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EIGENVALUE INEQUALITIES AND POINCARÉ DUALITY IN NONCOMMUTATIVE GEOMETRY

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Juillet 1996

IHES/M/96/51

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ABSTRACT. In the context of Connes' noncommutative geometry, eigenvalue inequalities of the type discovered by Vafa and Witten are shown to be a characteristic feature of those spectral geometric spaces of finite topological type that satisfy Poincaré duality in K -theory.

§1. Introduction

In a much noted paper [VW], Vafa and Witten have produced, by a surprising argument of a topological nature, uniform upper bounds for the first eigenvalues of the twisted Dirac operators on a compact Riemannian spin manifold, formed by coupling the fiducial Dirac operator $\mathcal{D}$ to an arbitrary background gauge potential A . Furthermore, for odd-dimensional manifolds they proved the existence of a constant $C > 0$, independent of the potential, such that every interval of length C contains at least one eigenvalue of $\mathcal{D}_A$. Their proof relies on ingenious and remarkably elementary manipulations with the index theorems of Atiyah-Singer [AS] and of Atiyah-Patodi-Singer [APS].

Our aim here is to point out that the explicit form of the index theorems is not essential and that, ultimately, the Vafa-Witten inequalities are a manifestation of Poincaré duality in K -theory. As such, they continue to hold in the full generality of Connes' noncommutative geometry (Theorem 1), for spectral geometric spaces of finite topological type that satisfy Poincaré duality in the sense of $[C_1, C_2]$. In particular, they hold for noncommutative spectral manifolds $[C_3]$, such as the noncommutative tori or the spectral triple describing

* J.S. Guggenheim Fellow.

Research supported in part by the U.S. National Science Foundation and the U.S.-Israel Binational Science Foundation.

the standard model of particle physics $[C^2]$, as well as for ordinary (topological) manifolds which may not be smooth but only Lipschitz. More remarkably, they remain true even when the growth of the eigenvalues is non-polynomial (Theorem 2), e.g. for duals of discrete, cocompact subgroups of $SO(n, 1)$ or $SU(n, 1)$ (Corollary 3).

§2. The general framework

In Connes' noncommutative geometry, a (compact) geometric space is given by a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive algebra with unit, represented in the Hilbert space $\mathcal{H}$ by bounded operators, and $D = D^*$ is an unbounded selfadjoint operator in $\mathcal{H}$ so that

- (1) the resolvent $(D + i)^{-1}$ is compact;
- (2) the commutators $[D, a] = Da - aD$ are bounded, for any $a \in \mathcal{A}$.

Such a triple is *even* if it is further equipped with a $\mathbb{Z}_2$ -grading $\gamma \in \mathcal{L}(\mathcal{H})$, $\gamma = \gamma^*$, $\gamma^2 = 1$, so that

- (3) $\forall a \in \mathcal{A}$, $\gamma a = a\gamma$, while $D\gamma = -\gamma D$.

Otherwise, the triple is *odd*.

The algebra $\mathcal{A}$ plays the role of the algebra coordinates on the “space” X , which in the commutative case is the spectrum of the C^* -algebra $\overline{\mathcal{A}}$, obtained as the norm closure of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$. In general, $\mathcal{A}$ may not be commutative, however we shall assume that

- (4) if $a \in \mathcal{A}$ is invertible in $\overline{\mathcal{A}}$, then $a^{-1} \in \mathcal{A}$.

This has as effect that the K -groups of $\mathcal{A}$ and $\overline{\mathcal{A}}$ coincide:

$$K_i(\mathcal{A}) = K_i(\overline{\mathcal{A}}) \quad , \quad i = 0, 1.$$

Let us recall that $K_0(\mathcal{A})$ classifies finite (i.e. finitely generated) projective modules over $\mathcal{A}$, or equivalently idempotents in $M_\infty(\mathcal{A})$, while $K_1(\mathcal{A}) = \pi_0 GL_\infty(\mathcal{A})$ is the group of connected components of $GL_\infty(\mathcal{A})$. In view of Bott periodicity, one has

$$\pi_n(GL_\infty(\mathcal{A})) \cong K_{n+1}(\mathcal{A}),$$

where n only matters mod $\mathbb{Z}/2\mathbb{Z}$.

