NLO CONFORMAL SYMMETRY IN THE REGGE LIMIT
OF QCD

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Abstract

We show that a scale invariant approximation to the next-to-leading order
BFKL kernel, constructed via transverse momentum diagrams, has a simple
conformally invariant representation in impact parameter space i.e.

\[ \tilde{K}(\rho_1, \rho_2, \rho_1', \rho_2') = g^4 N_c^2 \ln^4 \left[ \frac{|\rho_1 - \rho_1'|}{|\rho_1 - \rho_2|} \frac{|\rho_2 - \rho_2'|}{|\rho_2 - \rho_1'|} \right] \]

That a conformally invariant representation exists is shown first by relating
the kernel directly to Feynman diagrams contributing to two photon diffractive
dissociation.

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In the leading-log approximation, the Regge limit of QCD is described by the BFKL equation[1] and the resulting “BFKL Pomeron”. It is well-known that the forward limit ($q \to 0$) of the BFKL equation describes the small-$x$ evolution of parton distributions. An important property of the full equation is that, in impact parameter space, it is invariant under special conformal transformations. In next-to-leading-order (NLO) this property is expected to be lost as scale dependence enters the equation. The direct evaluation of NLO contributions to the BFKL equation is now close to completion[2].

It is nevertheless attractive to suppose that conformal symmetry could play a vital role in solving the full dynamical problem of the QCD Pomeron. It would be encouraging, if this is to be the case, to see that key properties of higher-order contributions are related to conformally symmetric interactions. Indeed it was conjectured in [4] that this could be the role of $t$-channel unitarity in determining higher-order contributions. The results of this paper are, perhaps, a first step in the right direction. That is we show explicit conformal symmetry for an interaction, derived first by $t$-channel arguments, that we anticipate to be an infra-red approximation to the NLO kernel of the BFKL equation.

Both the leading-order BFKL kernel and the NLO approximation (which we refer to as the $O(\alpha_s^4)$ kernel to indicate that no scale-dependence appears in the gauge coupling) have been derived by a reggeon diagram technique based only indirectly on $t$-channel unitarity[3]. More recently it has been shown[4] that, in part at least, these results can be obtained by a direct analysis of the $t$-channel unitarity equations, analytically continued in the complex $j$-plane. The same analysis, however, also shows that the overall normalization of the $O(\alpha_s^4)$ kernel depends on a rapidity scale (cut-off) which cannot be fixed directly from the analysis. A similar observation was made by Kirschner[5] who has discussed how the $O(\alpha_s^4)$ kernel emerges as an approximation when non-leading results are obtained using the leading-order $s$-channel multi-Regge effective action. It remains to directly connect this kernel to the explicit NLO calculations and so determine the missing normalization factor.

The $O(\alpha_s^4)$ kernel is expressed in terms of two-dimensional transverse momentum integrals that are directly scale invariant in transverse momentum space. At leading-order this scale invariance property leads to conformal invariance in impact parameter space. The main result of this paper is to show that the $O(\alpha_s^4)$ kernel actually has a remarkably simple representation in impact parameter space which is manifestly conformally symmetric. The kernel $\tilde{K}$ connects two initial points ($\rho_1, \rho_2,$)
to two final points \((\rho_1', \rho_2')\) and has the representation

\[
\tilde{K}(\rho_1, \rho_2, \rho_1', \rho_2') = g^4 N^2 \ln^4 |R|
\]

where

\[
R = \frac{|\rho_1 - \rho_1'| |\rho_2 - \rho_2'|}{|\rho_1 - \rho_2'| |\rho_2 - \rho_1'|}
\]

The above representation was actually found by realizing, as we will show below, that the same \(O(g^4)\) kernel can be found in a rather different context. In double diffractive dissociation of two virtual photons the color zero exchange is modeled by two t-channel gluons which after taking the square of the amplitude become four gluons. In this situation it is known how to impose conformal symmetry and so obtain a conformally symmetric interaction. The important result is that the diagrams with four gluons can be rearranged in a way to be identical with those occurring in the \(O(g^4)\) kernel. Here the overall normalization is actually fixed, but the whole contribution is of next-to-next-to-leading order (NNLO) and does not interfere with NLO BFKL-corrections.

Previously it had been shown\[6\] that, in momentum space, the forward \(O(g^4)\) kernel splits naturally into two components. A part proportional to the square of the \(O(g^2)\) BFKL kernel, and a new component that has an eigenvalue spectrum sharing many properties of the leading-order spectrum, in particular holomorphic factorization. Holomorphic factorization is closely related to conformal symmetry and also the square of the \(O(g^2)\) kernel has conformal properties. Therefore it was anticipated that the complete \(O(g^4)\) kernel could have some conformal symmetry property. The simplicity of (1.1) is nevertheless surprising.

