Non-Trivial Extensions of the $3D$–Poincaré Algebra
and Fractional Supersymmetry for Anyons

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Abstract

Non-trivial extensions of the three dimensional Poincaré algebra, beyond the supersymmetric one, are explicitly constructed. These algebraic structures are the natural three dimensional generalizations of fractional supersymmetry of order $F$ already considered in one and two dimensions. Representations of these algebras are exhibited, and unitarity is explicitly checked. It is then shown that these extensions generate symmetries which connect fractional spin states or anyons. Finally, a natural classification arises according to the decomposition of $F$ into its product of prime numbers leading to sub-systems with smaller symmetries.

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In $D$-dimensional spaces, particles are classified by irreducible representations of the Poincaré algebra. This algebra generates the space-time symmetries (Lorentz transformations and space-time translations), and after one has gauged the space-time translations we naturally obtain a theory of gravity. Therefore, in order to understand the fundamental interactions and the symmetries in particle physics, it is interesting to study all the possible extensions of the Poincaré symmetry. Quantum Field Theory restricts considerably the possible generalizations. If one imposes the unitarity of the $S$-matrix with a discrete spectrum of massive one particle states, then within the framework of Lie algebras, the Coleman and Mandula theorem [1] allows only internal symmetries, i.e. those commuting with the generators of the Poincaré algebra$^3$. However, if we go beyond Lie algebras, we can escape this no-go theorem. The well-known supersymmetric extension is generated by fermionic charges which, by the Haag, Lopuszanski and Sohnius theorem, are in the spinorial representation of $SO(1,D-1)$ [2]. So, it seems that there exists a unique non-trivial extension of the Poincaré algebra, up to the choice of the number $N$ of supercharges. Indeed, according to the Noether theorem, all these symmetries correspond to conserved currents, and are generated by charges which are expressed in terms of the fields. By the spin-statistics theorem we have two kinds of fields having integer or half-integer spin. The former will close with commutators and the latter with anticommutators leading respectively to Lie and super-Lie algebras.

The consideration of algebraic extensions, beyond the Poincaré algebra, is not new. Such a possibility was considered in [3, 4]. In the second paper, Wills Toro showed that the generators of the Poincaré algebra might themselves have non-trivial indices. In this paper we pursue a different possibility, namely the study of special dimensions. Particular dimensions can reveal exceptional behaviour. This opportunity to find “particular” dimensions has already been exploited with success and has led to generalizations of supersymmetry. Fractional supersymmetry (FSUSY) which was introduced in [5], is one such generalization. In one-dimensional spaces, where no rotation is available, this symmetry is generated by one generator which can be seen as the $F^{th}$ root of the time translation $(Q_t)^F = \partial_t$. $F = 2$ corresponds to the usual supersymmetry. A group theoretical justification was then given in [6, 7] and this symmetry was applied in the world-line formalism [7]. The second peculiar cases, are the two-dimensional spaces where, by use of conformal transformations, the (anti)holomorphic part of the fields transforms independently [8]. In [9], this situation was exploited to build a Conformal Field Theory with fractional conformal weight. The Virasoro algebra was extended by two generators satisfying $(Q_z)^F = \partial_z$ and $(Q_{\bar{z}})^F = \partial_{\bar{z}}$ and besides the stress-energy tensor, a conserved current of conformal weight $(1 + \frac{1}{F})$ was obtained. Several groups have also studied this symmetry in one [10] and two dimensions [11].

$^3$In the massless case, the Poincaré group can be promoted to the conformal one.
Finally, in $1+2$ dimensions particles with arbitrary spin and statistics exist. The so-called anyons were defined for the first time in [12]. In fact, studying the representations of the $3D$–Poincaré algebra $P_{1,2}$ the unitary irreducible representations divide into two classes: massive or massless. For the massive particles, we can consider a one-dimensional wave function with arbitrary real spin $-s$ (i.e. which picks up an arbitrary phase factor $\exp(2i\pi s)$ when rotated through $2\pi$). In the massless case, only two types of discrete spin exist [13]. Then a relativistic wave equation for anyons was formulated following different approaches in [14, 15].

The purpose of this letter is to build non-trivial extensions of the Poincaré algebra which go beyond supersymmetry (SUSY). We first give a fractional supersymmetric extension of the Poincaré algebra of any order $F$. Then, we study the representations of this algebra which turn out to contain anyonic fields with spin $(\lambda, \lambda - \frac{1}{F}, \cdots, \lambda - \frac{F-1}{F})$ (in the simplest case and with $\lambda$ an arbitrary real number). We also explicitly check that the representations we are considering are unitary.

It then appears that $3D$–FSUSY, like in $2D$, is a symmetry which connects the fractional spin states previously obtained. In this sense it is a natural generalization of SUSY. We also prove that the algebras so-obtained can be classified according to the decomposition of $F$ into its product of prime numbers.