We next recall that the datum of D as above, defines an additive *index map*

$$\text{Ind}_D : K_n(\mathcal{A}) \rightarrow \mathbb{Z}$$

as follows.

In the even case, one uses the grading γ to decompose $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, $D = D^+ \oplus D^-$, where

$$D^+ = \frac{1-\gamma}{2} D \frac{1+\gamma}{2} \quad , \quad \mathcal{H}^\pm = \frac{1 \pm \gamma}{2} \mathcal{H};$$

for any selfadjoint idempotent $e^* = e^2 = e \in M_q(\mathcal{A})$, the operator

$$e(D^+ \otimes I_q)e : e(\mathcal{H}^+ \otimes \mathbb{C}^q) \rightarrow e(\mathcal{H}^- \otimes \mathbb{C}^q),$$

is Fredholm and one defines

$$(5_{\text{ev}}) \quad \text{Ind}_D(e) = \text{Index } e(D^+ \otimes I_q)e.$$

In the odd case, given $U \in GL_q(\mathcal{A})$, $U^*U = UU^* = I_q$, one lets, similarly,

$$(5_{\text{odd}}) \quad \text{Ind}_D(U) = \text{Index}(P \otimes I_q \cdot U \cdot P \otimes I_q)$$

where $P = \frac{1+F}{2}$, $F = \text{Sign}(D)$.

It is important to remark at this point that

$$(5'_{\text{odd}}) \quad \text{Ind}_D(U) = Sf(D_t),$$

where D_t is the periodic one-parameter family of selfadjoint Fredholm operators

$$D_t = D \otimes I_q + tU[D \otimes I_q, U^*] \quad , \quad t \in [0, 1],$$

and the *spectral flow* $Sf(D_t)$ counts the net number of eigenvalues which cross the origin as the family turns once around the circle.

Note that, in view of assumption (4), we could afford to employ only selfadjoint elements in defining Ind_D .

A more geometric account of the preceding discussion can be given in terms of (Hermitian) finite projective modules over $\mathcal{A}$, which correspond to (Hermitian) vector bundles over the ghost space X .

To this end, we recall that, given a finite projective (right-)module $\mathcal{E}$ over $\mathcal{A}$, a *Hermitian structure* on $\mathcal{E}$ is a sesquilinear map

$$(\ , \) : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$$

satisfying the following conditions:

- (i) $(\xi a, \eta b) = a^*(\xi, \eta) b$, $\forall \xi, \eta \in \mathcal{E}$ and $\forall a, b \in \mathcal{A}$;
- (ii) $(\xi, \xi) \geq 0$;
- (iii) $\mathcal{E}$ is self-dual with respect to $(\ , \)$.

If we write $\mathcal{E}$ as a direct summand $\mathcal{E} = e \mathcal{A}^q$ of a free module $\mathcal{E}_0 = \mathcal{A}^q$, with $e^2 = e = e^* \in M_q(\mathcal{A})$, any Hermitian structure on $\mathcal{E}$ is isomorphic to the one obtained by restricting to $\mathcal{E}$ the Hermitian structure on $\mathcal{E}_0$ given by

$$(6) \quad (\xi, \eta) = \sum_1^q \xi_j^* \eta_j \quad , \quad \forall \xi = (\xi_j) \ , \ \eta = (\eta_j) \in \mathcal{E}_0 .$$

Before recalling the notion of connection (cf. [C₁]), we need to introduce the $\mathcal{A}$ -bimodule of bounded operators on $\mathcal{H}$

$$\Omega_D^1 \equiv \Omega_D^1(\mathcal{A}) = \left\{ \sum a_j [D, b_j] ; a_j, b_j \in \mathcal{A} \right\} ,$$

which plays the role of the 1-forms on X . Now a **connection** on the Hermitian finite projective module $\mathcal{E}$ is a $\mathbb{C}$ -linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ such that

- (iv) $\nabla(\xi a) = (\nabla \xi) a + \xi \otimes da$, $\forall \xi \in \mathcal{E}$, $a \in \mathcal{A}$,
- (v) $(\xi, \nabla \eta) - (\nabla \xi, \eta) = d(\xi, \eta)$, $\forall \xi, \eta \in \mathcal{E}$;

here $da = [D, a]$, $\forall a \in \mathcal{A}$, and if

$$\nabla \xi = \sum \xi_j \otimes \omega_j \quad , \quad \text{with } \xi_j \in \mathcal{E} \ , \ \omega_j \in \Omega_D^1$$

then

$$(\nabla \xi, \eta) = \sum \omega_j^*(\xi_j, \eta) .$$

An example of such is the **Grassmannian connection** ∇_0 on $\mathcal{E} = e \cdot \mathcal{A}^q$, given by

$$\nabla_0 \xi = e \eta \quad , \quad \text{where } (\eta_j) = (d \xi_j) .$$

Note also that any two connections only differ by a selfadjoint element of $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1)$.