The outline of the paper is as follows. We begin, in Section 2, by recalling the form of the \(O(g^4)\) kernel in momentum space. In Section 3 we discuss the construction of the kernel from Feynman diagrams contributing to virtual photon-photon scattering. This construction automatically gives a conformally invariant result, the interesting feature is that the previously determined \(O(g^4)\) kernel emerges. Section 4 is a brief discussion of how Ward identity properties and other features of the BFKL equation translate into impact parameter space. The direct evaluation of the \(O(g^4)\) kernel in impact parameter space is carried out in Section 5 and the above result obtained. In Section 6 we discuss operators corresponding to general powers of \(R\) and suggest obtaining the spectra of such operators from a generating function. We also give an intriguing formal representation of the BFKL kernel involving \(\ln^3 R\) and \(\ln^4 R\). We conclude with a brief summary of the results of the paper.
2. THE $O(g^4)$ KERNEL IN MOMENTUM SPACE

It will be helpful for our discussion to introduce transverse momentum dia-
grams, which we construct using the components illustrated in Fig. 2.1.

\[ \begin{array}{cc}
\begin{array}{cc}
\text{(a)} & \text{(b)} \\
\end{array}
\end{array} \]

Fig. 2.1 (a) vertices and (b) intermediate states in transverse momentum.

The rules for writing amplitudes corresponding to the diagrams are the following

- For each vertex, illustrated in Fig. 2.1(a), we write a factor
  \[ 16\pi^3 \delta^2(\sum k_i - \sum k'_i)(\sum k_i)^2 \]

- For each intermediate state, illustrated in Fig. 2.1(b), we write a factor
  \[ \Gamma^k_n = (16\pi^3)^{-n} \int d^2k_1...d^2k_n / k_1^2...k_n^2 \]

We denote dimensionless amplitudes with a hat and remove the hat to denote
the corresponding amplitude with the momentum conservation $\delta$-function removed, e.g. for a 2-2 kernel $\hat{K}_{2,2}$

\[ \hat{K}_{2,2}(k_1, k_2, k_1', k_2') = 16\pi^3 \delta^2(k_1 + k_2 - k_1' - k_2)K_{2,2}(k_1, k_2, k_1', k_2') \]

The dimensionless amplitudes are formally scale-invariant but, in general, are infra-
red divergent. The cancellation of such divergences is an essential pre-requisite for
defining a scale-invariant kernel.

In this paper we will be concerned with kernels that are symmetrized (or
antisymmetrized) with respect to initial and final momenta. The diagrammatic rep-
resentation of $\hat{K}_{2,2}^{(2)}$, the $O(g^2)$ symmetrized non-forward BFKL kernel, is as shown in
Fig. 2.2.
Fig. 2.2 Diagrammatic representation of $\hat{K}^{(2)}_{2,2}$.

The summation sign implies a sum over permutations of both the initial and final momenta.

The diagrammatic representation of the symmetric $O(g^4)$ kernel $K^{(4)}_{2,2}$ (obtained by considering the contribution of 4-particle nonsense states to the $t$-channel unitarity equations) is shown in Fig. 2.3. (The values of $a$ and $b$ will be discussed shortly.)

$$\sum \frac{1}{2} \left( a \begin{array}{c} \circlearrowleft \end{array} + b \begin{array}{c} \circlearrowleft \end{array} - \begin{array}{c} \circlearrowright \end{array} - \begin{array}{c} \circlearrowright \end{array} + \begin{array}{c} \circlearrowright \end{array} - \frac{1}{2} \begin{array}{c} \times \end{array} \right)$$

Fig. 2.3 The diagrammatic representation of $\hat{K}^{(4)}_{2,2}$.

Removing the momentum conservation $\delta$-function, we have

$$\frac{1}{(g^2 N)^2} K^{(4)}_{2,2}(k_1, k_2, k_1', k_2') = K^{(4)}_0 + K^{(4)}_1 + K^{(4)}_2 + K^{(4)}_3 + K^{(4)}_4 . \quad (2.1)$$

with

$$K^{(4)}_0 = \frac{a}{2} \sum k_1^4 k_2^4 J_1(k_1^2) J_1(k_2^2)(16\pi^3)\delta^2(k_2 - k_2') , \quad (2.2)$$

$$K^{(4)}_1 = \frac{b}{2} \sum k_1^4 J_2(k_1^2) k_2^3(16\pi^3)\delta^2(k_2 - k_2') , \quad (2.3)$$

$$K^{(4)}_2 = -\frac{1}{2} \sum \left( \frac{k_1^2 J_1(k_1^2) k_2^2 k_2' + k_1^2 k_2^2 J_1(k_2^2) k_1'}{k_1 - k_1'}^2 \right), \quad (2.4)$$