Introducing the generators of space-time translations $P^\alpha$ and the generators of Lorentz transformations $J^\alpha = \frac{1}{2} \eta^{\alpha\beta} \epsilon_{\gamma\delta} J^\gamma J^\delta$, we can rewrite the three dimensional Poincaré algebra as follows

\begin{align}
\left[ P^\alpha, P^\beta \right] &= 0 \\
\left[ J^\alpha, P^\beta \right] &= i \eta^{\alpha\gamma} \eta^{\beta\delta} \epsilon_{\gamma\delta\eta} P^\eta \\
\left[ J^\alpha, J^\beta \right] &= i \eta^{\alpha\gamma} \eta^{\beta\delta} \epsilon_{\gamma\delta\eta} J^\eta,
\end{align}

with $\eta_{\alpha\beta} = \text{diag}(1, -1, -1)$ the Minkowski metric and $\epsilon_{\beta\gamma\delta}$ the completely antisymmetric Levi-Civita tensor such that $\epsilon_{012} = 1$. Particles are then classified according to the values of the Casimir operators of the Poincaré algebra. More precisely, for a mass $m$ particle of positive/negative energy, the unitary irreducible representations are obtained by studying the little group leaving the rest-frame momentum $P^\alpha = (m, 0, 0)$ invariant. This stability group in $SO(1,2)$, the universal covering group of $SO(1,2)$, is simply the universal covering group $IR$ of the abelian sub-group of rotation $SO(2)$ (generated by $J^0$). As it is well-known, such a group is not quantized. This means that the substitution $J^0 \rightarrow J^0 + s$ leaves the $SO(2)$ part invariant. But the remarkable property of $SO(1,2)$, is that the concomitant transformation on the Lorentz boosts $J^J \rightarrow J^J + s \frac{P^\alpha}{m^2 + m}$ leaves the algebraic structure (1) unchanged. Anyway, following the method of induced representation for groups expressible as
a semi-direct product we find that unitary irreducible representations for a massive particles are one dimensional, and that the Lorentz generators are [14, 13] (for an arbitrary spin–s representation)

\[
\begin{align*}
J^0_s &= i \left( p^1 \frac{\partial}{\partial p_2} - p^2 \frac{\partial}{\partial p_1} \right) + s \\
J^1_s &= -i \left( p^2 \frac{\partial}{\partial p_0} - p^0 \frac{\partial}{\partial p_1} \right) + s \frac{p^1}{p^0 + m} \\
J^2_s &= -i \left( p^0 \frac{\partial}{\partial p_1} - p^1 \frac{\partial}{\partial p_0} \right) + s \frac{p^2}{p^0 + m},
\end{align*}
\]

with \( p^\alpha \) the eigenvalues of the operators \( P^\alpha \). This modification of the 3D Lorentz generators was pointed out in [16] and is not the most general one we can consider (see the last paper of [15]).

The main difference between \( SO(1,2) \), or more precisely the proper orthochronous Lorentz group, and \( SO(3) \) is that \( p^0 + m \) never vanishes with \( SO(1,2) \) and \( s \) does not need to be quantized.

In Ref.[14, 15], a relativistic wave equation for massive anyons was given. First, notice that the two Casimir operators are the two scalars \( P.P \) and \( P.J \) and their eigenvalues for a spin–s unitary irreducible representation are respectively \( m^2 \) and \( m s \). The equations of motion are then

\[
\begin{align*}
(P^2 - m^2) \Psi &= 0 \\
(P.J - s m) \Psi &= 0.
\end{align*}
\]

However, to obtain manifestly covariant equations one has to go beyond the mass-shell conditions (3) given by the induced representation. Therefore, we can start with a field which belongs to the appropriate spin–s representation of the full Lorentz group instead of the little group. When \( s \) is a negative integer, or a negative half-integer, this representation is not unitary and is \( 2|s| + 1 \) dimensional, and the solution of the relativistic wave equations reduces to the appropriate induced representation (see [13, 14] for an explicit calculation in the case \( |s| = 1, 1/2 \)). When \( s \) is an arbitrary number, the representation is infinite dimensional and belongs to the discrete series of \( SO(1,2) \) [17]. A relativistic wave equation for an anyon in the continuous series [17] was also considered in the third paper of [15]. Noting \( J_{s,\pm} = J^1_s \mp i J^2_s \) \( ([J^0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2 J^0) \) the Lorentz generators of the spin–s representation, and \( |s,n\rangle \) the states \( (n = 0, \ldots, \infty) \) we can build two spin–s representations; one bounded from below, noted \( D^+_s \)
\begin{align}
J^0_s|s_+,n⟩ &= (s + n)|s_+,n⟩ \\
J_{s,+}|s_+,n⟩ &= \sqrt{(2s + n)(n + 1)}|s_+,n + 1⟩ \\
J_{s,-}|s_+,n⟩ &= \sqrt{(2s + n - 1)n}|s_+,n - 1⟩, \\
\end{align}

and one bounded from above (\(D^+_s\))
\begin{align}
J^0_s|s_-,n⟩ &= -(s + n)|s_-,n⟩ \\
J_{s,+}|s_-,n⟩ &= -\sqrt{(2s + n - 1)n}|s_-,n - 1⟩ \\
J_{s,-}|s_-,n⟩ &= -\sqrt{(2s + n)(n + 1)}|s_-,n + 1⟩.
\end{align}

For both representations, the quadratic Casimir operator of the Lorentz group equals \(s(s-1)\). For the first representation we have \(J_{s,-}|s_+,0⟩ = 0\) and for the second \(J_{s,+}|s_-,0⟩ = 0\). Jackiw and Nair [14] and Plyushchay [15] were able to define an equation of motion (plus some subsidiary conditions) such that the solution of a spin\(-s\) anyonic equation decomposes into a direct sum of a positive energy solution in the representation bounded from below and a negative energy in the one bounded from above. In other words, a solution of a spin\(-s\) anyonic equation decomposes into a positive energy state of helicity \(h = s\) and a negative energy solution with \(h = -s\) : \(|s⟩ = |h = s, +⟩ \oplus |h = -s, −⟩\) and the two states are CP conjugate.