Given a Hermitian finite projective module $\mathcal{E}$ over $\mathcal{A}$, we can form the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$, by completing the algebraic tensor product with respect to the inner product

- (vi) $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, (\xi_1, \xi_2) \eta_2 \rangle$, $\forall \xi_j \in \mathcal{E}$, $\eta_j \in \mathcal{H}$.

If, in addition, $\mathcal{E}$ is equipped with a connection ∇ , then one can define a twisted operator $D_{\mathcal{E}, \nabla}$ on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ by setting

$$(vii) \quad D_{\mathcal{E}, \nabla}(\xi \otimes \eta) = \xi \otimes D\eta + (\nabla\xi)\eta, \quad \forall \xi \in \mathcal{E}, \eta \in \mathcal{H};$$

here, if $\nabla\xi = \sum \xi_j \otimes \omega_j$, then

$$(\nabla\xi)\eta = \sum \xi_j \otimes \omega_j(\eta) \in \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}.$$

It is straightforward to check that $D_{\mathcal{E}, \nabla}$ is selfadjoint and Fredholm. Indeed, for the Grassmannian connection one has

$$D_{\mathcal{E}, \nabla_0} = e(D \otimes I_q)e.$$

It then suffices to note that a (selfadjoint) element $T \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1)$ acts as a (selfadjoint) bounded operator on the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ via the composition

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \xrightarrow{T \otimes I} \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1 \otimes_{\mathcal{A}} \mathcal{H} \xrightarrow{I \otimes \pi} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H},$$

with

$$\pi(\omega \otimes \eta) = \omega(\eta) \quad , \quad \omega \in \Omega_D^1, \eta \in \mathcal{H}.$$

In this picture, the additive map $\text{Ind}_D : K_n(\mathcal{A}) \rightarrow \mathbb{Z}$ can be described as follows: in the even case

$$(6_{\text{ev}}) \quad \text{Ind}_D(e) = \text{Index } D_{\mathcal{E}, \nabla}^+,$$

where $\mathcal{E} = e\mathcal{A}^q$, equipped with a Hermitian structure and a connection ∇ ; in the odd case

$$(6_{\text{odd}}) \quad \text{Ind}_D(U) = Sf(D_t),$$

where the right hand side refers to the spectral flow of the one-parameter family

$$D_t = D_{\mathcal{E}_0, \nabla} + tU[D_{\mathcal{E}_0, \nabla}, U^*] \quad , \quad t \in [0, 1].$$

Here U is an unitary in $GL_q(\mathcal{A})$ and ∇ an arbitrary connection on $\mathcal{E}_0 = \mathcal{A}^q$ compatible with a given Hermitian structure. In the case when $\nabla = \nabla_0$ is the (trivial) Grassmannian connection in $\mathcal{E}_0$, the above family coincides with the one appearing in $(5'_{\text{odd}})$.

§3. Poincaré duality in K -theory

The formulation of Poincaré duality in noncommutative geometry involves, in general, a pair of two algebras. In its fullest expression, it also involves Kasparov's KK -bifunctor $[K_1]$. However, for the purposes of this paper, the appropriate version is a weaker, *rational* form of Poincaré duality in K -theory, which we proceed now to state following Connes $[C_1]$.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple describing the geometry of a noncommutative space X . Assume we are given another spectral triple $(\mathcal{B}, \mathcal{H}, D)$ such that:

- (α) $\mathcal{A}$ and $\mathcal{B}$ commute in $\mathcal{L}(\mathcal{H})$;
- (β) $[D, a]$ and b commute, for any $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Then $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D)$ forms again a spectral triple, and so we can consider the index map

$$\text{Ind}_D : K_n(\mathcal{A} \otimes \mathcal{B}) \rightarrow \mathbb{Z}.$$

Composing it with the natural biadditive map

$$K_i(\mathcal{A}) \otimes K_j(\mathcal{B}) \rightarrow K_{i+j}(\mathcal{A} \otimes \mathcal{B}),$$

one obtains a biadditive form

$$(\ , \)_D : K_i(\mathcal{A}) \times K_j(\mathcal{B}) \rightarrow \mathbb{Z}.$$

We shall say that $(\mathcal{A}, \mathcal{H}, D)$ and $(\mathcal{B}, \mathcal{H}, D)$ are in *rational Poincaré duality* if the form $(\ , \)_D$ is nondegenerate.

The prototypical example of this duality arises in the case of a (compact) Riemannian manifold M , when $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(M, \Lambda_{\mathbb{C}}^* T^*M)$ and D is the signature operator. Then (cf. $[C_1, \text{VI.4.}\beta]$), $(\mathcal{A}, \mathcal{H}, D)$ is in Poincaré duality with $(\mathcal{B}, \mathcal{H}, D)$, where $\mathcal{B}$ is the algebra of C^∞ -sections of the Clifford bundle $\text{Cliff}_{\mathbb{C}}(T^*M)$. If M is in addition a spin manifold, replacing the signature operator by the Dirac operator $\mathcal{D}$, which acts on $\mathcal{H} = L^2(M, S)$ with S denoting the spin bundle, one obtains a *self dual* spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

A simple noncommutative example is that of the noncommutative 2-torus $[C_1, \text{loc.cit}]$ $(\mathcal{A}_\theta, \mathcal{H}, D)$, $\theta \in \mathbb{R}/\mathbb{Z}$, which is in Poincaré duality with $(\mathcal{A}_\theta^0, \mathcal{H}, D)$, where $\mathcal{A}_\theta^0$ denotes the opposite algebra of $\mathcal{A}_\theta$. This is in fact a special case of the self-duality which arises in the presence of a *real structure* (cf. $[C_2]$).

§4. Eigenvalue Inequalities

We now fix a spectral space $(\mathcal{A}, \mathcal{H}, D)$, assumed to admit a rational Poincaré dual $(\mathcal{B}, \mathcal{H}, D)$. To state the main result, we need one more definition: the underlying noncommutative space is said to be of *finite topological type*, if

$$\dim K_*(\mathcal{A}) \otimes \mathbb{Q} = \dim K_*(\mathcal{B}) \otimes \mathbb{Q} < \infty.$$

THEOREM 1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral space of finite topological type, which admits a rational Poincaré dual $(\mathcal{B}, \mathcal{H}, D)$.*

1⁰. There exists a constant $C < \infty$ so that, if $\mathcal{E}$ is any Hermitian finite projective module over $\mathcal{A}$ with connection ∇ and if

$$|\lambda_1(\mathcal{E}, \nabla)| \leq |\lambda_2(\mathcal{E}, \nabla)| \leq \dots$$

denote the eigenvalues of the twisted operator $D_{\mathcal{E}, \nabla}$ indexed in the order of ascending absolute value, then

$$|\lambda_1(\mathcal{E}, \nabla)| \leq C.$$

2⁰. In the odd case, there also exists a constant $L > 0$ so that every interval of length L contains an eigenvalue of $D_{\mathcal{E}, \nabla}$, for any $(\mathcal{E}, \nabla)$ as above.

Proof. The proof will follow closely the original arguments of Vafa and Witten (cf. [VW], also [A]), with one exception. Instead of pulling back the Bott class from spheres, procedure which is not available in this generality, we shall make use of the finite topological type assumption and work with a fixed basis of $K_*(\mathcal{B}) \otimes \mathbb{Q}$.

