Light-cone quark model predictions for radiative meson decays

H.M. Choi$^{a, b}$ and Chueng-Ryong Ji$^a$

$^a$Department of Physics, North Carolina State University, Raleigh, N.C. 27695-8202, USA
$^b$CEBAF, 12000 Jefferson Avenue, Newport News, Virginia, 23606

Abstract

We investigate the radiative decays of pseudoscalar ($\pi, K, \eta, \eta'$), vector($\rho, K^*, \omega, \phi$) and axial vector ($A_1$) mesons using a simple relativistic constituent quark model. For both simplicity and relativity, we take advantage of the distinguished features in the light-cone quantization method: (1) the Fock-state expansion of meson wavefunctions are not contaminated by the vacuum fluctuation, (2) the problem of assigning quantum numbers $J^{PC}$ to mesons is circumvented by the Melosh transformation. Except the well-known constituent quark masses of ($u,d,s$) quarks and the spin-averaged meson masses, the only parameter in the model is the gaussian parameter $\beta$ which determines the broadness (or sharpness) of radial wavefunction. Our overall predictions of pseudoscalar, vector and axial vector meson radiative decay processes are remarkably in good agreement with the experimental data.
I. INTRODUCTION

While the nonperturbative QCD methods such as QCD sum-rule techniques [1-3] and lattice QCD calculations [4-6] are available, there is still growing interest in using simple relativistic quark models [7-12] to describe hadron properties. Our relativistic quark model [7,8] is based upon the Fock state decomposition of hadronic state which arises naturally in the 'light-cone quantization' of QCD. In this approach, a hadron is characterized by a set of Fock state wave functions, the probability amplitudes for finding different combinations of bare quarks and gluons in the hadron at a given light-cone time \( \tau = t + z/c \).

These wave functions provide the essential link between hadronic phenomena at short distances (perturbative) and at long distances (non-perturbative) [13].

The distinguished features in the light-cone approach are the simplicity of the vacuum except the zero modes and the dynamical property of rotation operators. The vacuum at equal \( \tau \) has dramatic differences compared to the vacuum at equal \( t \). For the particle which has the mass \( m \) and the four-momentum \( k = (k^0, k^i, k^j, k^k) \), the relativistic energy-momentum relation at equal \( \tau \) is given by

\[
k = \frac{\sqrt{k^2 + m^2}}{k^0}
\]

where the light-cone energy conjugate to \( \tau \) is given by \( k^\tau = k^0 - k^3 \) and the light-cone momenta \( k^\tau = k^0 + k^3 \) and \( k_\perp = (k^1, k^2) \) are orthogonal to \( k^\tau \) and form the light-cone three-momentum \( k = (k^\tau, k_\perp) \).

The rational relation given by Eq.(1.1) is drastically different from the irrational energy-momentum relation at equal \( t \) given by

\[
k^0 = \sqrt{k^2 + m^2},
\]

where the energy \( k^0 \) is conjugate to \( t \) and the three-momentum vector \( k \) is given by \( k = (k^i, k^j, k^k) \). The important point here is that signs of \( k^i \) and \( k^j \) are correlated while at equal \( t \) the signs of \( k^3 \) and \( k^0 \) are not correlated. Thus the momentum \( k^0 \) is always positive because only the positive energy \( k^0 \) makes the system evolve to the future direction (i.e., positive \( \tau \)), while the momentum \( k^3 \) can be either positive or negative even though \( k^0 \) is positive to evolve the system in the future direction (i.e., positive \( t \)). This provides a remarkable feature to the light-cone vacuum, namely, the Fock state vacuum is an eigenstate of the full Hamiltonian. Consequently, all bare quanta in a hadronic Fock state are associated with the hadron and none are disconnected elements of the vacuum [13].

Furthermore, the problem of boost operators at equal \( t \) changing particle numbers can be cured by this framework since the quantization surface \( \tau = 0 \) is invariant under both longitudinal and transverse boosts defined at equal \( \tau \). However, the quantization surface \( \tau = 0 \) is not invariant under the transverse rotation whose direction is perpendicular to the direction of the quantization axis \( z \) at equal \( \tau \) [14]. Thus, the transverse angular momentum operator involves the interaction that changes the particle number and it is not easy to specify the total angular momentum of a particular hadronic state. Also \( \tau \) is not invariant under parity [15]. We circumvent these problems by using the Melosh transformation of each constituents from equal \( t \) to equal \( \tau \).

The key approximation in the light-cone approach is the mock-hadron approximation [16] to saturate the Fock state expansion by the constituent quark and anti-quark and treat the saturated constituent Fock state as a free state as far as the spin-orbit part is concerned while the radial part is given by the ground state of the harmonic oscillator wavefunction. Then, the assignment of the quantum numbers such as angular momentum, parity and charge conjugation to the light-cone wavefunctions is given by the Melosh transformation [17]. For example, the meson state \( |M> \) is represented by

\[
|M> = \Psi_M^{QQ}|QQ>,
\]

where \( Q \) and \( \bar{Q} \) are the effective dressed quark and antiquark. The model wavefunction is then given by [7,8]

\[
\Psi_M^{QQ} = \Phi_M(z_i, k_{11}, \lambda_i) = \Phi_M(z_i, k_{11}) \chi_M(z_i, k_{11}, \lambda_i)
\]

where the radial wavefunction is given by

\[
\Phi_M(z_i, k_{11}) = A \exp \left[ -\frac{1}{2} \sum_i k_{1i}^2 + m_i^2/8B^2 \right],
\]

and the spin-orbit wavefunction \( \chi_M(z_i, k_{11}, \lambda_i) \) is obtained by the interaction independent Melosh transformation from the ordinary equal-time static spin-orbit wavefunction assigned by the quantum numbers \( J^{PC} \). These wavefunctions are represented by the Lorentz-invariant variables \( z_i = (p_i^0, p_i^\tau, k_{1i} = p_{1i} - z_i P_\perp, \lambda_i, \) \) where \( p^0 = (P^0, P^\tau, P_\perp) \), \( P_\perp \) is the momentum of the meson \( M \), and \( p_i^0 \) and \( \lambda_i \) are the momentum and the helicity of constituent quarks, respectively. Then, the light-cone spin-orbit wavefunction corresponding to the specific quantum number \( J^{PC} \) is given by

\[
\chi_M(z_i, k_{11}, \lambda_i) = \delta_{1i} \Gamma_{J^PC} \nu_{3i},
\]
where the operators $\Gamma_{\mu\nu}$ are given by [8]

$$\Gamma^{PC} = 0^+; \Gamma_{\mu} = (m_{\mu} + P) \gamma_{\mu},$$  \hspace{1cm} (1.7)
$$1^{-+}; \Gamma_{\mu} = m_{\mu} \gamma_{\mu} + \frac{[P, \gamma_{\mu}]}{2}$$ \hspace{1cm} (1.8)
$$1^{++}; \Gamma_{\mu} = (m_{\mu} + P) \left[ \frac{1}{m_{\mu}} \gamma_{\mu} + \frac{[P, \gamma_{\mu}]}{2} \right] \gamma_{\mu}. \hspace{1cm} (1.9)$$

Here the space components $\Gamma_{\mu}$ of the polarization four-vectors $\epsilon(\mu)$ in the rest frame have the components $\Gamma(\pm) = \mp(1, \pm i, 0)/\sqrt{2}, \Gamma(0) = (0, 0, 1, 1)$.

Using this model wavefunction Eq.(1.4), we have calculated various static properties of $\pi, K, \rho$, and $A_1$ mesons [7,8]. Since our model provided a remarkably good description of static properties for the pion and $K$-mesons and reproduced the basic features of the lattice QCD and the QCD sum rule results for $\pi, K, \rho$, and $A_1$ mesons [7,8], it is of our interest to investigate more observables with the same model. In this paper, we present a comprehensive study of the radiative decays of pseudoscalar($\pi, K, \eta, \eta'$), vector($\rho, K^*, \omega, \phi$) and axial vector($A_1$) mesons.