If \(s\) is a negative integer or a negative half-integer number we get a \(2|s| + 1\) dimensional representation, but for a general \(s\) we have an infinite number of states. Furthermore when \(s < 0\) the representation is non-unitary. Taking the spinorial representation as a guideline, we choose the case \(s = -1/F\) to build a non-trivial extension of the Poincaré algebra. If we observe the relations (5) and (4) with \(s = -1/F\), we see an ambiguity in the square root of \(-2/F\). So a priori we have four different representations for \(s = -1/F\), (two bounded from below/above) with the two choices \(\sqrt{-1} = \pm i\). We note \(D^{±}_{-1/F,±}\) (with obvious notations) these representations. Next, we can make the following identifications

- the dual representation of \(D^+_{-1/F,+}\) is obtained through the substitution \(J^a \rightarrow -(J^a)^t\) and is given by \(\left[D^+_{-1/F,+}\right]^* = D^-_{-1/F,+}\);
- the complex conjugate representation of \(D^+_{-1/F,+}\) is defined by \(J^a \rightarrow -(J^a)^*\) \(^4\) \(^5\) (we have to be careful when we do such a transformation because we have by definition \(J^\pm = J^1 \mp iJ^2,\) for any representation) is given by \(\overline{D^+_{-1/F,+}} = D^-_{-1/F,-}\).

\(^4\)In the mathematical literature because in the definition of Lie algebras there is no \(i\) factor –see equation (1)– we do not have a minus sign in the definition of this representation.

\(^5\)Note that, for a complex matrix \(X,\) \(X^*\) denotes the complex conjugate (and not the hermitian conjugate) matrix of \(X\); for a vector space \(V,\) \(V^*\) is its dual.
the dual of the complex conjugate representation of \( \mathcal{D}^{+}_{-1/F;+} \) is given by \( \left[ \mathcal{D}^{+}_{-1/F;+} \right]^* = \mathcal{D}^{+}_{-1/F;-} \).

If we note \( \psi_a \in \mathcal{D}^{+}_{-1/F;+}, \psi^a \in \mathcal{D}^{-}_{-1/F;+}, \bar{\psi}_a \in \mathcal{D}^{-}_{-1/F;-} \) and \( \bar{\psi}^a \in \mathcal{D}^{+}_{-1/F;-} \) then we have the following transformation laws:

\[
\psi'_a = S^b_a \psi_b \\
\psi'^a = \left(S^{-1}\right)^a_b \psi^b \\
\bar{\psi}'_a = \left(S^*\right)^b_a \bar{\psi}_b \\
\bar{\psi}'^a = \left((S^*)^{-1}\right)^a_b \bar{\psi}^b.
\]

Furthermore, if we define

\[
\psi^a = g^{a\bar{a}} \bar{\psi}_{\bar{a}},
\]

we can write the following scalar product

\[
\varphi^a \psi_a = -\varphi_0 \psi_0 + \sum_{a>0} \varphi_{\bar{a}} \psi_{a},
\]

where the infinite matrix \( g^{a\bar{a}} \) and its inverse \( g_{a\bar{a}} \) are given by \( \text{diag}(-1, 1, \cdots, 1) \). The reason why we have a pseudo-hermitian scalar product is because we are dealing with a non-unitary representation of a non-compact Lie group. The invariant scalar product gives an explicit isomorphism between the two representations bounded from below (or above) \( \left( (S^{-1})^a_b = g^{a\bar{a}} (S^*)^b_a g_{\bar{a}\bar{b}} \right) \).

From now on, we choose \( \sqrt{-2/F} = i \sqrt{-2/F} \) for representations bounded from below and \( \sqrt{-2/F} = -i \sqrt{-2/F} \) for those bounded from above.

Using the representations (4–5), and with the sign ambiguity resolved, we can define two series of operators, belonging to a non-trivial representation of the Poincaré algebra. We denote now \( \sqrt{-1} = i \). Note \( Q^+_{-1/F+n} \) those built from the representation bounded from below \( \mathcal{D}^+_{-1/F;+} \) and \( Q^-_{-1/F+n} \) the charges of the representation bounded from above \( \mathcal{D}^{-}_{-1/F;-} \). Using (5, 4) we get

\[
\begin{align*}
[J^0, Q^+_{-1/F+n}] &= (n - 1/F) Q^+_{-1/F+n} \\
[J^+, Q^-_{-1/F+n}] &= \sqrt{(-2/F + n)(n + 1)} Q^+_{-1/F+n+1} \\
[J^-, Q^+_{-1/F+n}] &= \sqrt{(-2/F + n - 1)n} Q^+_{-1/F+n-1} \\
[J^0, Q^-_{-1/F+n}] &= -(n - 1/F) Q^-_{-1/F+n}
\end{align*}
\]
\[
J_+, Q_{-1/F+n}^- = -\left(\sqrt{(-2/F + n - 1)n}\right) Q_{-1/F+n-1}
\]
\[
J_-, Q_{-1/F+n}^- = -\left(\sqrt{(-2/F + n)(n+1)}\right) Q_{-1/F+n+1}^-
\]

We want to combine this algebra (9) in a non-trivial way with the Poincaré algebra (1). With such a choice, \(Q_{-1/F+n}^\pm\) (resp. \(Q_{-1/F+n}^-\)) has a helicity \(h = -\frac{1}{F}\) (resp.) With the above choices for the square roots of the negative numbers we know that the representations are conjugate to each other i.e. \(Q_{-1/F+n}^+ \equiv Q_{-1/F+n}^-\).

