sqrt{3}\Lambda y^{-\frac{1}{2}} \left[ \frac{3(3\ln 3 - i\pi - 2)}{2\pi} 3F_2 \left( \frac{1}{3}, \frac{2}{3}, -\frac{1}{2}; 1, \frac{1}{2}; y \right) 
- \frac{3}{2\pi} 3F_2^* \left( \frac{1}{3}, \frac{2}{3}, -\frac{1}{2}; 1, \frac{1}{2}; y \right) \right],
\]

(4.17)

where \( 3F_2(a, b, c; 1, d, y) \) is the generalised hypergeometric function[13]

\[
3F_2(a, b, c; 1, d; y) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(n!)^2} y^n,
\]

(4.18)

and

\[
3F_2^*(a, b, c; 1, d; y) = 3F_2(a, b, c; 1, d; y) \ln y 
+ \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(n!)^2} y^n
\]

\[
\times \sum_{r=0}^{n-1} \left[ \frac{1}{a+r} + \frac{1}{b+r} + \frac{1}{c+r} - \frac{1}{d+r} - \frac{2}{1+r} \right],
\]

(4.19)

is other independent solutions of a generalized hypergeometric equation[13] around \( y = 0 \);

\[
y^2(1-y) \frac{d^3F}{dy^3} + \{(d+2)y - (3+a+b+c)y^2\} \frac{d^2F}{dy^2}
\]

\[
+ \{ d - (1+a+b+c+ab+bc+ca)y \} \frac{dF}{dy} - abcF = 0,
\]

(4.20)

which \( 3F_2(a, b, c; 1, d; y) \) obeys. Notice that Picard-Fuchs equation of \( N_f = 1 \) theory reduces to this equation when the theory has the conformal point. This equation has three regular singularities at \( y = 0, 1, \infty \). This is the reason why \( a \) and \( a_D \) are expressed by the special functions.

In order to obtain the expression around the conformal point \( u = 3\Lambda/4 \) from the expression (4.16) and (4.17), we have to perform the analytic continuation from the weak
coupling region. After that, the expressions for $a$ and $a_D$ contain no logarithmic terms,

$$a = -\frac{i\sqrt{2}}{8} 3\sqrt{3}\Lambda y^{-1/2} \left[ \frac{6}{5} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) (-y)^{-1/3} {}_3F_2\left(\frac{1}{3}, \frac{1}{3}, \frac{5}{6}, \frac{2}{3}, \frac{11}{6}; \frac{1}{y}\right) \right. \\
+ \frac{6}{7} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) (-y)^{-2/3} {}_3F_2\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{13}{6}, \frac{11}{6}; \frac{1}{y}\right) \right], \quad (4.21)$$

$$a_D = \frac{\sqrt{3} \sqrt{2}}{2} 3\sqrt{3}\Lambda y^{-1/2} \left[ \frac{6}{5} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) (-y)^{-1/3} {}_3F_2\left(\frac{1}{3}, \frac{1}{3}, \frac{5}{6}, \frac{2}{3}, \frac{11}{6}; \frac{1}{y}\right) \right. \\
- \frac{6}{7} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) (-y)^{-2/3} {}_3F_2\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{13}{6}, \frac{11}{6}; \frac{1}{y}\right) \right], \quad (4.22)$$

where $1/y = -16u/27\Lambda^2 + 4/9$. Thus the coupling constant $\tau$ which is the ratio of $\frac{\partial a_D}{\partial u}$ and $\frac{\partial a}{\partial u}$ has no logarithmic term, and the beta function on this conformal point vanishes.