Both in the past works and the present work, we use the spin averaged mass as argued by Dziembowski [9,10]. Before we perform the Melosh transformation, we assume that the starting ground state mesons are described by the harmonic oscillator wavefunctions. These equal $I$ wavenumbers are known to give a reasonable first approximation of the static properties [18]. Since these starting wavefunctions prior to the Melosh transformation are valid in a scheme without short-range hyperfine interactions, it is reasonable to assume the spin averaged mass [9]. The spin averaged masses of $\pi, K, \rho$, and $A_1$ mesons are given by [7,8] $m_{\pi} = m_{K} = 0.612$ GeV, $m_{K^*} = 0.793$ GeV and $m_{A_1}$ = 1.120 GeV. Since we consider now $\eta, \eta', \omega$ and $\phi$ mesons also in this work, we present the details of how we obtain the spin averaged values of $\eta, \eta', \omega$ and $\phi$ mesons in the Appendix A. The flavor assignment of $\eta$ and $\eta'$ mesons in the quark and antiquark basis are as follows:

$$\eta = X_\eta \sqrt{u+d} Y_{\eta^{\pm}s}, \hspace{1cm} (1.10)$$
$$\eta' = X_{\eta'} \sqrt{u+d} Y_{\eta'^{\pm}s}, \hspace{1cm} (1.11)$$

where $X_\eta = Y_\eta = -\sin \theta_{23}$ and $X_{\eta'} = Y_{\eta'} = \cos \theta_{23}$ with $\theta_{23} = \theta_{31} - \theta_{21} \approx \theta_{31} - 35^\circ$. Of particular interest are the values of mixing angle $\theta_{31} = -10^\circ$ (so called "perfect mixing") [19,20] and $-29^\circ$ that are used in our analysis. The spin averaged masses of $\eta$ and $\eta'$ for each scheme are shown in Table I. For $\omega$ and $\phi$ mesons, we use the scheme of ideal mixing [21] and obtain the spin averaged masses $m_{\omega} = 928$ MeV and $m_{\phi} = 799$ MeV. Once the spin averaged masses are fixed, then besides the well-known constituent quark masses of $(u,d,s)$ quarks, $m_u = m_d = 330$ MeV and $m_s = 450$ MeV, the only parameter in this model is the gaussian parameter $\beta$ which determines the broadness (or sharpness) of radial wavefunction. We will present our numerical results for a typical $\beta$ value of $\beta = 360$ MeV throughout the paper, unless stated otherwise. The predictions for other values of $\beta$ are summarized in Table II.

In the following section, Sec.II, we calculate the form factors, charge radii, magnetic and quadrupole moments of the $\rho$ and $A_1$ mesons and compare with the results of QCD sum rules [22]. In Sec.III, the transition form factors and the decay widths of the transitions $V \rightarrow P\gamma^*, P_5 \rightarrow V\gamma^*, (V = \rho, K^*, \omega, \phi$ and $I_5 = \pi, K, \eta, \eta')$ and $A_1 \rightarrow \pi\gamma^*$ are presented including the comparison with other theoretical results as well as the experimental data. In Sec.IV, we present the calculation of the transition form factors of the $\pi^+ \rightarrow \gamma^*, \pi^0 \rightarrow \gamma^*$, $\eta \rightarrow \gamma^*$ and $\eta' \rightarrow \gamma^*$ transitions and compare our results with the recent experimental data [23-25]. Summary and discussions of our major results are followed in Sec.V. In the Appendix A, the details of how we obtain the spin averaged masses of $\eta, \eta', \omega$ and $\phi$ are presented. In the Appendix B, we present the derivation of the formula used for the electromagnetic decay widths.

II. THE FORM FACTORS OF THE $\rho$ AND $A_1$ MESONS

A. The $\rho$ meson form factors.

Our analysis will be carried out using the standard light-cone frame (LCF) ($q^2 = q^0 \pm q^3$) [26]:

$$P = (P^+, P^-, P_z) = (P^+, M^2, 0_1),$$

$$q = (q^+, q^- q_z) = (0, q_1, q_z z), \hspace{1cm} (2.1)$$

where $M$ is the meson mass and the photon momentum is transverse to the direction of the incident spin-one system, with $q_1^2 = q_3^2 = -q^2$.

The Lorentz invariant electromagnetic form factors $F_i (i = 1, 2, 3)$ for a spin 1 particle are defined [27] by the matrix elements of the current operator $J^\mu$ between the initial $|P, \lambda>$ and the final $|P', \lambda'>$ eigenstates of momentum $P'$ and helicity $\lambda$ as follows:
\[ \langle P', \lambda' | J^\mu | P, \lambda \rangle = \epsilon'^* \epsilon \left[ -g_{\rho}^\mu (P + P')^\alpha F_{1}(Q^2) + (g_{\rho}^\mu q^\nu - g_{\rho}^\nu q^\mu) F_{3}(Q^2) \right. \\
\left. + \frac{q^\nu q^\mu}{2}(P + P')^\rho F_{5}(Q^2)/(2M^3) \right], \tag{2.2} \]

where \( q = P' - P \) and the polarization vectors of the initial and final mesons \( \epsilon \equiv \epsilon_\lambda \) and \( \epsilon' \equiv \epsilon_{\lambda'} \), respectively, are defined by:

\[
\epsilon(0) = \frac{1}{M}(P^+, -M^{-}, 0_1), \quad \epsilon'(0) = \frac{1}{M}(P'^+, -M'^{+} + Q^2, 0_1), \\
\epsilon(z) = \frac{1}{\sqrt{2}}(0, 0, 1, \pm 1), \quad \epsilon'(z) = \frac{1}{\sqrt{2}}(0, 2q_3, P^+, 1, \pm 1). \tag{2.3} \]

Also, the Lorentz invariant form factors \( F_i(Q^2) \) are related to the charge, magnetic and quadrupole form factors of a meson [27] as follows:

\[
F_C = F_1 + \frac{2}{3} F_2, \\
F_M = F_2 - \frac{1}{3} F_1, \\
F_Q = F_1 - F_3 + (1 + \kappa) F_5, \tag{2.4} \]

where \( \kappa = Q^2/4M^3 \) is a kinematic factor. At zero momentum transfer, these form factors are proportional to the usual static quantities of charge \( c \), magnetic moment \( \mu_1 \), and quadrupole moment \( Q_1 \):

\[
F_C(0) = 1, \quad \epsilon F_M(0) = 2M \mu_1, \quad \epsilon F_Q(0) = M^3 Q_1. \tag{2.5} \]

In the light-cone quark model, the matrix element can be calculated by the convolution of initial and final light-cone wavefunctions of a meson:

\[
\langle P', \lambda' | J^\mu | P, \lambda \rangle = \epsilon'^* \epsilon \int_0^1 dx \int \frac{d^4k_1}{16\pi^2} \Phi_M(x, k_1 + (1 - x)q_1) \Phi_M(x, k_1) \\
\times \sum_{\lambda, \lambda'} \chi_{\lambda}\chi'_{\lambda'}(x, k_1 + (1 - x)q_1) J^\mu \chi_{\lambda}\chi'_{\lambda'}(x, k_1) + \epsilon(1 \leftrightarrow 2 \text{ of the first term}), \tag{2.6} \]

where the spin covariant functions \( \chi(x, k_1) \) are given by Eqs.(1.7)-(1.9) and the vertex \( \Gamma^\mu \) is obtained from the expression \( \delta^\mu_{\rho'\rho}/(x^\rho x^{\rho'})[28] \).

The relationship in Eq.(2.2) between the covariant form factors and current matrix elements can be applied to spin 1 or greater, in addition to the parity and time-reversal invariance of the current operator \( J^\mu(0) \) [27, 31]. An additional constraint on the current operator comes from the rotational covariance requirement [29, 32, 33]. The angular condition for the spin 1 system is given by [32-34]:

\[
\Delta(Q^2) = (1 + 2\kappa) F^+_Q + F^-_Q - \sqrt{\kappa} F^+_M - F^-_M = 0. \tag{2.12} \]

As a matter of fact, the expressions of right-hand side in Eq.(2.7) are thus not unique because of the angular condition in Eq.(2.12). As pointed out by Refs. [33, 34], unless the exact Poincaré covariant current operator beyond one-body sector is used, the angular condition is in general violated (i.e., \( \Delta(Q^2) \neq 0 \)) and the calculation of the form factors \( F_i \) is dependent on the expressions of the r.h.s. of Eq.(2.7). Examples of different choices of current combinations can be found in Ref. [34] for the calculation of \( \rho \) meson form factors.
As shown in Fig.1(a), our result obtained for $\Delta(Q^2)$ is comparable with other choices given in Ref. [34].