Having set the values of \(s\), we have two reasons to close the algebra with the \(Q\)'s through a \(F\)-th order product. First of all, we would like the algebra to be a direct generalization of the one built in two-dimensions. Second, the charges we have introduced are in the spin \(-\frac{1}{F}\) representation of the Poincaré algebra, and so the \(Q\)'s pick up an \(\exp\left(-\frac{2\pi i}{F}\right)\) phase factor when rotated through \(2\pi\). They have a non-trivial \(\mathbb{Z}_F\) graduation, although the generators of the Poincaré algebra are trivial with respect to \(\mathbb{Z}_F\). The algebra splits then into an anyonic \(A\) and a bosonic \(B\) part. It can be written

\[
\{A, \ldots, A\}_F \sim B
\]
\[
[B, A] \sim A
\]
\[
[B, B] \sim B,
\]

with \(\{A_{s_1}, \ldots, A_{s_F}\}_F = \frac{1}{F!} \sum_{\sigma \in \Sigma_F} A_{i_{\sigma(1)}} \cdots A_{i_{\sigma(F)}}\) and \(\Sigma_F\) the permutation group with \(F\) elements. Equations (10) reveal the \(\mathbb{Z}_F\) structure of the algebraic extension of the Poincaré algebra we are considering. The bosonic part of the algebra is generated by \(J\) and \(P\) and has a graduation zero. The anyonic generators are the supercharges \(Q^\pm\) and have graduation \(\mp 1\) in \(\mathbb{Z}_F\). To close the algebra, both sides of the equation have to have the same graduation, justifying (10). In the case of the supersymmetric extension of the Poincaré algebra, (10) corresponds to a \(\mathbb{Z}_2\)-graded Lie algebra or a superalgebra.

Now, we want to identify the whole algebraic extension of \(P_{1,2}\). Part of this algebra is known (see eqs.(1) and (9)). Using adapted Jacobi identities, we calculate the remaining part of the algebra, and justify the use of a completely symmetric product in (10). Those involving three bosonic fields or two bosonic and one anyonic fields are the same as for superalgebras. Using the Leibniz rule of \(B\) with \(\{\ldots\}_F\) we get the third Jacobi identity and the last one is obtained by a direct calculation

\[
[[B_1, B_2], B_3] + [[B_2, B_3], B_1] + [[B_3, B_1], B_2] = 0
\]
\[
[[B_1, B_2], A_3] + [[B_2, A_3], B_1] + [[A_3, B_1], B_2] = 0
\]

6
\[ [\mathcal{B}, \{ A_1, \ldots, A_F \}]_F = \{ [\mathcal{B}, A_1], \ldots, [\mathcal{B}, A_F] \}_F + \ldots + \{ A_1, \ldots, [\mathcal{B}, A_F] \}_F \] (11)

\[ \sum_{i=1}^{F+1} \{ A_i, \{ A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_F \} \}_F = 0. \]

In order to identify the whole algebraic structure of the non-trivial extension of the Poincaré algebra, assume, as a first step, \([A_1, \cdots, A]_F = \alpha P + \beta J\) with \([\cdots]_F\) a symmetric product of charges to be defined. If we use the third Jacobi identity with \(B = P\), we obtain \(\beta = 0\) (\([P,Q] = 0\)), the same Jacobi identity with \(B = J^0\) proves that both sides of the equation have the same helicity. In other words, this equation just tells us that we need to build a mapping from a sub-space of \(S^F(D_{-1/2}^\pm)\) to the vectorial \((P)\) representation of \(SO(1,2)\) which is equivariant for the action of \(SO(1,2)\).

Now, we remark that there are primitive states in \(S^F(D_{-1/2}^\pm)\) from which we are able to construct the vector representation of \(SO(1,2)\):

\[ \begin{bmatrix} J^0, (Q^\pm_{-1/2})^F \end{bmatrix} = \mp (Q^\pm_{-1/2})^F \]
\[ \begin{bmatrix} J_{\mp}, (Q^\pm_{-1/2})^F \end{bmatrix} = 0 \]

From these relations, it follows that the sub-space

\[ D_{-1} = \left\{ (Q^\pm_{-1/2})^F, \left[ J_{\pm}, \left( Q^\pm_{-1/2} \right)^F \right], \left[ J_{\pm}, \left( Q^\pm_{-1/2} \right)^F \right] \right\} \]

of \(S^F(D_{-1/2}^\pm)\) is isomorphic to the vector representation of the Poincaré algebra. Note that this relations also imply that \(\begin{bmatrix} J_{\pm}, \left[ J_{\pm}, \left( Q^\pm_{-1/2} \right)^F \right] \end{bmatrix} = 0\).