$$K^{(4)}_3 = \frac{1}{2} \sum k_2^2 k_1' J_1((k_1 - k_1')^2) \quad (2.5)$$

and

$$K^{(4)}_4 = \frac{1}{4} \sum k_1^3 k_2^2 k_1' k_2' I(k_1, k_2, k_1', k_2'), \quad (2.6)$$

where

$$J_1(k^2) = \frac{1}{16\pi^3} \int \frac{d^2k'}{(k')^2(k' - k)^2} , \quad (2.7)$$
\[ J_2(k^2) = \frac{1}{16\pi^3} \int d^2q \frac{1}{(k-q)^2} J_1(q^2) \] (2.8)

and

\[ I(k_1, k_2, k_1', k_2') = \frac{1}{16\pi^3} \int d^2p \frac{1}{p^2(p+k_1)^2(p+k_1-k_1')^2(p+k_2')^2} \] (2.9)

The \( \Sigma \) again implies that we sum over permutations of both the initial and the final state.

In previous papers \( K_0^{(4)} \) and \( K_1^{(4)} \) have been defined by the values[3]

\[ a = 0, \quad b = \frac{2}{3} \] (2.10)

and also the values[4]

\[ a = 1, \quad b = -\frac{2}{3}. \] (2.11)

In [6] it was incorrectly argued that the latter values are determined uniquely by the cancellation, after integration, of all divergences of the complete kernel. Each of \( K_0^{(4)} \) and \( K_1^{(4)} \) contains single and double poles in \( \epsilon \) when dimensionally regularized and, at first sight, requiring the cancellation of both singularities, after integration of the full kernel, fixes \( a \) and \( b \). However, in [6] it is shown that

\[ \frac{3}{4} |k^2 J_1(k)|^2 = k^2 J_2(k^2) \]

\[ = 6\gamma^2 \pi^2 - \frac{\pi^4}{2} + \frac{12\pi^2}{\epsilon} + \frac{12\gamma \pi^2}{\epsilon} + 12\gamma \pi^2 \log(\pi) + \frac{12\pi^2 \log(\pi)}{\epsilon} + 6\pi^2 \log(\pi)^2 \]

\[ + 12\gamma \pi^2 \log(k^2) + \frac{12\pi^2 \log(k^2)}{\epsilon} + 12\pi^2 \log(\pi) \log(k^2) + 6\pi^2 \log(k^2)^2. \] (2.13)

and so the single and double poles occur with the same relation between the coefficients in both \( K_0^{(4)} \) and \( K_1^{(4)} \). As a result \( a \) and \( b \) are not determined uniquely by infra-red cancellations and it is sufficient only that

\[ a + \frac{3b}{4} = \frac{1}{2} \] (2.14)
which is, of course, satisfied by both (2.10) and (2.11). Below we shall find that conformal invariance determines uniquely that

\[ a = \frac{1}{4}, \quad b = \frac{1}{3}. \quad (2.15) \]

### 3. A FEYNMAN DIAGRAM CONSTRUCTION

We now show that the \( O(g^4) \) kernel can be identified with a certain set of Feynman diagrams as, for example, occur in the calculation of virtual photon-photon scattering with a large rapidity gap between the fragments. As illustrated in Fig. 3.1, (the square of the amplitude is shown), the two photons with virtuality \( Q^2 \) and \( Q'^2 \) dissociate into two quark-antiquark pairs which then interact via a color-zero exchange represented at lowest order by two gluons. It should be pointed out that the diagram in Fig. 3.1 is of next-to-next-to-leading order compared to the leading order BFKL-calculation and does not contribute at next-to-leading-order level as the \( O(g^4) \) kernel in its original form is meant to do. Nevertheless, as we discuss, it is understood how to introduce conformal symmetry directly in this context. The occurrence of the same structure in two different contributions has to be taken as pure coincidence, perhaps related to the general role of conformal symmetry.

![Feynman diagram](image)

**Fig. 3.1** Scattering of two virtual photons with rapidity gap.

Following ref.[7] the coupling of four gluons to the quark-loop can be summarized in seven terms:
\[ D^{(1;+,+)\, (4,0)}(k_1, k_2, k_3, Q^2) = g^2 N A \cdot \]
\[
\left\{ D_{(2,0)}(k_1, Q^2) + D_{(2,0)}(k_2, Q^2) + D_{(2,0)}(k_3, Q^2) + D_{(2,0)}(k_1 + k_2 + k_3, Q^2) \\
- D_{(2,0)}(k_1 + k_2, Q^2) - D_{(2,0)}(k_1 + k_3, Q^2) - D_{(2,0)}(k_2 + k_3, Q^2) \right\}
\] (3.1)

with an overall constant \(A\) which is not of relevance in this discussion. Although not needed explicitly, for completeness we give the two functional forms for \(D_{(2,0)}\) (depending on the polarization of the photon):