The magnetic (in unit of $e/2M$) and quadrupole moments (in unit of $e/M^2$) of the $\rho$ meson are obtained by Eq.(2.5):

$$
\mu_1 = 2.1, \quad Q_1 = 0.41.
$$

(2.13)

These values are not much different from the values of other model predictions presented in Ref. [34] ($\mu_1 = 2.26$, $Q_1 = 0.37$) and Ref. [33] ($\mu_1 = 2.3$, $Q_1 = 0.45$). We also calculated the electromagnetic radii associated with the form factors, $F^\pm$, $F^0$, $F^m$ and $F^q$ as $< r^2_C > = 14$ GeV$^{-2}$, $< r^2_M > = 22$ GeV$^{-2}$ and $< r^2_Q > = -5$ GeV$^{-2}$, respectively. The results of $F_C(Q^2)$, $F_M(Q^2)$ and $F_Q(Q^2)$ for $0 \leq Q^2 \leq 5$ GeV$^2$ are shown in Fig.1(b). To see the effect of the Melosh transformation (the measure of relativistic effects), we calculated the charge form factor by turning the Melosh rotations off and included the result in Fig.1(b). The charge radius from this nonrelativistic form factor is $< r^2 >_{\text{non-rel}} = 10.6$ GeV$^{-2}$ which is about 30% smaller than that of relativistic charge radius. In Fig.1(c), we also compared our results with the previous calculations of these form factors which were made in the framework of QCD sum rules by Ioffe and Shulga [22] 1.

B. The $A_1$ meson form factors.

The electromagnetic form factors of the $A_1$ meson are defined by Eq.(2.2) as in the case of the $\rho$ meson. Using the similar method taken in the $\rho$ meson case, we find the following helicity component of the $A_1$ form factors analogous to Eqs.(2.8)-(2.10):

$$
F^\pm = \frac{Q^2}{8\beta^2 N_{A_1}} \int_0^1 \frac{dx}{x(1-x)} \exp[-\frac{\xi^2 + \tilde{n}_1^2 + x(\tilde{n}_1^2 - \tilde{n}_2^2)}{x(1-x)}] [a_1 - a_2 + a_1^2 + 4a_2^2] R^2(1-x)^3,
$$

$$
F^0 = \frac{Q^2}{8N_{A_1}} \int_0^1 \frac{dx}{x} \exp[-\frac{\xi^2 + \tilde{n}_1^2 + x(\tilde{n}_1^2 - \tilde{n}_2^2)}{x(1-x)}] [a_1 - a_1^2 + a_2^2] R^2(1-x)^3,
$$

(2.14)

$$
F^m = \frac{Q^2}{8N_{A_1}} \int_0^1 \frac{dx}{x} \exp[\frac{\xi^2 + \tilde{n}_1^2 + x(\tilde{n}_1^2 - \tilde{n}_2^2)}{x(1-x)}] [\alpha_1 - \alpha_2 + \alpha_1^2 + 4\alpha_2^2] R^2(1-x)^3,
$$

(2.15)

As one can see in Fig.2(a), the behavior of each form factor looks similar to the $\rho$ meson case. In Fig.2(a), we also included $\Delta(Q^2)$ for $A_1$. The amount of deviation from the rotational covariance is almost the same as that of the $\rho$ meson case.

The electromagnetic radii associated with $F^\pm$, $F^0$, $F^m$ and $F^q$ are obtained as $< r^2_C > = 10$ GeV$^{-2}$, $< r^2_M > = 20$ GeV$^{-2}$ and $< r^2_Q > = -10$ GeV$^{-2}$, respectively. The magnetic and quadrupole moments of the $A_1$ meson are also obtained as

$$
\mu_1 = 2.16, \quad Q_1 = 1.34.
$$

(2.17)

While the magnetic moment of $A_1$ does not differ from that of the $\rho$ meson, the quadrupole moment of $A_1$ is about 4 times greater than that of the $\rho$ meson in accordance with the fact that the $A_1$ meson is a bound state with nonzero orbital angular momentum $l = 1$ [35]. In Fig.2(b), we compare our results with those of QCD sum rules [22].

III. THE TRANSITION FORM FACTORS OF $V \rightarrow B_{\gamma^*}$, $\eta' \rightarrow \rho(\omega)\gamma^*$, AND $A_1 \rightarrow \pi\gamma^*$

The transition form factors of $A \rightarrow B_{\gamma^*}$ ($A = (\rho, \pi), (\rho, \eta), (\pi, \sigma), (\omega, \pi), (\omega, \eta), (K^*, K)$, $(\eta', \rho), (\eta', \omega), (\phi, \rho), (\phi, \eta)$) and $A_1 \rightarrow \pi\gamma^*$ are defined by

$$
\langle B(P')|J^\mu(A(P, \lambda))|A(P, \lambda)\rangle = eG_{AB}(Q^2)\epsilon^{\mu\nu\rho\sigma}\epsilon_\nu(P')P^\rho P^\sigma,
$$

(3.1)

$$
\langle \pi(P')|J^\mu(A_1(P', \lambda))|A_1(P, \lambda)\rangle = \frac{1}{m_{A_1}} \left[ (P' - P) \cdot q_\rho q^\sigma - P' \cdot q_\rho q^\sigma - M_{A_1}^2 G_{12}(Q^2) \right] \epsilon_\nu(P', \lambda),
$$

(3.2)

where $e$ denotes the polarization vector of the initial particles and $\epsilon = P + P'$. As shown in the Appendix B, the width of the decay $A \rightarrow B_{\gamma^*}$ is given by

$$
\Gamma(A \rightarrow B_{\gamma^*}) = \frac{\alpha}{2S_A} \left[ G_{AB}(Q^2) \right] (M_A^2 - M_B^2)\lambda_A S_A
$$

(3.3)

where $\alpha$ is the fine structure constant, $S_A$ is the spin of the initial particle and $M_{AB}$ is the mass of the meson $A(B)$. In the case of $A_1 \rightarrow \pi\gamma$ transition, the decay width is expressed

$^3$This must be the physical mass rather than the spin-averaged mass because the phase factor is nothing to do with the model. Spin-averaged masses are used only for the calculation of form factors.
in terms of $G_A(0)$ (see Appendix B);
\[
\Gamma(A_1 \to \pi \gamma) = 4\alpha \frac{G_A(0)}{3} \left( -\frac{M_{A_1}^2 - M_{A_1}^2}{2M_{A_1}} \right)^3.
\]

(3.1)

Carrying out again a tedious but straightforward calculation, we find the following expressions for the form factors $G_\pi(Q^2)$ and $G_\omega(Q^2)$:
\[
G_\pi(Q^2) = (\epsilon_\pi + \epsilon_\omega) I_1(m_M, m_\pi, \beta),
\]
\[
G_\omega(Q^2) = (\epsilon_\omega - \epsilon_\pi) I_1(m_M, m_\pi, \beta).
\]

(3.5)

where
\[
I_1(m_M, m_\pi, \beta) = \frac{1}{2\pi} \sqrt{\frac{N_c N_f}{2}} \int_0^1 \frac{dx}{z} \exp \left[ - \frac{Q^2 + m_\pi^2 + x(m_M^2 - m_\pi^2)}{z(1-x)} \right]
\]
\[
\times \left[ a_1 x \left( x(1-x) - \xi^2 \right) + a_2 x \left( \xi^2 + a_1^2 + a_2^2 \right) \right]
\]
\[
+ a_1^2 \left( a_1^2 + a_2^2 \right).
\]

(3.6)

and $a_{ij} = (2m_{M_{ij}} + m_{M})/2\beta$ with i and j meaning incident and outgoing mesons. The normalization constants of the $\pi$ and $\omega$-mesons are given by $N_c = N_c/2 = 2(5/2\alpha_{em})^2$.

We obtained the following prediction of decay widths of the $\rho^0(770) \to \pi^\pm \gamma$ and $\omega(782) \to \pi^\pm \gamma$ transitions:
\[
\Gamma(\rho^0 \to \pi^\pm \gamma) = 69 \text{ keV} \quad (\Gamma_{\rho^0 \to \pi^\pm \gamma} = 68 \pm 8 \text{ keV}),
\]
\[
\Gamma(\omega \to \pi^\pm \gamma) = 708 \text{ keV} \quad (\Gamma_{\omega \to \pi^\pm \gamma} = 717 \pm 51 \text{ keV}).
\]

(3.7)

Our results for the decay widths are in very good agreement with the experimental data. The electromagnetic radii of these form factors are obtained as $< r_{\rho^0}^2 > = 7 \text{ GeV}^{-2}$ and $< r_{\omega}^2 > = 20 \text{ GeV}^{-2}$, respectively. In Fig.3(a), we present the transition form factor of $\rho \to \pi^* \gamma$ for $0 \leq Q^2 \leq 8 \text{ GeV}^2$ and compare with other model predictions of Refs. [34,36]. In Fig.3(b), our prediction (solid line) of $\omega \to \pi^* \gamma$ transition form factor in the space-like region is compared with the VDM (dashed line) with $F_{\pi^*} = 1/(1 + Q^2/M_{\rho^0}^2)$ and the pole fit (dotted line) of the experimental data in the time-like region [38,39].