Here we pause to discuss an interesting relation between the moduli space of this theory and the moduli space of 2-D $N=2$ superconformal field theory with central charge $c = 3$. Consider the complex projective space $\mathbb{P}^2$ with homogeneous coordinates $[x_0, x_1, x_2]$ and define the hypersurface $X$ by the equation

$$f = x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0. \quad (4.23)$$

Moduli space of the theory with $c = 3$ is described by $\tau$, the ratio of two independent period integrals of holomorphic one-form $\Omega$ over the cycle on $X$,

$$\tau = \int_\gamma \Omega / \int_{\gamma'} \Omega. \quad (4.24)$$

It is known that this period satisfy the Picard-Fuchs equation which reduce to

$$\left(z \frac{d}{dz}\right)^2 f(z) - z \left(z \frac{d}{dz} + \frac{1}{3}\right) \left(z \frac{d}{dz} + \frac{2}{3}\right) f(z) = 0, \quad (4.25)$$

where $z = \psi^{-3}$. This is a hypergeometric differential equation and $f(z)$ is obtained as a linear combination of $F(1/3, 2/3; 1; z)$ and $F^*(1/3, 2/3; 1; z)$. By comparing this solution to (4.13) and (4.14), we deduce an identification $\psi^3 = -16u/27\Lambda^2 + 4/9$, and that the conformal point ($u = 3\Lambda^2/4$) of 4-D $SU(2) N_f = 1$ super QCD corresponds to the Landau-Ginzburg point ($\psi = 0$) of 2-D SCFT with $c = 3$. It seems interesting to use this identification to investigate the theory at the conformal fixed point.
4.2 $N_f = 2$ theory

We consider the theory with $N_f = 2$, $m_1 = m_2 = m$ whose curve and discriminant are given by

$$
y^2 = \left( x^2 - u + \frac{\Lambda^2}{8} \right) - \Lambda^2(x + m)^2 = \left( x^2 - \Lambda x - \Lambda m - u + \frac{\Lambda^2}{8} \right) \left( x^2 + \Lambda x + \Lambda m - u + \frac{\Lambda^2}{8} \right)
$$

$$
\Delta = \frac{\Lambda^2}{16} \left( 8u - 8m^2 - \Lambda^2 \right)^2 \left( 8u + 8\Lambda m + \Lambda^2 \right) \left( 8u - 8\Lambda m + \Lambda^2 \right). \tag{4.26}
$$

In this case, we can use the formula of section 3.1 because of the factorized form of the curve. Reading $z'$, $e_2 - e_1$, $e_4 - e_3$ from the coefficients of the curve,

$$
(e_2 - e_1)^2 = 4u + 4\Lambda m + \frac{\Lambda^2}{2},
$$

$$
(e_4 - e_3)^2 = 4u - 4\Lambda m + \frac{\Lambda^2}{2}, \tag{4.27}
$$

$$
z' = \frac{\Lambda^2(u - m^2 - \frac{\Lambda^2}{8})}{(u + \Lambda m + \frac{\Lambda^2}{8})(u - \Lambda m + \frac{\Lambda^2}{8})}.
$$

Substituting these to (3.5) and (3.6), we can obtain $a$ and $a_D$ after expansion around $u = \infty$ and integration with respect $u$. The prepotential in the weak coupling region is

$$
\mathcal{F}(\tilde{a}) = i\tilde{a}^2 \left[ \frac{1}{2} \ln \left( \frac{\tilde{a}^2}{\Lambda^2} \right) + \left( -1 + \frac{i\pi}{2} + \frac{5\ln 2}{2} \right) - \frac{\sqrt{2\pi}}{2i\tilde{a}} (n'm) 
- \ln \left( \frac{\tilde{a}}{\Lambda} \right) \frac{m^2}{2\tilde{a}^2} + \sum_{i=2}^{\infty} \mathcal{F}_i \tilde{a}^{-2i} \right]. \tag{4.28}
$$