\[
D^t_{(2,0)}(k, Q^2) = \sum_f e_f^2 \alpha_s \frac{\sqrt{N^2 - 1}}{2\pi} \int_0^1 d\alpha \int_0^1 dy \frac{[1 - 2\alpha(1 - \alpha)][1 - 2y(1 - y)] k^2}{y(1 - y)k^2 + \alpha(1 - \alpha)Q^2} \] (3.2)

\[
D^l_{(2,0)}(k, Q^2) = \sum_f e_f^2 \alpha_s \frac{\sqrt{N^2 - 1}}{2\pi} \int_0^1 d\alpha \int_0^1 dy \frac{[2\alpha(1 - \alpha)][2y(1 - y)] k^2}{y(1 - y)k^2 + \alpha(1 - \alpha)Q^2} . \] (3.3)

The index \((1;+,+)\) stands for the color projection and symmetry of the gluon pair above and below the central cut in Fig. 3.1. In this example ‘1’ means color singlet and ‘+’ means even under interchange of the two gluons. A very important property of \(D^{(1;+,+)\, (4,0)}\) is the fact that it vanishes whenever one of the momenta \(k_1, \ldots, k_4\) becomes zero. This property, which we will refer to as color cancellation, still holds in the case of an even color octet state. The odd color octet configuration, however, fails to provide complete color cancellation and diagrams like that in Fig. 3.1 with odd signature are not infrared safe. The infrared singularities are supposed to cancel with those from contributions at next-to-next-to-leading order real gluon emission.

In our case where we required color singlet exchange, no infrared problems occur. The singularities due to the gluon propagators for \(k_i = 0\) are cancelled. We get the \(O(g^4)\) kernel by taking “the square” of expression (3.1), i.e. by adding the propagators and integrating over the transverse phase space:

\[
\frac{1}{(16\pi^3)^3} \int \frac{d^2k_1}{k_1^2} \frac{d^2k_2}{k_2^2} \frac{d^2k_3}{k_3^2} \frac{d^2k_4}{k_4^2} \delta^2(k_1 + k_2 + k_3 + k_4) \\
D^{(1;+,+)\, (4,0)}(k_1, k_2, k_3, Q^2) D^{(1;+,+)\, (4,0)}(k_1, k_2, k_3, Q^2) \] (3.4)

\[
= \frac{A^2}{(16\pi^3)^2} \int \frac{d^2k}{k^2} \frac{d^2k'}{k'^2} D_{(2,0)}(k, Q^2) D_{(2,0)}(k', Q^2) \frac{1}{24} K^{(4)}_{2,2}(k, -k, k', -k') .
\]

The expansion of the lhs of eq.(3.4) is straightforward. A quick check can be done by adding the absolute values of the coefficient in eqs. (2.2)-(2.6) and multiplying the
result by 24 (note that (2.4) contains two diagrams and should be counted twice).
Provided that the coefficient $a$ is equal to 1/4 and the coefficient $b$ equal to 1/3 one finds (2.1) agrees directly with the 7x7 terms of the lhs of eq.(3.4).

After having established the relation between the $O(g^4)$ kernel and the diagram in Fig. 3.1, we know from ref.[9] that due to color cancellation the gluon propagators in impact parameter space may be rewritten in a conformally invariant way. (We illustrate this for two gluon exchange in the next Section). This implies that a conformally invariant representation of the $O(g^4)$ kernel necessarily exists. It remains only to find the precise form. This we do via the explicit calculations below.

4. STRUCTURE OF THE INTEGRAL EQUATION

We first write the BFKL equation in momentum space in terms of the Green’s function $f$:

$$f(\omega, k_1, k_2, k_{1'}, k_{2'}) = f^0(\omega, k_1, k_2, k_{1'}, k_{1'})$$

$$+ \frac{1}{\omega} \frac{1}{16\pi^3} \int \frac{d^2k_{1''}'}{k_{1''}^2} \frac{d^2k_{2''}}{k_{2''}^2} \hat{K}^{(2)}_{2,2}(k_1, k_2, k_{1''}, k_{2''}) f(\omega, k_{1''}, k_{2''}, k_{1'}, k_{2'})$$

(4.1)

where, as we have discussed in Section 2, $\hat{K}^{(2)}_{2,2}(k_1, k_2, k_{1'}, k_{2'})$ is a dimensionless kernel including a momentum conserving $\delta$-function. The inhomogeneous term $f^0$ in (4.1) is given by the two gluon propagator, i.e. the starting configuration for a gluon-ladder:

$$f^0(\omega, k_1, k_2, k_{1'}, k_{2'}) = \frac{1}{\omega} \frac{\delta^2(k_1 - k_{1'}) \delta^2(k_2 - k_{2'})}{k_1^2 k_2^2}. \quad (4.2)$$

In the complete scattering amplitude $f^0$ is sandwiched between two colorless states described by “wave functions” $F$ and $F'$ with the following “Ward-identity” property:

$$F(k_1, k_2) \rightarrow 0 \quad i = 1, 2 \quad k_i \rightarrow 0$$

and

$$F'(k_{1'}, k_{2'}) \rightarrow 0 \quad i' = 1', 2' \quad k_{i'} \rightarrow 0$$

(4.3)

This property is due to color cancellation and ensures an infrared-stable result.