So, we obtain the following algebra (we have to be careful with the normalization appearing in the bracket \(\cdots\), for instance \(\left( Q^\pm_{1/2} \right)^F \)).

\[ \left\{ Q^\pm_{-1/2}, \ldots, Q^\pm_{-1/2} \right\}_F = P_+ \]
\[ \left\{ Q^\pm_{1/2}, \ldots, Q^\pm_{1/2}, Q^\pm_{1/2} \right\}_F = \pm i \sqrt{\frac{2}{F}} P^0 \]
\[ -(F-1) \left\{ Q^\pm_{-1/2}, \ldots, Q^\pm_{-1/2}, Q^\pm_{1/2} \right\}_F \pm i \sqrt{F-2} \left\{ Q^\pm_{-1/2}, \ldots, Q^\pm_{2/2} \right\}_F = P_+ \]

\[ \]
with \( P_\pm = P^1 \mp iP^2 \). The normalization of the R.H.S. of eq.(13) comes from the definition of the bracket \{\cdots\}_F and (1,9). Now, we can address the question of the remaining brackets? In fact, it is impossible to find a decomposition\(^6\)

\[
S^F \left( D_{-1/F}^\pm \right) = D_{-1} \oplus V,
\]

where \( V \) is stable under \( SO(1,2) \). Indeed, if there were such a decomposition there would be a \( SO(1,2) \) equivariant projection

\[
\pi: S^F \left( D_{-1/F}^\pm \right) \rightarrow D_{-1}.
\]

But then \( X_\pm = \pi \left( S^F \left( Q_{-1/F}^\pm, \cdots, Q_{-3/F}^\pm \right) \right) \in D_{-1} \) satisfies (see 9)

\[
[J_\mp, [J_\mp, [J_\mp, X_\mp]]] = \pm i \sqrt{2/F} \sqrt{2(1-2/F)} \sqrt{3(2-2/F)} P_- \neq 0,
\]

and this is impossible because in the vector representation \( D_{-1} \), \( J^3 \) acts as zero.

Finally, we can note that direct calculation easily shows that equations (13) are stable under hermitian conjugation.

In this family of algebras, noted \( FSP_{1,2} \) if we take \( F = 2 \) we are in an exceptional situation. First, instead of having an infinite number of charges we have only two. Secondly, the two representations \( Q^\pm \) are equivalent whereas the two series of charges are inequivalent representations of \( SO(1,2) \) when \( F \neq 2 \). In the case \( F = 2 \), with one series of supercharges \( Q \) we obtain the well-known supersymmetric extension of the Poincaré algebra, and (9), (13) can be easily rewritten with the Pauli matrices. For more details on this algebra, one can see, for example, the book of Wess and Bagger [19]. The algebra we have obtained is then a direct generalization of the super-Poincaré one. It is remarkable that the supersymmetric algebra, which can be generalized easily in one and two-dimensional spaces, can also be considered in \( 1 + 2 \) dimensions. This is a consequence of the special feature of \( SO(1,2) \) which allows to define states with fractional statistics, \textit{i.e.} anyons. If we try to go beyond, and to build an extension of SUSY for higher dimensional spaces, one immediately faces an obstruction. Indeed, when \( D \geq 4 \) one just has bosonic or fermionic states and supersymmetry is the unique non-trivial extension of the Poincaré algebra one can build.

Finally, let us mention that, the similarity of the algebra (13) and the SUSY algebra does not stop at this point. If one considers now \( N \) series of charges \( Q^+ \) and \( Q^- \) we obtain, as in SUSY, algebraic extensions with central charges.

\(^6\)We thank the referee for pointing this to us.
Before studying the representations of the algebra (13) we can address some general properties. First, \( P^2 \) commutes with all the generators so that all states in an irreducible representation have the same mass. Secondly, if we define an anyonic-number operator \( \exp(2i\pi N_A) \) which gives the phase \( e^{2i\pi s} \) on a spin\(-s\) anyon we have \( \text{tr} \exp(2i\pi N_A) = 0 \) showing that in each irreducible representation there are \( F \) possible statistics \((s, s-\frac{1}{F}, \ldots, s-\frac{F-1}{F})\), where \( s \) will be specified later) and the dimension of the space with a given statistics is always the same. This can be checked proving by that \( \exp(2i\pi N_A) Q_s = e^{2i\pi s} Q_s \exp(2i\pi N_A) \) and using cyclicity of the trace

\[
\text{tr} \left( \exp(2i\pi N_A) \left\{ Q^+_1, \ldots, Q^+_1, Q^+_{1-F} \right\} \right) = 1/F \times \sum_{a=0}^{F-1} e^{2i\pi a/F} \left( Q^+_1 \right)^a \left( Q^+_{1-F} \right) = 0.
\]

Of course because we are dealing with infinite dimensional algebras the construction of the trace should be done with care. However, we will explicitly see, by constructing the unitary representations, that \( \text{tr} \exp(2i\pi N_A) = 0 \).

Having defined the anyonic extensions of the Poincaré algebra, we now look at the massive representations of (1), (9) and (13). Up to now we have written the algebra in such a way that there is still one ambiguity: we do not know whether we can choose an algebraic extension of the Poincaré algebra using only one series of supercharges \((Q^+_r, Q^-_r)\) or whether we need both. In fact the unitarity of the representation will force us to take both simultaneously. Let us first concentrate on the case where one series of supercharges is involved, say \( Q^+_r \). For the Poincaré as well as for its supersymmetric extension, the irreducible representations are obtained, using the Wigner method of induced representation. Then, the massive representations \( \rho^a p_a = m^2 \) are constructed by studying the sub-algebra leaving the rest-momentum \( p^a = (m,0,0) \) invariant. Similarly, within the framework of the FSUSY algebras, the one particle-states are characterised by the eigenvalue of the rotation in the \((x^1, x^2)\) plane \( i.e. \) by the helicity. In other words, all the representations are obtained by studying the sub-algebra where \( P_\pm, J_\pm \) are set to zero. On the level of the charges, a similar assumption will be made (valid only on shell): if we are looking at eqs.(13) only one fundamental bracket does not vanish, \( i.e. \) the one involving \((F-1)\) times the charge \( Q^+_{1-F} \) and the one involving \( Q^+_{1-F} \) once. All brackets involving \( Q^+_{n-F} \) with \( n > 1 \) acts trivially on the rest-frame states (the R.H.S. always vanishes), so those charges can be represented by 0 (this is not a new feature and this already appears in usual SUSY, and for instance in four dimensions, in the massless case, the surcharges \( Q_2 \) and \( Q_2 \) vanish)). After an appropriate
normalization (13) becomes

\[ \left\{ Q_{-\frac{1}{F}}^+, \ldots, Q_{-\frac{1}{F}}^+, Q_{1-\frac{1}{F}}^+ \right\}_F = 1/F \]
\[ \left\{ Q_{s_1}^+, \ldots, Q_{s_F}^+ \right\}_F = 0, \quad i_1, \cdots, i_F = -1/F, 1 - 1/F \quad \text{and} \quad i_1 + \ldots + i_F \neq 0. \]