These $\mathcal{F}_i$ agree with the result up to known orders [12]. We can calculate the prepotential in $m_1 \neq m_2$ case. Let us compare the results of $a$ and $a_D$ for massless case to the previous results[10]. The variable $z$ can be written in the form $z' = 4w(1-w)$ if we set $x = \Lambda^2/8u$ and $w = 2x/(x+1)$. Therefore, we transform $z'$ to $w$ by using the identity (3.2) and $w$ to $x^2$ by using the identity

$$
F(a, b; 2b; w) = \left( 1 - \frac{z}{2} \right)^{-a} F \left( \frac{a}{2}; \frac{1}{2}; \frac{a}{2}; b + \frac{1}{2}; \frac{w^2}{(w-2)^2} \right), \tag{4.29}
$$
where \( a = b = 1/2 \) and \( w^2/(w - 2)^2 = x^2 \). Thus we get the expression for the massless case;

\[
\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} \frac{1}{2\sqrt{u}} F\left(1, \frac{3}{4}, \frac{1}{4}, x^2\right),
\]

\[
\frac{\partial a}{\partial u} = \frac{i\sqrt{2}}{2} \frac{1}{2\sqrt{u}} \left[ \frac{3\ln 4}{2\pi} F\left(1, \frac{3}{4}, \frac{1}{4}, x^2\right) - \frac{1}{2\pi} F^*\left(1, \frac{3}{4}, \frac{1}{4}, x^2\right) \right].
\]

Integrating with respect to \( u \), we can recover the previous result for \( a \) and \( a_D \) [10].

Next we consider the case where the same factor appears in the denominator and the numerator of \( z' \). This is satisfied if \( m = \Lambda/2 \). In this case the theory has conformal point[14, 15] at \( u = 3\Lambda^2/8 \) where the elliptic curve is factorized as

\[
y^2 = \left(x + \frac{\Lambda}{2}\right)^3 \left(x - \frac{3\Lambda}{2}\right).
\]

Main difference between the massless theory and the massive theory is the existence of this conformal point. Usual massive \( N_f = 2 \) theory has five singularity points where additional two singularity points are coming from the two bare mass parameter. In this subsection we set \( m_1 = m_2 \), so the number of the singularity is four. When the theory has conformal point, two of four singularity points coincide and this point becomes a conformal point[14, 15]. On the other hand, in our representation since the pole and the zero of the variable \( z' \) of the hypergeometric function correspond to the singularity points, the theory has a conformal point when the pole and the zero of \( z' \) coincide and this pole becomes a conformal point.

To obtain \( a \) and \( a_D \) of this theory, we substitute \( m = \Lambda/2 \) to (3.5) and (3.6), and use the identity (4.12), we obtain \( \frac{\partial a}{\partial u}, \frac{\partial a_D}{\partial u} \)

\[
\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} (-\Lambda^2)^{-1/2} y^{1/2} F\left(1, \frac{3}{4}, \frac{1}{4}, y\right)
\]

\[
\frac{\partial a_D}{\partial u} = \frac{i\sqrt{2}}{2} (-\Lambda^2)^{-1/2} y^{1/2} \left[ \frac{(6\ln 4 - i\pi)}{2\pi} F\left(1, \frac{3}{4}, \frac{1}{4}, y\right) - \frac{1}{\pi} F^*\left(1, \frac{3}{4}, \frac{1}{4}, y\right) \right],
\]

where

\[
y = \frac{8\Lambda^2}{-8u + 3\Lambda^2},
\]

(4.35)
Integrate with respect to $u$, we get $a$ and $a_D$ in the weak coupling region as

$$a = -\frac{\sqrt{2}}{2} (-1)^{3/4} \Lambda y^{-1/2} F_2 \left( \frac{1}{4}, -\frac{1}{2}; 1, \frac{1}{2}; y \right),$$  \hspace{1cm} (4.36)$$

$$a_D = -i \frac{\sqrt{2}}{2} (-1)^{3/4} \Lambda y^{-1/2} \left[ \frac{6 \ln 4 - i\pi - 4}{2\pi} F_2 \left( \frac{1}{4}, -\frac{1}{2}; 1, \frac{1}{2}; y \right) \right],$$  \hspace{1cm} (4.37)$$