The corresponding form of the BFKL equation in impact parameter space is
then
\[ \tilde{f}(\omega, \rho_1, \rho_2, \rho_1', \rho_2') = \tilde{f}^0(\omega, \rho_1, \rho_2, \rho_1', \rho_2') \]
\[ + \frac{1}{\omega} \int d\rho_1'' d\rho_2'' d\rho_2''' d\rho_2''' \tilde{K}^{(2)}_{2,2}(\rho_1, \rho_2, \rho_1'', \rho_2'') |\partial_{\rho_1''}|^2 |\partial_{\rho_2''}|^2 \tilde{f}(\omega, \rho_1'', \rho_2'', \rho_1', \rho_2') \]
(4.4)

(\partial_{\rho''} = \partial/\partial \rho_{\rho''} \text{ etc.}) with a conformally invariant kernel \(\tilde{K}^{(2)}_{2,2}(\rho_1, \rho_2, \rho_1'', \rho_2'')\). Explicit representations can be found in [8, 9]. In this context \(\tilde{K}^{(2)}_{2,2}(\rho_1, \rho_2, \rho_1'', \rho_2'')\) is simply defined in a formal way to be the fourier transform of the “reduced” kernel
\[ \tilde{K}^{(2)}_{2,2}(k_1, k_2, k_1'', k_2'') / k_1^2 k_2^2 k_1'' k_2'' \]
(4.5)

The momentum factors had to be added to compensate for the dimensional dependence introduced by the fourier transform.

The “Ward-identity” property (4.3) translates into impact parameter space as
\[ \int d^2 \rho_1 \tilde{F}(\rho_1, \rho_2) = 0 \quad i = 1, 2 \]
and
\[ \int d^2 \rho_1' \tilde{F}'(\rho_1', \rho_2') = 0 \quad i' = 1', 2' \quad (4.6) \]

In order to write down the inhomogeneous term \(\tilde{f}^0\) in impact parameter space we need to know the Fourier transform of a simple propagator. We use the formula
\[ \int d^2 k \frac{e^{ik.\rho}}{(k^2 + m^2)} = K_0(m|\rho|) \]
(4.7)
where \(K_0\) is the modified Bessel function, and take the limit \(m \to 0\):
\[ K_0(m|\rho|) \xrightarrow{m \to 0} -\ln|m|\rho/2| + \psi(1) + O(m) \].
(4.8)

By means of eq.(4.6) the \(\ln|m/2|\) terms drop out and we are left with:
\[ \tilde{f}^0(\omega, \rho_1, \rho_2, \rho_1', \rho_2') = \frac{4}{(2\pi)^4 \omega} \ln |\rho_{11'}| \ln |\rho_{22'}| \]
(4.9)
with the notation \(\rho_1 - \rho_{1'} = \rho_{11'}\).

One can now directly show, again with property (4.6), that the rhs of eq.(4.2) is conformally invariant. Equivalently we can simply simply exploit (4.6 to add extra
terms which lead to an explicitly conformally invariant expression [9] (this is what we will do for the four gluon kernel in the next Section). The result is

\[ \tilde{f}_0(\omega, \rho_1, \rho_2, \rho_1', \rho_2') = \frac{4}{(2\pi)^4\omega} \ln^2 \left| \frac{\rho_{11'}\rho_{22'}}{\rho_{12}\rho_{12'}} \right| \]

\[ = \frac{4}{(2\pi)^4\omega} \ln^2 R \tag{4.10} \]

which differs from (4.9) only by terms which give zero contribution by virtue of (4.6). \( R \) is the harmonic ratio previously defined in (1.2). Note that we have defined the Green’s function to be symmetric under interchange of 1 and 2 or 1’ and 2’.

5. \( O(g^4) \) KERNEL IN IMPACT PARAMETER SPACE

We now discuss the fourier transforms of each term in \( \hat{K}^{(4)}_{2,2} \). We shall need just the one basic transform, i.e. (4.7) of the previous section.