Let us stress some properties of the algebras defined by (16). This kind of algebra is known to mathematicians, and is called the Clifford algebra of the polynomial \( x^{F-1}y \) [20]. Indeed, using (16) we obtain (developing explicitly the \( F^{th} \) power) \( (xQ_{-\frac{1}{F}} + yQ_{1-\frac{1}{F}})^F = x^{F-1}y \). Hence, the algebra generated by the two charges \( Q_{-\frac{1}{F}} \) and \( Q_{1-\frac{1}{F}} \) is associated with the linearization of the polynomial \( x^{F-1}y \) and constitute a generalization of usual Clifford algebras. This procedure can be considered for any polynomial. However, this algebra does not admit a finite dimensional faithful representation. This means that, using a faithful representation, we are able to build representations with an infinite number of states. It was shown in [23] that the Clifford algebra of a polynomial of degree greater than 2 admits a non-trivial, finite but not faithful representation. For \( F = 2 \), the situation is slightly different because Clifford algebras admit a finite dimensional faithful representation in terms of the Dirac \( \gamma \)-matrices. Because we want to have a representation which contains a finite number of states, we consider non-faithful representations.

An extensive study of the representations of Clifford algebras of cubic polynomials was undertaken by Revoy [22] and a family of representations can be obtained. This result can be generalized for \( F \geq 4 \). To obtain the irreducible representations for arbitrary \( F \) we first observe that \( F \) is the first power of \( Q_{-\frac{1}{F}} \) which is equal to zero (in other words the rank of \( Q_{-\frac{1}{F}} \) is \( F - 1 \)). Indeed, if one assumes \( Q_{-\frac{1}{F}}^{F-b} = 0 \) (with \( b > 1 \)), and multiplies the first equation of (16) by \( Q_{-\frac{1}{F}} \) on the left and \( Q_{1-\frac{1}{F}}^{F-b-2} \) on the right, one gets a contradiction. Using the Jordan decomposition and the property that all eigenvalues of \( Q_{-\frac{1}{F}} \) are zero, we can write

\[ Q_{-\frac{1}{F}}^+ = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_{1-\frac{1}{F}}^+ = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}. \] (17)

The matrix representation of \( Q_{1-\frac{1}{F}} \) has been obtained, solving (16). When \( F = 3 \) the matrix given in (17) for \( Q_{1-\frac{1}{F}} \) is not the only possibility [22], and probably other representations can be obtained when \( F \geq 4 \). However, the matrices given in (17) are the only ones

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\( ^7 \)This property was pointed out to us by Ph. Revoy.
consistent with the Poincaré algebra: if some of the matrix elements which are equal to zero in (17) are different from zero, we obtain equations where both sides do not have the same helicity (see below). Finally, using the property that the dimensions of the representations of Clifford algebras are a multiple of the degree of the polynomial [24] \((F\) in this case), by similar arguments we can prove that the other representations are reducible and are built with the two matrices given in (17).

However, the matrices exhibited are not convenient to prove that the representations of the FSUSY algebra are unitary. Indeed, we need quadratic relations upon the matrices \(Q^+\) and \(Q^- = (Q^+)^\dagger\). So, instead of the two \(Q\)'s given on (17) (and their hermitian conjugate matrices) we would prefer more suitable matrices obtained after a rescaling. At least two interesting solutions have been found (the second was suggested by the referee)

\[
Q^+_{1-\frac{1}{F}} = \begin{cases}
0 & 0 & \ldots & 0 & 0 \\
\sqrt{[1]} & 0 & \ldots & 0 & 0 \\
0 & \sqrt{[2]} & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & \sqrt{[F-1]} & 0
\end{cases}
\quad Q^+_{-\frac{1}{F}} = \begin{cases}
0 & 0 & \ldots & 0 & \{\sqrt{[F-1]}\}^{-1} \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 1/\sqrt{[F-1]} \\
0 & 0 & \ldots & 0 & 0 
\end{cases}
\]

(18)

with \([a] = \frac{q^{-a/2} - q^{a/2}}{q^{1/2} - q^{-1/2}},\ [F - 1]\ = [F - 1][F - 2] \cdots [2][1]\) and \(q = \exp(2i\pi/F)\). Of course the three sets of matrices given in (17) and (18) are related by a conjugation transformation (or a rescalling of the vectors which belong to the representation –see after–).

From the basic conjugation we obtain immediately the associated representation for the \(Q^-\) charges

\[
Q^-_{1-\frac{1}{F}} = \left(Q^+_{1-\frac{1}{F}}\right)^\dagger \quad (Q^-_{-\frac{1}{F}} = \left(Q^+_{-\frac{1}{F}}\right)^\dagger)
\]

There are two consequences of the exhibited representations.