As $N_f = 1$ theory, after the use of the analytic continuation from the weak coupling region, we obtain $a$ and $a_D$ around conformal point $u = 3\Lambda^2/8$ as follows;

$$a = -\frac{\sqrt{2}}{2} (-1)^{3/4} \Lambda y^{-1/2} \left[ \frac{4}{3} \Gamma \left( \frac{1}{4} \right)^2 (y) - \frac{1}{4} \right] F_2 \left( \frac{1}{4}, 1, \frac{3}{4}, \frac{1}{2}; 2, \frac{1}{2}; y \right)$$

$$+ \frac{4}{5} \Gamma \left( \frac{1}{4} \right)^2 (y) - \frac{3}{4} \right] F_2 \left( \frac{3}{4}, 3, 5, 3, 9, 1; 2, \frac{1}{2}; y \right),$$  \hspace{1cm} (4.38)$$

$$a_D = -i \frac{\sqrt{2}}{2} (-1)^{3/4} \Lambda y^{-1/2} \left[ \frac{4}{3} \Gamma \left( \frac{1}{4} \right)^2 (y) - \frac{1}{4} \right] F_2 \left( \frac{1}{4}, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}; 2, \frac{1}{2}; y \right)$$

$$- \frac{4}{5} \Gamma \left( \frac{1}{4} \right)^2 (y) - \frac{3}{4} \right] F_2 \left( \frac{3}{4}, 3, 5, 3, 9, 1; \frac{1}{2}, \frac{1}{2}; y \right).$$  \hspace{1cm} (4.39)$$

### 4.3 $N_f = 3$ theory

As in $N_f = 1$ theory we read $\Delta$ and $D$ from the curve although they are much more complicated because of many bare mass parameter $m_i$. After substituting these to (3.18) and (3.19) and using similar manner as $N_f = 1, 2$ case, we get the prepotential in the weak coupling region as

$$F(\tilde{a}) = \frac{ia^2}{\pi} \left[ \frac{1}{4} \ln \left( \frac{\tilde{a}}{\Lambda} \right)^2 - \frac{1}{4} (9 \ln 2 - 2 - \pi i) - \frac{\sqrt{2\pi}}{4i\tilde{a}} \sum_{\bar{i}=1}^{3} n_i' m_i \right]$$

$$- \frac{1}{4a^2} \ln \left( \frac{\tilde{a}}{\Lambda} \right) \sum_{i=1}^{3} m_i^2 + \sum_{i=2}^{\infty} F_i \tilde{a}^{-2i}.$$  \hspace{1cm} (4.40)$$

These $F_i$ agree the result up to known orders [12].

Let us consider massless case where $\Delta$ and $D$ are given by

$$\Delta = -\Lambda^2 u^4 (-\Lambda^2 + 256u), \hspace{1cm} D = \frac{-\Lambda^4 + 256\Lambda^2 u - 4096u^2}{256},$$

$$z'' = \frac{27(256)^3 \Lambda^2 u^4 (\Lambda^2 - 256u)}{4(\Lambda^4 - 256\Lambda^2 u + 4096)^2}.$$  \hspace{1cm} (4.41)$$
We set $y = \Lambda^2/256u$ and $w = 4y(1 - y)$, then $z'' = 27w/(4w - 1)^3$, so we use the other cubic transformaion (3.14) and the quadratic transformation (3.2) subsequently, we get the expression for the massless case,

\[
\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} \left( \frac{1}{2} \right) \frac{1}{2\sqrt{u}} F \left( \frac{1}{2} \cdot \frac{5}{6}, \frac{1}{2}; 1; y \right),
\]

\[
\frac{\partial a_D}{\partial u} = i \frac{\sqrt{2}}{2} \left( \frac{1}{2} \right) \frac{1}{2\sqrt{u}} \left[ \left( 2\ln 4 - i\pi \right) \frac{2\pi}{2\pi} F \left( \frac{1}{2} \cdot \frac{5}{6}, \frac{1}{2}; 1; y \right) - \frac{1}{2\pi} F^* \left( \frac{1}{2} \cdot \frac{5}{6}, \frac{1}{2}; 1, y \right) \right].
\] (4.42)

Integrate with respect to $u$, we can recover the previous result for $a$ and $a_D$ [10]. Expression in the strong coupling region can be obtained quite similarly.