We begin with the most complicated case i.e. the box diagram \( K^{(4)}_4 \). We use the notation of Fig. 4.1

Fig. 4.1 Notation for the crossed box diagram.

and consider the transform of “reduced” diagram as discussed in the previous Section i.e.

\[ \tilde{K}^{(4)}_4(\rho_1, \rho_2, \rho_1', \rho_2') = \frac{1}{4} \sum \int d^2k_1 d^2k_2 d^2k_1' d^2k_2' e^{ik_1 \cdot \rho_1 + ik_2 \cdot \rho_2 - ik_1' \cdot \rho_1' - ik_2' \cdot \rho_2'} \]

\[ \delta^2(k_1 + k_2 - k_1' - k_2') I(k_1, k_2, k_1', k_2') \tag{5.1} \]
where once again summation is over permutation of the initial and final momenta. (In this case, since \( I(k_1, k_2, k_{1'}, k_{2'}) \) is completely symmetric, the factor of 1/4 simply cancels the effect of the summation.) Using the \( \delta \)-function to perform the \( k_{2'} \) integration and inserting (2.9) we obtain, using (4.7) and (4.8),

\[
\tilde{K}_4^{(4)}(\rho_1, \rho_2, \rho_{1'}, \rho_{2'}) = \int d^2k_1d^2k_2d^2k_{1'}d^2p \ e^{-i(\rho_1-\rho_{2'})p-i(\rho_{1'}-\rho_1)(p+k_1)}
\times e^{-i(\rho_2-\rho_{1'})(p-k_{1'}+k_1-i(\rho_{2'}-\rho_2)(p-k_{1'}+k_1+k_2)}
\times \frac{1}{p^2(p+k_1)^2(p+k_1-k_{1'})^2(p+k_1+k_{1'}+k_2)^2}

= [-ln|\rho_1 - \rho_{2'}| - ln|m| + ln2 + \psi(1) + O(m)]
\times [-ln|\rho_2 - \rho_{1'}| - ln|m| + ...]
\times [-ln|\rho_1 - \rho_{1'}| - ...][-ln|\rho_2 - \rho_{2'}| + ...]

\equiv ln|\rho_{12'}|ln|\rho_{21'}|ln|\rho_{11'}|ln|\rho_{22'}| + ....
\]

where, in the last line, the omitted terms involve factors of \( ln|m|, \psi(1) \) etc..

Consider next the disconnected diagrams contained in \( K_0^{(4)} \). For the reduced diagram we simply have, using the notation of Fig. 4.2,

\[
\tilde{K}_0^{(4)}(\rho_1, \rho_2, \rho_{1'}, \rho_{2'}) = \frac{a}{2} \sum \int d^2k_1d^2k_2d^2k_{1'}d^2k_{2'}
\times \delta^2(k_1 - k_{1'}) \delta^2(k_2 - k_{2'}) J_1(k_1) J_1(k_2)
\]

Using the \( \delta \)-functions to perform the \( k_{1'} \) and \( k_{2'} \) integrations we obtain

\[
\tilde{K}_0^{(4)}(\rho_1, \rho_2, \rho_{1'}, \rho_{2'}) = \frac{a}{2} \sum \int d^2k_1d^2k_2d^2p_1d^2p_2 \ e^{i(\rho_1-\rho_{1'})(k_1+p_1)+i(\rho_{1'}-\rho_1)p_1}
\times e^{i(\rho_2-\rho_{2'})(k_2+p_2)+i(\rho_{2'}-\rho_2)p_2}
\times \frac{1}{p_1^2p_2^2(p_1+k_1)^2(p_1+k_2)^2}

= \frac{a}{2} \sum [-ln|\rho_1 - \rho_{1'}| - ln|m| + ln2 + \psi(1) + O(m)]^2
\times [-ln|\rho_2 - \rho_{2'}| - ln|m| + ...]^2

\equiv \frac{a}{2} \sum ln^2|\rho_{11'}|ln^2|\rho_{22'}| + ....
\]

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Proceeding in the same manner we obtain

\[
\tilde{K}_1^{(4)}(\rho_1, \rho_2, \rho_{1'}, \rho_{2'}) = \frac{b}{2} \sum \ln^3 |\rho_{11'}| |\ln| |\rho_{22'}| + .... \tag{5.5}
\]

\[
\tilde{K}_2^{(4)}(\rho_1, \rho_2, \rho_{1'}, \rho_{2'}) = -\frac{1}{2} \sum \ln^2 |\rho_{11'}| (|\ln| |\rho_{12'}| |\ln| |\rho_{22'}| + .... + \ln^2 |\rho_{11'}| |\ln| |\rho_{22'}| |\ln| |\rho_{22'}| + ....) \tag{5.6}
\]

\[
\tilde{K}_3^{(4)}(\rho_1, \rho_2, \rho_{1'}, \rho_{2'}) = \frac{1}{2} \sum \ln |\rho_{11'}| \ln^2 |\rho_{12'}| |\ln| |\rho_{22'}| + .... \tag{5.7}
\]