1. A direct calculation shows that the two charges \(Q^+_{-1/F}\) and \(Q^-_{-1/F}\) satisfy quadratic relations.

   (a) In the case of the first series we obtain the \(q\)-oscillator algebra introduced by Biedenharn and Macfarlane [25]

   \[
   Q^-_{-1/F}Q^+_{-1/F} - q^{\pm 1/2}Q^+_{-1/F}Q^-_{-1/F} = q^{\pm N/2}
   \]

11
\[ [N, Q^\pm_{-1/F}] = Q^\pm_{-1/F} \]

\[ [N, Q^-_{-1/F}] = -Q^-_{-1/F}, \]

with \( N = \text{diag}(0, 1, \cdots, F-1) \) the number operator (which can be expressed with \( Q^\pm_{-1/F} \)).

(b) For the second choice we have

\[ [Q^-_{-1/F}, Q^\pm_{-1/F}] = N = \text{diag}(F-1, F-3, \cdots, 1-F) \quad (21) \]

\[ [N, Q^\pm_{-1/F}] = \pm 2 Q^\pm_{-1/F}, \]

showing that the \( Q \) generate the \( F \)-dimensional representation of \( sl(2, \mathbb{R}) \).

Among those two matrix representation of the FSUSY algebra (and eventually others) we were not able to find arguments to select one rather the other i.e. to obtain naturally and independently of any matrix realization a quadratic relation among \( Q^\pm_{-1/F} \) and \( Q^-_{-1/F} \) which characterizes the structure of the FSUSY algebra. Some indications in this direction should be given. We can first notice the property that the usual superspace construction of SUSY, by the help of Grassmann variables, can be generalized, and an adapted version has already been built within the framework of FSUSY, at least when \( D = 1, 2 \) \[7, 9, 10, 11\]. Secondly, we can observe that the quantization of the algebra generated by \( Q^\pm_{-1/F} \) and its conjugate \( Q^-_{-1/F} \) (variables fulfilling \( \theta^F = 0 \) and generalizing the well-known Grassmann variables) is related with the \( q \)-deformed Heisenberg algebra \[26\]. In other words we might have relations like \( Q^+_{-1/F} \sim \theta, \)

\( Q^-_{-1/F} \sim \partial_\theta \) and \( \partial_\theta \theta - q \theta \partial_\theta \sim 1 \). Furthermore it is known that the algebra generated by \( \theta \) and \( \partial_\theta \) is equivalent to the \( q \)-oscillators \[25\]. These two remarks are surely related and can be compared with the fact that the quantization of the Grassmann algebra is the Clifford algebra.

As a consequence, the representation built with the \( Q \)'s is unitary. Indeed, the quadratic relations (20) or (21) enable us to prove that the norm of the vector \( (Q^+_{-1/F})^n |0 \rangle \), with \( n = 0, \cdots, F-1 \) and \( |0 \rangle \) the primitive vector on which the representation span by \( Q^\pm_{-1/F} \) is built, is positive. This result can be obtained even more simply, using the results of the \( q \)-oscillators for the first series \[25\], or by proving that the matrices given in (21) can be mapped to the \( F \times F \) hermician matrices of \( SU(2) \), which generate unitary representation (see after). The deep reason for the emergence of a quadratic structure is the non-faithfulness of the representation. Indeed, relations (16) are not strong enough to order the monomials in such a way that, say \( Q^+_{-1/F} \), is always on the left of \( Q^+_{1-1/F} \), and the number of monomials increase with their degree. If we have a finite-dimensional representation then it means that we have obtained quadratic relations: this allows us to order the monomials.
2. We can observe directly that
\[ Q^{+}_{1-1/F} = \frac{1}{[F-1]!} \left( Q^{-}_{-1/F} \right)^{F-1}, \]
for the first choice, and
\[ Q^{+}_{1-1/F} = \frac{1}{((F-1)!)^2} \left( Q^{-}_{-1/F} \right)^{F-1} \]
for the second. Because of this constraint, the \( Q^{\pm}_{-1/F} \) alone span the representation of the FSUSY algebra.

We note that the representations built with the matrices \( Q_{-1/F} \) and \( Q_{1-1/F} \) can be obtained in a way similar to the way one obtains representations of SUSY [27]. We start with a vacuum \( \Omega_\lambda \) in the spin\( -\lambda \) representation of \( SO(1,2) \). On-shell, using the results established in [14, 15], we have the following decomposition
\[ \Omega_\lambda = \Omega^+_{h=\lambda} \oplus \Omega^-_{h=-\lambda}, \]
with two states of helicity \( \pm \lambda \) and positive/negative energy. These two vacua are \( CP \)-conjugate and allow us to build a \( CP \)-invariant representation. This constraint of \( CP \) invariance is very strong, because as soon as we have chosen the representation built from \( \Omega_{h=\lambda,+} \), the one built from \( \Omega_{h=-\lambda,-} \) is not arbitrary. Altogether, with (18) and (19) we get the representation \( (Q^{-}_{-1/F}\Omega_{h=\lambda}=0, Q^{+}_{-1/F}\Omega_{h=-\lambda}=0, \) and for our normalization we have chosen the first choice for the \( Q \)'s)