As the example of the theory which has the conformal points, we treat two cases where $\Delta$ become factorized multiple; one is the theory with $m_1 = m_2 = m_3 = \Lambda/8$[14, 15] and another one is $m_1 = m_2 = 0$, $m_3 = \Lambda/16$ case. Of course other possibilities exist but we will not consider these possibilities for simplicity.

In $m_1 = m_2 = m_3 = \Lambda/8$ case, $\Delta = 0$ has a 4-fold root as

\[
\Delta = -\frac{\Lambda^2}{2^6} (32u - \Lambda^2)^4 (256u + 19\Lambda^2), \quad D = -\frac{(32u - \Lambda^2)^2}{64},
\] (4.43)

\[
z'' = -\frac{27\Lambda^2 (256u + 19\Lambda^2)}{16 (32u - \Lambda^2)^2}.
\] (4.44)

If we take

\[
y = \frac{-27\Lambda^2}{256u - 8\Lambda^2},
\] (4.45)

then $z'' = 4y(1 - y)$. Using the quadratic transformation (3.2) , we get $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$

\[
\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} \left( -\frac{27\Lambda^2}{256} \right)^{1/2} y^{1/2} F \left( \frac{1}{6}, \frac{5}{6}, 1; 1, y \right),
\] (4.46)

\[
\frac{\partial a_D}{\partial u} = i \frac{\sqrt{2}}{2} \left( -\frac{27\Lambda^2}{256} \right)^{1/2} y^{1/2} \left[ \left( 3\ln 3 + 2\ln 4 - i\pi \right) \frac{2\pi}{2\pi} F \left( \frac{1}{6}, \frac{5}{6}, 1; 1, y \right) - \frac{1}{2\pi} F^* \left( \frac{1}{6}, \frac{5}{6}, \frac{1}{2}, 1, y \right) \right].
\] (4.47)

Integrating with respect to $u$, we get $a$ and $a_D$ in the weak coupling region as

\[
a = -\frac{\sqrt{2}}{2} \left( -\frac{27\Lambda^2}{256} \right)^{1/2} y^{-1/3} F_2 \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{2}; 1, y \right),
\] (4.48)
\[ a_D = -i \frac{\sqrt{2}}{2} \left( -27\Lambda^2 \right)^{1/2} y^{-1/2} \left[ \frac{3}{2} \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{5}{6} \right)} (-y)^{-1/6} 3F_2 \left( \frac{1}{6}, \frac{3}{6}, \frac{1}{3}; \frac{1}{2}, \frac{1}{2}; y \right) \right. \]
\[ \left. - \frac{1}{2\pi} F^* \left( \frac{1}{2} \frac{3}{4}, \frac{1}{2}; y \right) \right]. \]

By using the analytic continuation from the weak coupling region, we obtain the expression around the conformal point \( u = \Lambda^2/32 \) as follows;

\[ a = -\frac{\sqrt{2}}{2} \left( -27\Lambda^2 \right)^{1/2} y^{-1/2} \left[ \frac{3}{2} \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{5}{6} \right)} (-y)^{-1/6} 3F_2 \left( \frac{1}{6}, \frac{3}{6}, \frac{1}{3}; \frac{1}{2}, \frac{1}{2}; y \right) \right. \]
\[ \left. + \frac{3}{4} \frac{\Gamma \left( -\frac{2}{3} \right)}{\Gamma \left( \frac{1}{6} \right) \Gamma \left( \frac{1}{6} \right)} (-y)^{-5/6} 3F_2 \left( \frac{5}{6}, \frac{5}{6}, \frac{7}{3}; \frac{5}{3}, \frac{5}{3}; y \right) \right], \]
\[ a_D = -i \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} \left( -27\Lambda^2 \right)^{1/2} y^{-1/2} \left[ \frac{3}{2} \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{5}{6} \right)} (-y)^{-1/6} 3F_2 \left( \frac{1}{6}, \frac{3}{6}, \frac{1}{3}; \frac{1}{2}, \frac{1}{2}; y \right) \right. \]
\[ \left. - \frac{3}{4} \frac{\Gamma \left( -\frac{2}{3} \right)}{\Gamma \left( \frac{1}{6} \right) \Gamma \left( \frac{1}{6} \right)} (-y)^{-5/6} 3F_2 \left( \frac{5}{6}, \frac{5}{6}, \frac{7}{3}; \frac{5}{3}, \frac{5}{3}; y \right) \right]. \]