Similarly, we calculated all other radiative decay processes between vector (1$$) and pseudoscalar mesons(0$$) using the one loop integral formula $I_1(m_M, m_\pi, \beta)$ given by Eq.(3.6). We needed to change only the spin-averaged meson masses $m_M$ and constituent quark masses $m_q(t = 1, 2)$, accordingly. Thus, for the $\rho \to \eta \gamma$ and $\omega \to \eta \gamma$ decays, we obtain the following transition form factors
\[
G_{\rho\eta}(Q^2) = X_\rho (\epsilon_\rho - \epsilon_\eta) I_1(m_M, m_\pi, \beta),
\]
\[
G_{\omega\eta}(Q^2) = X_\omega (\epsilon_\omega - \epsilon_\eta) I_1(m_M, m_\pi, \beta).
\]

(3.8)

Using the two different mixing schemes, i.e., $\theta_{12} = -10^\circ$ and $-23^\circ$, we obtain the decay widths of the transitions $\rho \to \eta \gamma$ and $\omega \to \eta \gamma$ as $\Gamma_{\rho \to \eta \gamma} = 56 \text{ keV}$, $\Gamma_{\omega \to \eta \gamma} = 6.4 \text{ keV}$ and $\Gamma_{\omega \to \eta \gamma}^{23^\circ} = 65 \text{ keV}$, $\Gamma_{\omega \to \eta \gamma}^{23^\circ} = 7.4 \text{ keV}$, respectively. Both schemes are in excellent agreement with the experimental data of $\Gamma_{\rho \to \eta \gamma} = 58 \pm 10 \text{ keV}$ and $\Gamma_{\omega \to \eta \gamma}^{23^\circ} = 7.0 \pm 1.8 \text{ keV}$. The electromagnetic charge radii of $\rho \to \eta \gamma$ transition are also predicted as $< r_{\rho \to \eta \gamma}^2 > = 16 \text{ GeV}^{-2}$ for $-10^\circ$ and $18 \text{ GeV}^{-2}$ for $-23^\circ$ mixing angle. The charge radii for $\omega \to \eta \gamma$ transition are $< r_{\omega \to \eta \gamma}^2 > = 5 \text{ GeV}^{-2}$ for $-10^\circ$ and $6 \text{ GeV}^{-2}$ for $-23^\circ$, respectively. The form factors of the $\eta' \to \rho^0 \gamma$ and $\pi^0 \to \pi^+ \gamma$ are given by
\[
G_{\eta'\rho}(Q^2) = X_{\eta'} (\epsilon_\eta - \epsilon_{\rho^0}) I_1(m_M, m_{\rho^0}, \beta),
\]
\[
G_{\eta'\pi}(Q^2) = X_{\eta'} (\epsilon_\eta + \epsilon_{\pi^0}) I_1(m_M, m_{\pi^0}, \beta).
\]

(3.9)

Our predictions of the decay widths are given by $\Gamma_{\eta'\rho}(Q^2) = 117 \text{ keV}$, $\Gamma_{\eta'\pi}(Q^2) = 9.7 \text{ keV}$ and $\Gamma_{\omega \to \eta \gamma}^{23^\circ} = 72 \text{ keV}$, $\Gamma_{\omega \to \eta \gamma}^{23^\circ} = 6.0 \text{ keV}$ and $\Gamma_{\omega \to \eta \gamma}^{23^\circ} = 5.9 \pm 0.9 \text{ keV}$. It is interesting to note that in case of transitions involving $\eta'$, the result of $-23^\circ$ mixing scheme is much better than that of $-10^\circ$ mixing scheme. The electromagnetic charge radii for $-23^\circ$ mixing angle are predicted as $< r_{\rho \to \eta \gamma}^2 > = 16 \text{ GeV}^{-2}$ and $< r_{\omega \to \eta \gamma}^2 > = 4 \text{ GeV}^{-2}$, respectively.

The transitions of the $K^+ \to K^0 \gamma$ and $K^0 \to K^0 \gamma$ in which the constituent quarks have unequal masses are also interesting processes to test our model predictions. As shown in our previous works [7], the predictions for the kaon charge radius $< r_K^2 > > 17\text{f}_{K}$, the kaon form factor $F_K$, and the decay constant $f_K$ are consistent with available experimental data. The transition form factors of the charged and neutral $K^* - K^0$ decays are given by
\[
G_{K^+ \to K^0 \gamma}(Q^2) = \pm (\epsilon_\rho + \epsilon_\omega) I_1(m_M, m_{K^0}, \beta),
\]
\[
G_{K^- \to K^0 \gamma}(Q^2) = (\epsilon_\pi + \epsilon_\omega) I_1(m_M, m_{K^0}, \beta).
\]

(3.10)

Using the spin averaged masses of $m_{K^+} = 792.1 \text{ MeV}$ and $m_{K^0} = 796.5 \text{ MeV}$ and the constituent quark mass $m_q = 0.45 \text{ GeV}$ [7], we obtain the decay widths for these charged
and neutral vector kaon decay processes as $\Gamma_{K^*\pi^0\pi^0} = 53$ keV, $\Gamma_{K^*\pi^0\pi^0} = 122$ keV, respectively. The experimental data are $\Gamma_{K^*\pi^0\pi^0} = 50 \pm 5$ keV, $\Gamma_{K^*\pi^0\pi^0} = 117 \pm 10$ keV. The electromagnetic transition charge radii are predicted as $\langle r_{1s2s}^2 \rangle$ and $\langle r_{2s3s}^2 \rangle = 11$ GeV$^{-2}$ and $\langle r_{2s3s}^2 \rangle = 10$ GeV$^{-2}$. In Fig. 4, we reported $Q^2$-dependence of transition form factor of charged and neutral vector kaon, i.e., $G_{K^*\pi^0\pi^0}(Q^2)$ and $G_{K^*\pi^0\pi^0}(Q^2)$, respectively, for $0 \leq Q^2 \leq 10$ GeV$^2$. Even though we showed the result of $\beta = 0.36$ GeV only, we note that our results become much closer to those of Bethe-Salpeter (B-S) qark model prediction [39] with smaller values of $\beta$, i.e., $\beta \sim 0.3$ GeV.

For the decay processes of $\phi \rightarrow \eta(\phi')\gamma'$, the transition form factors of $\phi \rightarrow \eta \gamma$ and $\phi \rightarrow \eta' \gamma'$ are given by

$$G_{\phi\eta}(Q^2) = Y_2^\phi Y_4^\eta \langle m_{\phi}, m_{\eta}, \beta \rangle$$

$$G_{\phi\eta'}(Q^2) = Y_4^\phi Y_2^\eta \langle m_{\phi}, m_{\eta}, \beta \rangle.$$  

(3.11)

As shown in the Appendix A, we obtain the spin averaged mass of $\phi$ meson as $m_{\phi} = 799$ GeV. Our predictions of the decay widths of the transitions $\phi \rightarrow \eta \gamma$ and $\phi \rightarrow \eta' \gamma$ with $\beta_{\phi\eta}(2D) = -10^9$ and $-23^8$ are given by $\Gamma_{\phi\eta}(2D) = 61$ keV, $\Gamma_{\phi\eta'}(2D) = 28$ keV and $\Gamma_{\phi\eta}(2D) = 1.45$ keV, $\Gamma_{\phi\eta'}(2D) = 0.45$ keV, respectively. The current experimental data are $\Gamma_{\phi\eta}(2D) = 56.9 \pm 4.9$ keV and $\Gamma_{\phi\eta'}(2D) = 1.8 \pm 0.8$ keV. It will be very interesting to compare our results, especially for $\phi \rightarrow \eta \gamma$, with the more precise measurements envisioned at TJNAF. The electromagnetic charge radii for $-10^9$ mixing angle are also predicted as $\langle r_{1s2s}^2 \rangle > 5$ $\text{GeV}^{-2}$ and $\langle r_{2s3s}^2 \rangle = 5.5$ GeV$^{-2}$.