<table>
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<tbody>
<tr>
<td>( \Omega^+_{\lambda} )</td>
<td>( \lambda )</td>
<td>( \Omega^-_{-\lambda} )</td>
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</tr>
<tr>
<td>( Q^{+}<em>{-1/F}\Omega</em>{\lambda} )</td>
<td>( \lambda - 1/F )</td>
<td>( Q^{-}<em>{-1/F}\Omega^-</em>{-\lambda} )</td>
<td>( -\lambda + 1/F )</td>
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<tr>
<td>( \cdots )</td>
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</tr>
<tr>
<td>( \frac{(Q^{+}<em>{-1/F})^a}{\sqrt{[a]!}}\Omega^+</em>{\lambda} )</td>
<td>( \lambda - a/F )</td>
<td>( \frac{(Q^{-}<em>{-1/F})^a}{\sqrt{[a]!}}\Omega^-</em>{-\lambda} )</td>
<td>( -\lambda + a/F )</td>
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</tr>
<tr>
<td>( \frac{(Q^{+}<em>{-1/F})^{F-1}}{\sqrt{[F-1]!}}\Omega^+</em>{\lambda} )</td>
<td>( \lambda - (F-1)/F )</td>
<td>( \frac{(Q^{-}<em>{-1/F})^{F-1}}{\sqrt{[F-1]!}}\Omega^-</em>{-\lambda} )</td>
<td>( -\lambda + (F-1)/F )</td>
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The states of positive energy and helicity \( (\lambda, \lambda - \frac{1}{F}, \ldots, \lambda - \frac{F-1}{F}) \) are \( CP \)-conjugate to the states of negative negative energy and helicity \( (-\lambda, -\lambda + \frac{1}{F}, \ldots, -\lambda + \frac{F-1}{F}) \), and following the remarks given here above it is known that the representation is unitary. An interesting consequence of the second choice for the \( Q \)-matrices is the fact that the representation
of the FSUSY algebra belong to a $F$–dimensional representation of $SU(2)$. Indeed, it is easy to check that the matrices $K_1 = 1/2 \left( Q_{-1/F}^+ + Q_{-1/F}^- \right), K_2 = i/2 \left( Q_{-1/F}^+ - Q_{-1/F}^- \right)$ and $K_3 = N/2$ are unitary and generate the $SU(2)$ algebra.

Hence, FSUSY is a direct generalization of SUSY in the sense that these fractional spin states or anyons are connected by FSUSY transformations. The next step would be to construct explicitly a Lagrangian invariant under a FSUSY transformation which mixes these states, as has been done in one and two dimensions [5, 6, 7, 9, 10, 11]. As a starting point, one could use the lagrangian formulation of anyonic fields given in [14, 15].

To conclude this general study of the algebra, it is of great interest to mention some properties when $F$ is not a prime number. Assuming $F = F_1 F_2$, we have $F_1 SP_{1,2} \subset FSP_{1,2}$. This property was already observed in two dimensions in the second paper of [9]. So, this inclusion (which can also be proven in one dimension) is a general property of FSUSY and does not depend on the dimension. To prove this statement, we focus on the case where we have only the $Q^+$ charges and we omit the $+$ superscript.

If we define $\left( Q_{-\frac{1}{F}}^- \right)^F = Q_{-\frac{1}{F_1}}^-$, using the algebra we can build, from the spin$-\frac{1}{F}$ representation, a spin$-\frac{1}{F_1}$ representation of $SO(1, 2) : Q_{n-\frac{1}{F_1}} \sim \left[ J_+, \ldots, J_+, Q_{-\frac{1}{F_1}}^- \right], \ldots$ where $J_+$ has been applied $n$–times. Using the Jacobi identities (11), we can construct an algebraic generalization of (13) which mixes the spin$-\frac{1}{F}$ and spin$-\frac{1}{F_1}$ anyonic operators.

The case where $F$ is an even number is special because the spin$-1/2$ representation is finite, so we have the same constraints as before for (13). From these inclusions of algebras, we are able to build sub-algebras with smaller symmetries when $F$ is not a prime number. In such a situation, the $F$–multiplet of $FSP_{1,2}$ splits into $F_2$ $F_1$–multiplets of $F_1 SP_{1,2}$

$$\Phi^{(F)}_{\lambda} = \bigoplus_{a=0}^{F_2-1} \Phi^{(F_1)}_{\lambda+\frac{a}{F}}.$$ 

The $F_1$–multiplet $\Phi^{(F_1)}_{\lambda+\frac{a}{F}}$ is built from the vacuum $\Omega_{\lambda+\frac{a}{F}}$. This can be checked directly from the definitions and using the representations ) or the matrices (18) and (17).

In this letter, we have explicitly constructed non-trivial algebraic extensions of the 3D Poincaré algebra that go beyond the supersymmetric ones. The study of their representations enables us to show that these symmetries connect the fractional spin states given in (17-18). We have pointed out an interesting classification of these algebras by means of the decomposition of $F$ (the order of FSUSY) as a product of prime numbers. This leads to sub-systems with smaller symmetries. A first application of these algebras, would be to build a Lagrangian formulation where FSUSY, among anyonic fields, is manifest. This could lead to some generalizations of the well known Wess-Zumino model [28]. A further application would be to gauge FSUSY along the lines given in [7], after having studied the massless representations of the algebra (1),(9) and (13).
Recently, a very interesting interpretation of supersymmetry and fractional supersymmetry in one dimension was given as an appropriate limit of the braided line [29]. Is it possible to understand, along these lines, how supersymmetry and fractional supersymmetry emerge in two and three dimensions and to prove that when the dimension is higher than three only SUSY is allowed?

Finally, it should be interesting to understand the consequences of the FSUSY extensions of the Poincaré algebra, in relation with three dimensional physics.

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References