Next we note that \(\ln^4 R\) has the expansion

\[
\ln^4 \left[ \left| \frac{\rho_{11'}}{\rho_{11'}} \right| \frac{\rho_{22'}}{\rho_{22'}} \right] = (\ln |\rho_{12'}| + \ln |\rho_{21'}| - \ln |\rho_{11'}| - \ln |\rho_{22'}|)^4 
\]

\[
= 6 \ln^2 |\rho_{11'}| \ln^2 |\rho_{22'}| + 6 \ln^2 |\rho_{12'}| \ln^2 |\rho_{21'}| + 4 \ln^3 |\rho_{11'}| |\ln| |\rho_{22'}| \\
+ 4 \ln |\rho_{11'}| \ln^3 |\rho_{22'}| + 4 \ln^3 |\rho_{12'}| |\ln| |\rho_{21'}| + 4 \ln |\rho_{12'}| \ln^3 |\rho_{21'}| \\
- 12 \ln^2 |\rho_{11'}| |\ln| |\rho_{12'}| |\ln| |\rho_{22'}| - 12 \ln^2 |\rho_{11'}| |\ln| |\rho_{21'}| |\ln| |\rho_{22'}| \\
- 12 \ln |\rho_{11'}| |\ln| |\rho_{12'}| \ln^2 |\rho_{22'}| - 12 \ln^2 |\rho_{11'}| |\ln| |\rho_{21'}| \ln^2 |\rho_{22'}| \\
- 12 \ln |\rho_{12'}| |\ln| |\rho_{11'}| \ln^2 |\rho_{22'}| - 12 \ln^2 |\rho_{12'}| |\ln| |\rho_{21'}| \ln^2 |\rho_{22'}| \\
- 12 \ln |\rho_{12'}| |\ln| |\rho_{12'}| \ln^2 |\rho_{21'}| - 12 \ln^2 |\rho_{12'}| |\ln| |\rho_{22'}| \ln^2 |\rho_{21'}| \\
+ 12 \ln |\rho_{11'}| \ln^2 |\rho_{12'}| |\ln| |\rho_{22'}| + 12 \ln |\rho_{11'}| \ln^2 |\rho_{21'}| |\ln| |\rho_{22'}| \\
+ 12 \ln |\rho_{12'}| \ln^2 |\rho_{11'}| |\ln| |\rho_{22'}| + 12 \ln |\rho_{12'}| \ln^2 |\rho_{21'}| |\ln| |\rho_{22'}| \\
+ 24 (\ln |\rho_{12'}| |\ln| |\rho_{21'}| |\ln| |\rho_{11'}| |\ln| |\rho_{22'}| + ....) \tag{5.8}
\]

where the omitted terms are all independent of one (or more) of \(\rho_1, \rho_2, \rho_{1'}\) and \(\rho_{2'}\).
We can rewrite (5.8) as

\[
\frac{1}{24} \ln^4 \left[ \left| \frac{\rho_{11}}{\rho_{11}'} \right| \left| \frac{\rho_{22}}{\rho_{22}'} \right| \right] = \frac{1}{8} \sum \ln^2 |\rho_{11}| \ln^2 |\rho_{22}| + \frac{1}{6} \sum \ln^3 |\rho_{12}| \ln |\rho_{21}| \\
- \frac{1}{2} \sum (\ln^2 |\rho_{11}| \ln |\rho_{12}| \ln |\rho_{22}| - \ln^2 |\rho_{11}| \ln |\rho_{21}| \ln |\rho_{22}|) \\
+ \frac{1}{2} \sum \ln |\rho_{11}'| \ln^2 |\rho_{12}'| \ln |\rho_{22}| \\
+ \frac{1}{4} \sum (\ln |\rho_{12}'| \ln |\rho_{21}'| \ln |\rho_{11}'| \ln |\rho_{22}'|) + ....
\]  

(5.9)

From (5.2) - (5.7), it is then clear that, with the values of \( a \) and \( b \) given by (2.15), (5.8) gives the fourier transform of the complete sum of terms in \( \tilde{K}_{2,2}^{(4)} \), apart from an overall multiplicative factor and apart from terms that are independent of one (or more) of \( \rho_1, \rho_2, \rho_{1'}, \) and, or, involve factors of \( \ln |m|, \ln 2 \) and \( \psi(1) \). In fact the infra-red cancellation of the factors of \( \ln |m| \) in \( \tilde{K}_{2,2}^{(4)} \) implies that the factors of \( \ln 2 \) and \( \psi(1) \) must also cancel. The terms that are independent of one of \( \rho_1, \rho_2, \rho_{1'} \) and \( \rho_{2'} \) can be dropped by virtue of (4.6) and the following discussion. It follows that (1.1) can be used directly as the fourier transform \( \tilde{K}_{2,2}^{(4)} \) of \( \tilde{K}_{2,2}^{(4)} \) in the BFKL equation (4.4).