A. We start with the proof in the even case. Once and for all, we fix a basis $\{\beta_1, \dots, \beta_r\}$ of $K_0(\mathcal{B}) \otimes \mathbb{Q}$. Each β_j , $j = 1, \dots, r$, is of the form

$$\beta_j = [\mathcal{F}_j] - [\mathcal{A}^{q_j}].$$

Given $(\mathcal{E}, \nabla)$ as in 1⁰, by virtue of the Poincaré duality hypothesis, there exists $\beta \in \{\beta_1, \dots, \beta_r\}$, $\beta = [\mathcal{F}'] - [\mathcal{B}^q]$, so that

$$(1) \quad ([\mathcal{E}], \beta)_D \neq 0.$$

We may assume

$$(2) \quad \text{Index } D_{\mathcal{E}, \nabla}^+ = 0,$$

the statement being otherwise trivially satisfied. Together with (1), this implies

$$(3) \quad ([\mathcal{E}], [\mathcal{F}'])_D \neq 0.$$

Let us write $\mathcal{F}'$ as $f\mathcal{B}^N$, with $f^2 = f = f^* \in M_N(\mathcal{B})$ and fix a connection ∇' , e.g. the Grassmannian connection, on $\mathcal{F}'$.

By the very definitions of the index map (§2) and of the intersection form (§3), we can express $([\mathcal{E}], [\mathcal{F}'])_D$ as the index of the operator $D_{\mathcal{E} \otimes \mathcal{F}'}^+$ defined as follows.

We view $\mathcal{H}$, which is an $\mathcal{A} \otimes \mathcal{B}$ left module, as an $(\mathcal{A}, \mathcal{B}^0)$ -bimodule, where $\mathcal{B}^0$ is the opposite algebra to $\mathcal{B}$, acting on $\mathcal{H}$ on the right:

$$\eta \cdot b^0 = b\eta \quad , \quad \eta \in \mathcal{H} \quad , \quad b \in \mathcal{B}.$$

Similarly, we regard $\mathcal{F}'$ as a left $\mathcal{B}^0$ -module:

$$b^0 \cdot \zeta' = \zeta' b \quad , \quad \zeta' \in \mathcal{F}' \quad , \quad b \in \mathcal{B}.$$

We can now form the Hilbert space

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{B}^0} \mathcal{F}'$$

as in §2, and then define the selfadjoint operator $D_{\mathcal{E} \otimes \mathcal{F}'}$ as follows:

$$(4) \quad D_{\mathcal{E} \otimes \mathcal{F}'}(\xi \otimes \eta \otimes \zeta') = \nabla \xi \otimes \eta \otimes \zeta' + \xi \otimes D\eta \otimes \zeta' + \xi \otimes \eta \otimes \nabla' \zeta'.$$

Then

$$([\mathcal{E}], [\mathcal{F}'])_D = \text{Index } D_{\mathcal{E} \otimes \mathcal{F}'}^+,$$

and because of (3) one has

$$(5) \quad \text{Ker } D_{\mathcal{E} \otimes \mathcal{F}'} \neq 0.$$

Consider now the complementary module

$$\mathcal{F}'' = (1 - f)\mathcal{B}^N,$$

equipped with its (Grassmannian) Hermitian structure and connection ∇'' , and let $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}'' \cong \mathcal{B}^N$ denote their direct sum equipped with the direct sum Hermitian structure and connection $\nabla' \oplus \nabla''$. We can form

$$D_{\mathcal{E} \otimes \mathcal{F}} = D_{\mathcal{E} \otimes \mathcal{F}'} \oplus D_{\mathcal{E} \otimes \mathcal{F}''}$$

using $\nabla' \oplus \nabla''$, and also

$$D_{\mathcal{E} \otimes \mathcal{F}_0} \cong \underbrace{D_{\mathcal{E}, \nabla} \oplus \cdots \oplus D_{\mathcal{E}, \nabla}}_{N\text{-times}}$$

using the trivial connection ∇_0 on $\mathcal{F}_0 = \mathcal{B}^N$. From (4) it is obvious that the difference

$$B = D_{\mathcal{E} \otimes \mathcal{F}} - D_{\mathcal{E} \otimes \mathcal{F}_0}$$

depends only on the difference of connections on $\mathcal{F} \cong \mathcal{F}_0$:

$$T_0 = \nabla' \oplus \nabla'' - \nabla_0 \in \Omega_D^1(\mathcal{B})^N,$$

which is bounded and independent of $(\mathcal{E}, \nabla)$.

To complete the proof of the even case, it remains to notice that (5) implies that

$$\text{Ker } D_{\mathcal{E} \otimes \mathcal{F}} \neq 0,$$

while $D_{\mathcal{E} \otimes \mathcal{F}_0}$ has the same eigenvalues as $D_{\mathcal{E}, \nabla}$.