In the case of transition $A_1 \rightarrow \pi^+ \pi^-$, the kinematical factors in front of the form factor $G_2$ in Eq.(3.2) yields zero for a certain $Q^2$ in the standard Drell-Yan frame ($P_1 = 0, Q^2 = 0$) as discussed in Ref. [35]. This leads to a technical difficulty in extracting the form factors numerically from Eq.(3.2). Thus, as pointed out in Ref. [35], we use the following symmetric coordinate frame for the calculation of $A_1 \rightarrow \pi^+ \pi^-$ transition form factors:

$$p = (P, -m_{\pi^+}^2 + \frac{1}{2} Q^2, -\frac{1}{2} Q^2), \quad p' = (P, m_{\pi^+}^2 + \frac{1}{2} Q^2, \frac{1}{2} Q^2),$$

$$q = (0, m_{\pi^-}^2 - m_{\pi^+}^2, Q^2).$$

(3.12)

Then, we find the following expressions for the $A_1$ transition form factors:

$$G_1(Q^2) = \frac{m_{\pi^+}^2}{P^2 (m_{\pi^+}^2 - m_{\pi^-}^2)} \left[ G_0^2 + \frac{(m_{\pi^+}^2 - m_{\pi^-}^2)^2 + Q^4}{2m_{\pi^-} P^2} \right].$$

(3.13)

$$G_2(Q^2) = \frac{2m_{\pi^+}^2}{(m_{\pi^+}^2 - m_{\pi^-}^2) Q} \left[ G_0^2 - \frac{(m_{\pi^+}^2 - m_{\pi^-}^2)^2}{2m_{\pi^-} P^2} \right].$$

(3.14)

where $G_0^2 \equiv < S(P') | M_0 | S(P) >$ and its matrix elements are given by

$$G_0^2 = \frac{Q}{2 \beta^2 \sqrt{\alpha \beta^2}} \int_0^\infty \frac{z dz}{z(x(1-z) + \frac{Q^2}{x} - 2x(1-z))}$$

$$G_0^2 = \frac{2\alpha \beta^2}{Q^2 \sqrt{\alpha \beta^2}} \int_0^\infty \frac{z dz}{z(x(1-z) + \frac{Q^2}{x} - 2x(1-z))}$$

$$G_0^2 = \frac{Q}{2 \beta^2 \sqrt{\alpha \beta^2}} \int_0^\infty \frac{z dz}{z(x(1-z) + \frac{Q^2}{x} - 2x(1-z))}.$$  

(3.15)

(3.16)

(3.17)

with a common factor $G_0$ in the longitudinal component of $G$:

$$G_0 = 2\beta^2 \int_0^\infty \frac{z dz}{z(x(1-z) + \frac{Q^2}{x} - 2x(1-z))}.$$  

(3.18)

In Fig.5, we show the result of $A_1$ transition form factors, $G_1(Q^2)$ (solid line) and $G_2(Q^2)$ (dashed line). While the form factor $G_2(Q^2)$ agrees with the predictions of the QCD sum-rule calculations [22] in the region $1 \leq Q^2 \lesssim 3 \text{ GeV}^2$, the form factor $G_2(Q^2)$ seems to be quite different from the QCD sum-rule result (dotted line) [22]. However, there are other QCD sum rule calculations of $G_1$ and $G_2$ [40,41] and the results are rather different from each other. The authors of the QCD sum rules [22] also pointed out that their predictions for the transition $A_1 \rightarrow \pi^+ \pi^-$ are of semiqualitative results. The electromagnetic radii corresponding to these form factors are evaluated as $\langle r_{1s2s}^2 \rangle = 15 \text{ GeV}^{-2}$ and $\langle r_{2s3s}^2 \rangle = -30 \text{ GeV}^{-2}$, respectively.

The decay width of $A_1(1260) \rightarrow \pi^+ \pi^-$ is obtained from Eq.(3.4) as

$$\Gamma(A_1 \rightarrow \pi^+ \pi^-) = 705 \text{ keV}.$$  

(3.19)

The prediction of VDM [42,43] is given by $\Gamma_{A_1 \rightarrow \pi^+ \pi^-} = 1000 - 1500 \text{ keV}$ and the experimental value is reported as $[44,45]$ for $\Gamma(A_1 \rightarrow \pi^+ \pi^-) = 610 \pm 246 \text{ keV}$ using the measurement of Primakoff production of the $A_1$ resonance. Thus, our predicted width is quite consistent with the corresponding experimental data. In Refs. [35,37], the same decay width was calculated using the invariant mass of $A_1$ meson instead of the spin averaged mass. Their results $\Gamma(A_1 \rightarrow \pi^+ \pi^-) = 319 \text{ keV}$ [35] and 250 keV [37] seem to have rather large discrepancy from the experimental data.

IV. THE FORM FACTORS OF $s^0 \rightarrow \gamma \gamma' \eta \rightarrow \gamma \gamma' \gamma$ AND $\eta' \rightarrow \gamma \gamma'$ TRANSITIONS

The transition form factor of $P_5 \rightarrow \gamma \gamma' \eta(P_5 = s^0, \eta, \eta')$ is defined from the matrix element of electromagnetic current $\gamma_\mu = \gamma(P + q) | J_{\mu} | P_5(P) >$ as follows:

12
\[ \Gamma_{\mu} = i e^2 G_{\mu\nu}(Q^2) \kappa_{\tau \nu} P^\nu e^\tau, \]

(4.1)

where \( P \) and \( \eta \) are the momenta of the incident pseudoscalar meson and virtual photon, respectively, and \( \epsilon \) is the polarization vector of the final (on-shell) photon.

The \( Q^2 \) dependent decay rate for \( P_5 \to \gamma \gamma \) is given by

\[ \Gamma_{P_5 \to \gamma \gamma}(Q^2) = \frac{\alpha^3 M_{P_5}^3 G_{P_5}^2(Q^2)}{4}, \]

(4.2)

where \( M_{P_5} \) is the mass of \( P_5 = \pi^0, \eta, \) and \( \eta' \). Here, the decay width is given by \( \Gamma_{P_5 \to \gamma \gamma} \) at \( Q^2 = 0 \). If we choose the 't-channel' component of the current, the vertex factor in Eq. (4.1) is given by

\[ \Gamma^t = \sqrt{\alpha} \frac{G_\mu}{\sqrt{2}} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\sqrt{2i}} \psi_{P_5}(z_1, k_1) \left[ \frac{\bar{u}_q(z_2, k_2) \gamma_\mu u_q(z_1, k_1 + q)}{\sqrt{2i}} \right] \times \frac{\bar{u}_q(z_1, k_1 + q) \gamma_\mu u_q(z_1, k_1)}{\sqrt{2i}} \frac{1}{Q^2 - (k_1 - k_2)^2 - i\epsilon} \left( 1 \to 2 \right). \]

(4.3)

Here, only anti-parallel helicities of constituents contribute to the integrations and our model wave function \( \psi_{P_5}(x_1, k_1) \) for anti-parallel helicities is given by

\[ \psi_{P_5}(x_1, k_1) = \frac{x}{2\beta^0}(N_{P_5})^2 \frac{g_{P_5}(k_1^2) + m^2}{2x} \exp\left( -\frac{k_1^2 + m^2}{8\alpha(1 - z)\beta^2} \right). \]

(4.4)

A straightforward calculation for the transition \( \pi^0 \to \gamma \gamma \) gives the following result

\[ \Gamma_{\pi^0 \to \gamma \gamma} = \sqrt{\alpha} \frac{G_\mu}{\sqrt{2}} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\sqrt{2i}} \psi_{\pi^0}(z_1, k_1) \left[ \frac{\bar{u}_q(z_2, k_2) \gamma_\mu u_q(z_1, k_1 + q)}{\sqrt{2i}} \right] \times \frac{\bar{u}_q(z_1, k_1 + q) \gamma_\mu u_q(z_1, k_1)}{\sqrt{2i}} \frac{1}{Q^2 - (k_1 - k_2)^2 - i\epsilon} \left( 1 \to 2 \right). \]

(4.5)

where

\[ I_2(m_{\pi^0}, m_{\pi^0}) = \frac{\sqrt{N_{P_5}}}{\pi}\int \frac{d^4 k}{(2\pi)^4} \frac{1}{\sqrt{2i}} \exp\left( -\frac{m^2}{2(1 - z)} \right) \left( 2(1 - z) \right) \exp\left( \frac{4k_1^2 + m^2}{2x(1 - z)} \right) \times \frac{(a_{12} + m^2 + 4z)}{\sqrt{2i}} \frac{1}{Q^2 - (k_1 - k_2)^2 - i\epsilon} \left( 1 \to 2 \right). \]

(4.6)