Next we consider \( m_1 = m_2 = 0, m_3 = \Lambda/16 \) case. In this case \( \Delta = 0 \) has one triple root and one double root as

\[ \Delta = \frac{\Lambda^2}{227} (\Lambda^2 - 128u)(\Lambda^2 + 128u)^2, \quad D = -\frac{(7\Lambda^2 - 128u)(\Lambda^2 - 128u)}{1024}, \]
\[ z'' = \frac{54\Lambda^2 (\Lambda^2 + 128u)^2}{(7\Lambda^2 - 128u)^3}. \]

If we take

\[ w = \frac{2\Lambda^2}{128u + \Lambda^2}, \]

then \( z'' = 27w/(4w - 1)^3 \). So we use the cubic transformation (3.14), and use the identity (4.12), we obtain \( \frac{\partial a}{\partial u} \) and \( \frac{\partial a_D}{\partial u} \)

\[ \frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2} \left( -\frac{\Lambda^2}{64} \right)^{1/2} y^{1/2} F \left( \frac{1}{4}, \frac{3}{4}, 1, y \right) \]
\[ \frac{\partial a_D}{\partial u} = -i \frac{\sqrt{2}}{2} \left( -\frac{\Lambda^2}{64} \right)^{1/2} y^{1/2} \left[ \frac{3\ln 4 - i\pi}{2\pi} F \left( \frac{1}{4}, \frac{3}{4}, 1, y \right) \right. \]
\[ \left. - \frac{1}{2\pi} F^* \left( \frac{1}{2} \frac{3}{4}, 1, y \right) \right], \]
where
\[ y = \frac{2\Lambda^2}{-128u + \Lambda^2}. \] (4.56)

Integrate with respect to \( u \), we obtain \( a \) and \( a_D \) in the weak coupling region as
\[
a = -\frac{\sqrt{2}}{2} \left( -\frac{\Lambda^2}{64} \right)^{\frac{1}{2}} y^{-\frac{1}{2}} {}_3F_2 \left( \frac{1}{4}, \frac{3}{4}, -\frac{1}{2}; 1, \frac{1}{2}; y \right),
\] (4.57)
\[
a_D = -i \frac{\sqrt{2}}{2} \left( -\frac{\Lambda^2}{64} \right)^{\frac{1}{2}} y^{-\frac{1}{2}} \left[ \frac{(3 \ln 4 - i\pi - 4)}{2\pi} {}_3F_2 \left( \frac{1}{4}, \frac{3}{4}, -\frac{1}{2}; 1, \frac{1}{2}; y \right) \right. \\
\left. - \frac{1}{2\pi} {}_3F_2^* \left( \frac{1}{4}, \frac{3}{4}, -\frac{1}{2}; 1, \frac{1}{2}; y \right) \right].
\] (4.58)

Since this expression is the same as \( N_f = 2 \) case (4.36) and (4.37 except the argument \( y \), we obtain the expression around the conformal point \( u = \Lambda^2/128 \) by replacing the argument of (4.32) and (4.34) to \( y = 2\Lambda^2/(-128u + \Lambda^2) \).

## 5 Summary

We have derived a formula for the periods of \( N = 2 \) supersymmetric \( SU(2) \) Yang-Mills theory with massive hypermultiplets both in the weak coupling region and in the strong coupling region by using the identities of the hypergeometric functions. We also show how to deal with the theories with conformal points by using the formula.

The approach to evaluate the integral is useful when Picard-Fuchs equation is not solved by any special functions. Similar situation occurs when we consider the theories having higher rank gauge groups. In these case, we no longer expect that the similar transformations exist. We should know how to evaluate the dual pair of fields by another method, which will be reported in a separate paper[16].
References


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