B. Passing now to the odd case, we fix a basis $\{[U_1], \dots, [U_r]\}$ of $K_1(\mathcal{B}) \otimes \mathbb{Q}$ and note that

$$(6) \quad ([\mathcal{E}], [U])_D \neq 0$$

for some unitary $U \in \{U_1, \dots, U_r\}$, $U^*U = UU^* = I$, $U \in GL_N(\mathcal{B})$.

Again appealing to the definitions of the index map and intersection form, $([\mathcal{E}], [U])_D$ can be expressed as the spectral flow of a one-parameter family of selfadjoint operators, as follows. On the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{B}^0} \mathcal{F}_0$, where $\mathcal{F}_0 = \mathcal{B}^N$ equipped with the Grassmannian connection ∇_0 , we consider the operators

$$D_0 = D_{\mathcal{E} \otimes \mathcal{F}_0} \simeq N D_{\mathcal{E}, \nabla}$$

and

$$D_1 = (I \otimes U) \cdot D_0 \cdot (I \otimes U^*).$$

Then $([\mathcal{E}], [U])_D$ is the spectral flow of the family

$$(7) \quad D_t = (1-t) D_0 + t D_1, \quad t \in [0, 1].$$

Because of (6), there must be some $t_0 \in [0, 1]$ such that D_{t_0} has a zero-eigenvalue.

But $D_{t_0} - D_0 = t_0 B$, with $B = D_1 - D_0$ involving only the bounded operator $U[D \otimes I_N, U^*]$.

This proves 1^0 in the odd case, and therefore achieves the proof of the first part of the theorem.

C. To prove 2^0 it suffices to note that the fact that the family (7) has a nonzero spectral flow actually implies that for *any* $\lambda \in \mathbb{R}$, there is a $t \in [0, 1]$ so that D_t has eigenvalue λ . Thus, D_0 must have an eigenvalue within $\|B\|$ of λ . $\square$

Remark. The statement is symmetric with respect to the algebras $\mathcal{A}$ and $\mathcal{B}$.

§5. Non-smooth and non-polynomial growth examples

The first comment we want to make is to emphasize the topological nature of the result itself, not just of its proof. This is put in evidence by the fact that the original proof of Vafa-Witten applies *verbatim* to the signature operator D on a Lipschitz manifold M , if instead of the Atiyah-Singer index theorem one uses its extension to Lipschitz manifolds (cf. [T]). In particular, the uniform estimates for the higher eigenvalues

$$(1) \quad |\lambda_N(\mathcal{E}, \nabla)| \leq C N^{1/n} \quad , \quad n = \dim M \quad ,$$

with $C < \infty$ independent on the (Lipschitz) Hermitian bundle with connection $(\mathcal{E}, \nabla)$, still hold.

Incidentally, the Lipschitz manifold example can be easily fitted in the noncommutative framework discussed above. Thus, one can take $\mathcal{A}$ as being the algebra of Lipschitz functions on M , $\mathcal{H}$ as being the Hilbert space of L^2 -forms on M and D the signature operator, as in [H]. As in the smooth case $[C_1]$, the commutant of the algebra generated by $\mathcal{A}$ and $[D, a]$, with $a \in \mathcal{A}$, is canonically isomorphic to the algebra of bounded measurable forms on M , equipped with the *Clifford multiplication*. Taking as $\mathcal{B}$ the subalgebra of Lipschitz forms (relative to the same Clifford multiplication), one can check with the help of the results in [H] that $(\mathcal{A}, \mathcal{H}, D)$ and $(\mathcal{B}, \mathcal{H}, D)$ are in Poincaré duality.

The second and more substantive remark is that, unlike the higher eigenvalue estimates (1) which *a fortiori* imply finite-dimensionality, the properties stated in Theorem 1 *do not* represent a strictly finite-dimensional phenomenon. To illustrate this, we shall produce below a class of non-polynomial growth, θ -summable, noncommutative spaces (duals of discrete countable groups) which satisfy the hypotheses of Theorem 1.

We begin by recalling Connes' construction of a geometric structure on a dual of a discrete subgroup Γ of a Lie group G (see [C₁, IV.9.α]).

To simplify the discussion, we shall assume that G is connected and semisimple and that Γ is torsion-free and cocompact. In particular $B\Gamma$ can be identified with the symmetric space $X = G/K$, where K is the maximal compact subgroup of G . We shall equip X with the canonical G -invariant Riemannian metric.