Using Eqs. (4.2)-(4.6), the decay widths for \( P_5 \to \gamma \gamma \) are obtained as

\[ \Gamma(\pi^0 \to \gamma \gamma) = 6.50 \text{ eV}, \]

(4.7)

\[ \Gamma(\eta \to \gamma \gamma) = 0.47 \text{[0.65]} \text{ keV}, \]

\[ \Gamma(\eta' \to \gamma \gamma) = 7.95 \text{[0.65]} \text{ keV}, \]

\[ \text{where \ the \ values \ for \ } \eta \to \gamma \gamma \text{ and } \eta' \to \gamma \gamma \text{ are obtained for the -10^2[23]\text{ mixing scheme. The experimental data [47] are given by } \Gamma_{\pi^0}^{\text{exp}} = 7.8 \pm 0.5 \text{ eV}, \Gamma_{\eta}^{\text{exp}} = 0.47 \pm 0.05 \text{ keV and } \Gamma_{\eta'}^{\text{exp}} = 4.3 \pm 0.6 \text{ keV. The agreement of our results with the experimental data is not unreasonable. In Figs. 8-9, the } Q^2 \text{ dependence of the decay rate } \Gamma_{P_5}(Q^2) \text{ for } P_5 = \pi^0, \eta, \text{ and } \eta' \text{ are shown and compared with the recent experimental data [23-25]. Our predictions for all of these processes are overall in a good agreement with the experimental data up to a rather large } Q^2. \text{ In Table II, our predictions of the two photon decay widths for various } \beta \text{ values are included.} \]

V. SUMMARY AND DISCUSSION

In this paper, we have investigated the radiative decays of pseudoscalar(\( \pi, K, \eta, \eta' \)), vector(\( \rho, K^*, \omega, \phi \)) and axial vector(\( A_1 \)) mesons as well as the form factors of \( \rho \) and \( A_1 \) mesons using a simple relativistic constituent quark model. The use of a spin averaged hadron mass in the lightcone quark model has of course a wide literature [7-12]. The kinematical constraints following from the proper Poincaré-group invariance has been studied in detail [10] and it is shown that our light-cone quark model wavefunctions given by Eq.(1.4) are Lorentz invariant. Also, in our method, each constituent is individually Melosh transformed and thus the orthogonality of the constructed meson wavefunctions is not spoiled by the Melosh transformation. Since we started from the orthogonal wave functions of pseudoscalar, vector and axial vector mesons, the constructed wavefunctions by the Melosh transformation are clearly orthogonal.

We summarized all of our predictions on the meson decay widths for various values of \( \beta \) (\( \beta = 0.32, 0.34, 0.36, 0.38 \) GeV in Table II). Remarkably, most of our predictions with the parameter \( \beta \) in this region are within the experimental errors. We have also investigated the sensitivity of our results by varying quark masses. For \( \pm 10\% \) variation of the nonstrange quark masses, the decay width \( \Gamma \) and the transition charge radius \( <q^2>^{1/2} \) of the radiative meson decay change by 3.4% and 2.4%, respectively. Changing strange quark mass by \( \pm 10\% \) for the process of \( K^* \to K^* \gamma \) yields 13-15% and 3-4% difference in the decay width and the transition charge radius, respectively.

Overall, our numerical results for all the meson radiative decays that we investigated in this model are remarkably in a good agreement with the experimental data. In our point of view, the success of this model hinges upon the simplicity of the light-cone vacuum. The
recent lattice QCD results [48] indicate that the mass difference between \( \eta' \) and pseudoscalar octet mesons due to the complicated nontrivial vacuum effect increases (or decreases) as the quark mass \( m_q \) decreases (or increases), i.e., the effect of the topological charge contribution should be small as \( m_q \) increases. This makes us to believe that the complicated nontrivial vacuum effect can be traded off by the constituent quark masses. This may mean that there is a suppression of complicate zero mode contribution [49] from the light-cone vacuum in our model due to rather large constituent quark masses.

Our approach in this work was obviously to model the wavefunction rather than to model the potential. However, we have attempted to compare our results with various other available theoretical results [32-43] including the potential models and the QCD sum rules. At the very least, our results seem to be quite comparable with the results from modding the potential. The results on the angular condition are also not drastically different from the result of the potential models (See Fig. 1(a)). Furthermore, the agreement with the QCD sum rule results is not unreasonable. Therefore, the light-cone quark model presented here should be distinguished from the so called naive quark model. Our model has a predictive power and more experimentally meaningful quantities should be calculated and compared with data. A particularly interesting prediction from our model is the branching ratio of \( \phi \rightarrow \eta' \) estimated as \( 10^{-4} \) for the \( \eta - \eta' \) mixing angle of \( -23^\circ \). For the mixing angle \( -10^\circ \), the estimation for this branching ratio is reduced by 60% from that of \( -23^\circ \). Thus, it will be very interesting to compare our results with the precise measurements envisioned at TJNAP.

VI. ACKNOWLEDGEMENT

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APPENDIX A: SPIN AVERAGED MESON MASSES OF \( \eta, \eta', \omega \) AND \( \phi \)

Here, let’s first list our flavor wave functions of the neutral pseudoscalar \( \eta, \eta' \) and vector \( \omega, \phi \) nonet states to show explicitly how we obtained the values of spin averaged mass used in this paper;

\[
\eta = \frac{1}{\sqrt{6}}(uu + dd - 2ss),
\]

\[
\eta' = \frac{1}{\sqrt{3}}(uu + dd + ss),
\]

\[
\omega = \frac{1}{\sqrt{2}}(uu + dd),
\]

\[
\phi = ss.
\]

For ideally mixed isoscalar and isovector mesons, we take the flavor wave function as

\[
\chi^{N(V)}_{\phi} = \frac{1}{\sqrt{2}}(uu + dd),
\]

\[
\chi^{i(N(V))} = ss,
\]

where \( P_0(V) \) denotes pseudoscalar (vector) meson states. In terms of this basis, the neutral meson nonet states are given by

\[
\eta_\tau = \sqrt{\frac{2}{3}} \chi^{N(V)}_\omega + \frac{1}{\sqrt{3}} \chi^{i(N)}_\omega, \quad \omega_\tau = \sqrt{\frac{2}{3}} \omega + \frac{1}{\sqrt{3}} \phi,
\]

\[
\eta_\omega = \frac{1}{\sqrt{3}} \chi^{N(V)}_\omega - \sqrt{\frac{2}{3}} \chi^{i(N)}_\omega, \quad \omega_\omega = \frac{1}{\sqrt{3}} \omega - \sqrt{\frac{2}{3}} \phi.
\]

where \( \eta = \eta_\tau, \eta' = \eta_\omega, \omega = \chi_\omega, \) and \( \phi = \chi_\phi \). We define the spin averaged masses \( \bar{M} \) of \( \chi_\omega \) and \( \chi_\phi \) as follows;

\[
\bar{M}_{\omega} = \frac{1}{4} M_{\omega}^N + \frac{3}{4} M_{\omega}^i,
\]

\[
\bar{M}_{\phi} = \frac{1}{4} M_{\phi}^N + \frac{3}{4} M_{\phi}^i.
\]

Using Eqs. (A3), (A4), (A5) and (A6), we assign

\[
M_{\eta}^{pp} = \frac{1}{3} M_{\eta}^N + \frac{2}{3} M_{\eta}^i,
\]

\[
M_{\eta'}^{pp} = \frac{2}{3} M_{\eta'}^N + \frac{1}{3} M_{\eta'}^i,
\]

\[
M_{\omega}^{pp} = M_{\omega}^N,
\]

\[
M_{\phi}^{pp} = M_{\phi}^i.
\]
Then, we obtain the following spin averaged masses of $\eta$ and $\eta'$;

$$M_\eta = \frac{1}{3} M_{\omega} + \frac{2}{3} M_\pi = 842\text{MeV},$$

$$M_{\eta'} = \frac{2}{3} M_{\omega} + \frac{1}{3} M_\pi = 885\text{MeV}. \quad (A9)$$

Likewise, from Eq.(A6), we obtain

$$M_{\omega} = \frac{1}{3} M_\pi + \frac{2}{3} M_\phi,$$ \hspace{0.5cm} (A11)

$$M_{\eta'} = \frac{1}{3} M_\pi + \frac{2}{3} M_\phi.$$ \hspace{0.5cm} (A12)

Using $M_\omega = M_{\omega}$ and $M_{\eta'} = M_{\eta}$, we can then evaluate the spin averaged masses of $\omega$ and $\phi$ as $M_\omega = 928\text{MeV}$ and $M_\phi = 799\text{MeV}$, respectively.

To calculate the spin averaged meson masses depending on the schemes of flavor mixing, let's consider the "perfect mixing"($SU(3) = -10^{-2}$) $\tilde{\eta}$ and $\tilde{\eta}'$ states defined by [19,20],

$$\tilde{\eta} = \frac{1}{\sqrt{2}}(\eta^2_\pi - \eta^2_\rho), \quad \tilde{\eta}' = \frac{1}{\sqrt{2}}(\eta^2_\rho + \eta^2_\pi). \quad (A13)$$

Using Eqs.(A13) and Eq.(A6), we obtain

$$\eta_1 = \frac{\sqrt{2} - 1}{\sqrt{6}} \eta + \frac{\sqrt{2} + 1}{\sqrt{6}} \eta',$$

$$\eta_2 = \frac{1 + \sqrt{2}}{\sqrt{6}} \eta + \frac{1 - \sqrt{2}}{\sqrt{6}} \eta'. \quad (A14)$$

Thus, the spin averaged masses $m_\alpha$ and $m_{\alpha'}$ in "perfect mixing" scheme are related to $M_\omega$ and $M_{\eta'}$ calculated in our scheme (see Eq.(A9) and (A10)) respectively, as follows;

$$M_\omega = \frac{(\sqrt{2} - 1)^2}{6} m_\omega + \frac{\sqrt{2} + 1}{6} m_{\eta'},$$

$$m_{\omega} = \frac{(\sqrt{2} + 1)^2}{6} m_\pi + \frac{\sqrt{2} - 1}{6} m_{\phi}.$$ \hspace{0.5cm} (A15)

From Eq.(A15), the spin averaged masses for the "perfect mixing" states are given by $m_\omega = 843\text{MeV}$ and $m_{\omega'} = 884\text{MeV}$, respectively. Using the same method in the above, for the $SU(3) = -23$ mixing scheme, we obtain the following spin averaged meson masses: $m_\pi = 838\text{MeV}$ and $m_{\eta'} = 873\text{MeV}$, respectively.

**APPENDIX B: THE ELECTROMAGNETIC DECAY WIDTH $\Gamma(A \rightarrow B + \gamma)$**

In this appendix, we derive the decay widths of $A \rightarrow B + \gamma$ ($A = \rho, \pi^*, \omega, \phi$, and $B = \pi, K, \eta, \eta'$) and $A_1 \rightarrow \pi + \gamma$.

Consider the electromagnetic decay process $A \rightarrow B + \gamma$. If the masses of $A$ and $B$ mesons are given by $m_\pi$ and $m_B$, respectively, then the decay rate $\Gamma(A \rightarrow B + \gamma)$ in the rest frame of $A$ is given by ($k = e = 1$);

$$\Gamma(A \rightarrow B + \gamma) = \frac{1}{2} \frac{S}{2E_B} \int \frac{d^3p_B}{2\pi} \int \frac{d^3p_\gamma}{2\pi} |M|^2 \left(2\pi\right)^4 \delta(m_A - E_B - E_\gamma), \quad (B1)$$

where $S = 1/j!$ for each group of $j$ identical particles in the final states. Here, $M$ is the transition matrix element defined by

$$M = e^2(\lambda_3) < B(P) | J^\mu | A(P, \lambda) >.$$ \hspace{0.5cm} (B2)

To allow decays to all possible spin configurations, we consider the replacement

$$|\mathcal{M}|^2 \rightarrow \left| \mathcal{M} \right|^2 \equiv \frac{1}{2S_A + 1} \sum_{\lambda_\mu = \pm 1} |\mathcal{M}|^2, \quad (B3)$$

where $S_A$ is the spin of particle $A$ and $\lambda_\mu$ corresponds to the state of transverse polarization of the emitted photon. The above replacement means the average over the initial spin and the sum over the helicities of the emitted photon. Then, from Eq.(B1), we get

$$\Gamma(A \rightarrow B + C) = \frac{S}{2(4\pi)^2 m_A} \int \frac{d^3p_B}{2\pi} \int \frac{d^3p_\gamma}{2\pi} \frac{\delta(m_A - \sqrt{m_B^2 + p_\gamma^2})}{|p_B| \sqrt{m_B^2 + p_\gamma^2}} \times \sum_{\lambda_\mu = \pm 1} |\mathcal{M}|^2, \quad (B4)$$

If we let $|p_B| = \rho$ and $E \equiv \sqrt{m_B^2 + \rho^2}$, then

$$\Gamma(A \rightarrow B + \gamma) = \frac{S}{8\pi m_A} \int_0^\infty \frac{1}{m_A} \sum_{\lambda_\mu = \pm 1} |\mathcal{M}|^2, \quad (B5)$$

Therefore, we get

$$\Gamma(A \rightarrow B + \gamma) = \frac{S}{8\pi m_A} \int_0^\infty \frac{1}{m_A} \sum_{\lambda_\mu = \pm 1} |\mathcal{M}|^2, \quad (B6)$$

where

$$S = 1 \quad \text{if} \quad B \neq \gamma,$$

$$S = \frac{1}{2} \quad \text{if} \quad B = \gamma. \quad (B7)$$

Here, $\rho_B$ is the value of $\rho$ when $E = m_A$, i.e., $\rho_B = (m_A^2 - m_B^2)/2m_A$. In the rest frame of $A$, i.e., $p_A = (m_A, 0)$, $p_B = (m_B, p_\gamma)$ and $P_A = (|p_B| - p_B)$, the invariant amplitude square is given by
\[ \sum_{\lambda\mu} |\mathcal{M}|^2 = \sum_{\lambda_{\mu},k_{\mu}} |eG_{\mu\nu}(Q^2) e_{\lambda} e_{\mu}(\lambda,\mu,\nu,\lambda,\mu)|^2 \]
\[ = 2|e|^2 G_{\mu\nu}(Q^2)^2 \frac{m_A^2 - m_B^2}{2m_A^2} \]
\[ = \frac{(m_A^2 - m_B^2)}{2m_A^2} \frac{m_A^2}{2m_A^2} \]
\[ = \frac{\alpha}{2} |G_{\mu\nu}(Q^2)|^2. \] 

(B8)

Therefore, we get the following decay width \( \Gamma(A \to B + \gamma) \)
\[ \Gamma(A \to B + \gamma) = \frac{\alpha}{2\mu_A + 1} |G_{\mu\nu}(0)|^2 \left( \frac{m_A^2 - m_B^2}{2m_A^2} \right), \] 

(B9)

where \( \alpha = e^2/4\pi \) is the fine structure constant.

The decay width of \( A_1 \to \pi\gamma \) can be calculated in the same manner using our definition of the transition matrix element given by Eq.(3.2) as follows;

\[ \sum_{\lambda_{\mu},k_{\mu}} |\mathcal{M}|^2 = |e_{\lambda} e_{\mu}(\lambda,\mu,\nu,\lambda,\mu)|^2 \]
\[ = 2|e|^2 G_{\mu\nu}(Q^2) \frac{m_A^2 - m_A^2}{2m_A^2} \]
\[ = 2|e|^2 \frac{G_{\mu\nu}(Q^2)}{m_A} |p_A - m_A|^2 \]
\[ = 2|e|^2 \frac{G_{\mu\nu}(Q^2)}{m_A} \left( 2m_A^2 + m_A p_A \right)^2 \]
\[ = 2\alpha \frac{G_{\mu\nu}(Q^2)}{m_A} \left( 2m_A^2 + m_A p_A \right)^2, \] 

(B10)

and the result is given by

\[ \Gamma(A_1 \to \pi\gamma) = \frac{4\alpha}{3} \frac{G_{\mu\nu}(0)}{m_A} \left( \frac{m_A^2 - m_A^2}{2m_A^2} \right). \] 

(B11)

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[24] CELLO coll., V.Savinov et al., these proceedings.
Fig. 1(a). The form factors of $\rho$ meson with parameter $\beta = 0.36$ GeV. The solid, dotted, and dashed lines correspond to $F_1(Q^2)$, $F_2(Q^2)$, and $F_0(Q^2)$, respectively. The dashed-dotted line is the result of non-relativistic limit of $F_2(Q^2)$ by turning off the McShan transformation.

Fig. 1(b). The transition form factor of $\rho^+ \rightarrow \pi^+ \gamma$ multiplied by $Q^4$ is shown as a function of $Q^2$. The solid lines are our results with $\beta = 0.32$ and 0.36 GeV. The dotted line, dashed, and dashed-dotted lines correspond to the results obtained using $\omega T$ in Ref. [34], the BS approach of Ref. [36] and the predictions of VDM model with $G_{\omega\gamma} = 1/(1 + Q^2/M_\rho^2)$.