Let $\mathcal{H}$ be the Hilbert space of L^2 -forms on X , on which Γ acts by left translations. Consider the Morse function 2φ , given by the square of the geodesic distance to the base point $\{K\} \in X$, then define (cf. [W])

$$D_\tau = d_\tau + d_\tau^*,$$

where $d_\tau = e^{-\tau\varphi} de^{\tau\varphi}$, with $\tau \neq 0$. We let $\mathcal{A}$ be the closure under the holomorphic functional calculus of the group algebra $C\Gamma$ in the reduced C^* -algebra $C_r^*(\Gamma)$, and $\mathcal{B} = C^\infty(\Gamma \backslash X)$, acting by multiplication operators on $\mathcal{H}$.

At this point, we need to recall that there is a natural *analytic assembly map* (cf. [C₁, Chapter II])

$$\mu_r^\Gamma : K_i(B\Gamma) \rightarrow K_i(C_r^*(\Gamma)),$$

which was conjectured by Baum and Connes to be an isomorphism. This conjecture has already been verified for discrete subgroups of $SO(n, 1)$ (cf. [K₂]) and of $SU(n, 1)$ (cf. [J-K]).

With these preparations, we are ready to state the main result of this section.

THEOREM 2. *Let Γ be a discrete, torsion-free, cocompact subgroup of a connected, noncompact, semisimple Lie group G . Assume that the assembly map*

$$\mu_r^\Gamma : K_i(B\Gamma) \rightarrow K_i(C_r^*(\Gamma))$$

is a rational isomorphism. Then:

1⁰. *$(\mathcal{A}, \mathcal{H}, D_\tau)$, with the $\mathbb{Z}/2\mathbb{Z}$ -grading γ given by the even/odd grading on forms, is an even spectral triple of finite topological type with rational Poincaré dual $(\mathcal{B}, \mathcal{H}, D_\tau)$;*

2⁰. *D_τ satisfies the θ -summability property*

$$(2) \quad \text{Trace } e^{-\beta D_\tau^2} < \infty \quad , \quad \forall \beta > 0,$$

without being finitely-summable:

$$(3) \quad \text{Trace}(1 + D_\tau^2)^{-p} = \infty \quad , \quad \forall p < \infty .$$

Proof. The fact that $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D_\tau)$ is a θ -summable spectral triple satisfying the order 1 property

$$(4) \quad [[D, a], b] = 0 \quad , \quad \forall a \in \mathcal{A} \quad , \quad b \in \mathcal{B}$$

is the content of Prop. 3 in [C₁, IV.9.α].

Given the assumption of rational isomorphism of the analytic assembly map μ_r^Γ , the nondegeneracy of the pairing

$$(\quad, \quad)_{D_\tau} : K_i(\mathcal{A}) \times K_i(\mathcal{B}) \rightarrow \mathbb{Z}$$

is a consequence of the following index formula of Connes (Theorem 4, [C₁, loc.cit]):

$$(\mu_r^\Gamma(x), y)_{D_\tau} = \langle ch_*(x), ch^*(y) \rangle ,$$

for any K -homology class $x \in K_i(B\Gamma)$ and any K -theory class $y \in K_i(\mathcal{B}) = K^i(\Gamma \backslash X)$.

Finally, since Γ cannot be amenable, the non-finite summability property (3) follows from Theorem 1 in [C₁, loc.cit]. $\square$

We conclude with a consequence which can be stated without any reference to non-commutative geometry.

COROLLARY 3. *Let Γ be a discrete, torsion-free, cocompact subgroup of $G = SO_0(n, 1)$ or $SU(n, 1)$. There exists a constant $C < \infty$ so that if E is any Γ -equivariant Hermitian vector bundle on $X = G/K$ equipped with a Γ -equivariant connection ∇ , and if $|\lambda_1(E, \nabla)| \leq |\lambda_2(E, \nabla)| \leq \dots$ denote the eigenvalues of the twisted operator $D_{\tau, \varepsilon, \nabla}$, then*

$$|\lambda_1(E, \nabla)| \leq C .$$

On the other hand, for any $p < \infty$,

$$\sum_{n=1}^{\infty} |\lambda_n(E, \nabla)|^{-p} = \infty .$$

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