Fig. 1(c). The transition form factor of $\omega \rightarrow \gamma \gamma$. Our result with $\beta = 0.36$ GeV(solid line) is compared with VDM with $G_{\omega\gamma} = 1/(1 + Q^2/M_\rho^2)$ and the pole fit of the experimental data(dotted line) [38].

Fig. 4. The normalized form factors of $K^{*+} \rightarrow K^{*}\gamma$(solid line) and $K^0 \rightarrow K^\gamma$(dotted line) transitions with the parameter $\beta = 0.36$ GeV. The dotted and dashed lines correspond to charged and neutral vector kaon transition form factors by Ref. [39]. The dashed-dotted line corresponds to the VDM prediction.

Fig. 5. The form factors of $A_1^* \rightarrow \pi^* \gamma$ transition with the parameter $\beta = 0.36$ GeV. $G_1(Q^2)$ and $G_2(Q^2)$/10 are our results represented by the solid and dashed lines, respectively. For comparison, we show also the QCD sum-rule results: $G_1(Q^2)$(dotted line) and $G_2(Q^2)$/10(dotted-dashed line).

Fig. 6. The decay rate for $\pi^0 \rightarrow \gamma \gamma$ transition with the parameter $\beta = 0.36$ GeV. Data are taken from Ref. [23,24].

Fig. 7. The decay rate for $\eta \rightarrow \gamma \gamma$ transition with the parameter $\beta = 0.36$ GeV. The solid and dotted lines correspond to the $\theta_{SU(3)} = -10^\circ$ and $-23^\circ$ mixing schemes, respectively. Data are taken from Ref. [23-25].

Fig. 8. The decay rate for $\rho^0 \rightarrow \gamma \gamma$ transition with the parameter $\beta = 0.36$ GeV. The solid and dotted lines correspond to the $\theta_{SU(3)} = -10^\circ$ and $-23^\circ$ mixing schemes, respectively. Data are taken from Ref. [23-25].
### TABLE I. Three different mixing schemes for η and η' and the corresponding spin averaged masses.

<table>
<thead>
<tr>
<th>$\theta_{SU(3)}$</th>
<th>$X_\eta$</th>
<th>$Y_\eta$</th>
<th>$X_{\eta'}$</th>
<th>$Y_{\eta'}$</th>
<th>$m_\eta [\text{MeV}]$</th>
<th>$m_{\eta'} [\text{MeV}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>$\sqrt{1/3}$</td>
<td>$\sqrt{1/3}$</td>
<td>$\sqrt{1/3}$</td>
<td>$\sqrt{1/3}$</td>
<td>842</td>
<td>885</td>
</tr>
<tr>
<td>-10°</td>
<td>$\sqrt{1/2}$</td>
<td>$\sqrt{1/2}$</td>
<td>$\sqrt{1/2}$</td>
<td>$\sqrt{1/2}$</td>
<td>843</td>
<td>884</td>
</tr>
<tr>
<td>-23°</td>
<td>0.85</td>
<td>0.53</td>
<td>0.53</td>
<td>0.85</td>
<td>834</td>
<td>873</td>
</tr>
</tbody>
</table>

### TABLE II. Decay widths for $V \rightarrow p_5 \gamma, p_5 \rightarrow \gamma \gamma, A_1 \rightarrow \pi \gamma$ and $p_5 \rightarrow 2\gamma$ with $p_5 = \pi, K, \eta, \eta'$ and $V = \rho, K^*, \omega, \phi$ for various model parameters $\beta$.

<table>
<thead>
<tr>
<th>$\beta$ (GeV)</th>
<th>0.32</th>
<th>0.34</th>
<th>0.36</th>
<th>0.38</th>
<th>ST*</th>
<th>Experiment*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma(\rho^0 \rightarrow \pi^0 \gamma)$</td>
<td>78</td>
<td>73</td>
<td>69</td>
<td>64</td>
<td>60 ± 8 [keV]*</td>
<td></td>
</tr>
<tr>
<td>$\Gamma(\omega \rightarrow \gamma \gamma)$</td>
<td>775</td>
<td>742</td>
<td>708</td>
<td>674</td>
<td>717 ± 51</td>
<td></td>
</tr>
<tr>
<td>$\Gamma(K^{*0} \rightarrow K^0 \gamma)$</td>
<td>69</td>
<td>57</td>
<td>53</td>
<td>50</td>
<td>50 ± 5</td>
<td></td>
</tr>
<tr>
<td>$\Gamma(K^{*0} \rightarrow K^0 \gamma)$</td>
<td>134</td>
<td>128</td>
<td>122</td>
<td>116</td>
<td>117 ± 10</td>
<td></td>
</tr>
<tr>
<td>$\Gamma(\rho \rightarrow \gamma \gamma)$</td>
<td>66[77]</td>
<td>60[70]</td>
<td>56[65]</td>
<td>51[60]</td>
<td>40</td>
<td>58 ± 10</td>
</tr>
<tr>
<td>$\Gamma(\omega \rightarrow \gamma \gamma)$</td>
<td>7.4[8.5]</td>
<td>6.8[7.9]</td>
<td>6.4[7.4]</td>
<td>6.0[6.8]</td>
<td>4.6</td>
<td>7.0 ± 1.8</td>
</tr>
<tr>
<td>$\Gamma(\gamma' \rightarrow \gamma \gamma)$</td>
<td>112[7.3]</td>
<td>10.4[6.6]</td>
<td>9.7[6.0]</td>
<td>9.1[5.6]</td>
<td>12.0</td>
<td>5.9 ± 0.9</td>
</tr>
<tr>
<td>$\Gamma(\phi \rightarrow \gamma \gamma)$</td>
<td>5.4[40]</td>
<td>58[42]</td>
<td>64[45]</td>
<td>65[47]</td>
<td>7.1</td>
<td>56.9 ± 2.9</td>
</tr>
<tr>
<td>$\Gamma(\phi \rightarrow \eta' \gamma)$</td>
<td>0.26[0.44]</td>
<td>0.27[0.44]</td>
<td>0.28[0.45]</td>
<td>0.29[0.46]</td>
<td>0.23</td>
<td>&lt;1.8</td>
</tr>
<tr>
<td>$\Gamma(A_1 \rightarrow \pi \gamma)$</td>
<td>620</td>
<td>664</td>
<td>705</td>
<td>742</td>
<td>640 ± 246 [44,45]</td>
<td></td>
</tr>
<tr>
<td>$\Gamma(\pi^0 \rightarrow 2\gamma)$</td>
<td>7.58</td>
<td>7.06</td>
<td>6.50</td>
<td>5.91</td>
<td>7.8 ± 0.5 [eV]</td>
<td></td>
</tr>
<tr>
<td>$\Gamma(\eta \rightarrow 2\gamma)$</td>
<td>0.61[0.78]</td>
<td>0.53[0.71]</td>
<td>0.47[0.63]</td>
<td>0.42[0.58]</td>
<td>0.44</td>
<td>0.47 ± 0.05</td>
</tr>
<tr>
<td>$\Gamma(\eta' \rightarrow 2\gamma)$</td>
<td>8.6[6.5]</td>
<td>8.2[6.1]</td>
<td>7.9[5.6]</td>
<td>7.3[5.1]</td>
<td>9.0</td>
<td>4.3 ± 0.6</td>
</tr>
</tbody>
</table>

* ST = standard mixing ($\theta_{SU(3)} = 0^\circ$) for $\beta = 0.36$ GeV.

* From Ref. [47], unless otherwise noted.

* The unit of decay width is [keV], unless otherwise noted.

* The values are the result from $\theta_{SU(3)} = -10^\circ [-23^\circ]$ mixing scheme.
Fig. 3(b)

\[ G_\text{eff}(Q^2)/G_\text{eff}(0) \]

- Our prediction
- VDM
- Experimental pole fit (Ref. [38])

\[ Q^2 [\text{GeV}^2] \]

0.0 0.2 0.4 0.6 0.8 1.0

10^0 10^{-1} 10^{-2} 10^{-3}

Fig. 4

\[ |G_{\text{eff}}(Q^2)/G_{\text{eff}}(0)| \]

- Our prediction
- VDM
- B–S
- \( K\bar{K} \)
- \( K\bar{K}' \)

\[ Q^2 [\text{GeV}^2] \]

0 2 4 6 8 10