6. HIGHER-ORDER GENERALISATIONS AND SPECTRA

It is interesting to note that the distinctive terms in (5.9) each corresponds to one of the diagrams of Fig. 5.1, which are impact parameter analogs of the transverse momentum diagrams originally used to define the kernel. Each line in these diagrams corresponds to a logarithm (propagator) of the relative impact parameter of the two points joined. We have shown above that the sum of all diagrams in which two pairs of points are joined by four propagators is associated with the conformally invariant operator \( \ln^4 R \). We have also seen in Section 3 that the sum of diagrams in which the two pairs are joined by two propagators (i.e. the two gluon propagator) is associated with \( \ln^2 R \).
As a generalization, we can expect that for arbitrary $m$ we can associate $\ln^m R$ with diagrams in which the two pairs of points are joined by $m$ propagators. It is natural to conjecture that such interactions will appear at the appropriate order in the BFKL kernel in some form of conformal approximation. This conjecture was made for the corresponding transverse momentum diagrams in [10]. We should note, however, that it is far from clear that the appropriate $t$-channel unitarity construction can actually be carried through. It is possible that the explicit relationship of the $O(g^4)$ kernel to the exact NLO calculations could provide a better understanding of whether such higher-order interactions should be expected to appear.

At first sight it might be thought that $\ln^3 R$ is directly related to the BFKL kernel. However, the antisymmetry under $1' \leftrightarrow 2'$ implies that only the “odd” part of the kernel is obtained. Similarly the $\ln^2 R$ and $\ln^4 R$ kernels are necessarily symmetric. In general the kernels are odd (even) under $1' \leftrightarrow 2'$ when $m$ is odd(even).

The eigenvalue spectrum of the kernels we are discussing are defined with respect to eigenfunctions labeled by $(\nu,n)$, $-\infty < \nu < \infty$, $n = 0, \pm 1, \pm 2, \ldots$ (in momentum space, for $q = 0$, the eigenfunctions are $\phi_{\nu,n} = |k|^\nu e^{i2\theta}$). Symmetry or antisymmetry under $1' \leftrightarrow 2'$ determines $n$ to be even or odd respectively. In [6] it was shown that the spectrum of $K_{2,2}^{(4)}$ (which is even in $n$) can be written in the form

$$\mathcal{E}(\nu,n) = \frac{1}{\pi} [\chi(\nu,n)]^2 - \Lambda(\nu,n).$$

(6.1)

where $\chi(\nu,n)$ are the eigenvalues of the (even part of) the BFKL kernel and

$$\Lambda(\nu,n) = -\frac{1}{4\pi} \left( \beta'\left(\frac{|n|+1}{2} + i\nu\right) + \beta'\left(\frac{|n|+1}{2} - i\nu\right) \right).$$

(6.2)

$\beta(x)$ is the incomplete beta function, i.e.

$$\beta(x) = \int_0^1 dy \, y^{x-1}[1 + y]^{-1}$$

(6.3)
The $\Lambda(\nu, n)$ are the eigenvalues of a separately infra-red finite component $K_2$ of $K_2^{(4)}$ extracted from the box diagram $K_4^{(4)}$. The $\Lambda(\nu, n)$ also have a holomorphic factorisation property. From the above discussion, and from (6.1) in particular, it appears that the BFKL kernel $K_{BFKL}$ has the formal representation

$$K_{BFKL} = c_1 \ln^3 R + c_2 [\ln^4 R - K_2]^\frac{3}{2},$$

(6.4)

where $c_1$ and $c_2$ can easily be calculated. $\ln^3 R$ provides the antisymmetric part of the kernel and $[\ln^4 R - K_2]^\frac{3}{2}$ the symmetric part. The representation of the symmetric part, however, is non-trivial and needs more investigation. The hope is that with $K_2$ suitably defined in impact parameter space one finds a simple form similar to the antisymmetric part.

It is interesting that all the kernels $\ln^m R$ can be simultaneously studied by considering derivatives (with respect to $\delta$) of the generating function

$$G(\mathcal{R}, \delta) = \mathcal{R}^\delta$$

(6.5)

It appears that a number of interesting spectra and relationships between operators, including (6.4), can be derived from this starting point. This subject is currently under study and will be discussed in a future publication.

7 Conclusions

By finding the surprisingly simple representation (1.1), we have shown explicitly the conformal invariance of a contribution to the NLO BFKL kernel constructed via $t$-channel unitarity. This lends some support to the conjecture[6] that $t$-channel unitarity determines conformally invariant interactions underlying non-leading log contributions to the Regge limit of QCD. We have also generated a further set of candidate interactions, i.e.

$$\ln^m R = \ln^m \left[ \frac{|\rho_1 - \rho'_1|}{|\rho_1 - \rho_2'|} \frac{|\rho_2 - \rho_2'|}{|\rho_2 - \rho_1'|} \right]$$

(7.6)

and suggested a method for their study.

It is important that the representation (1.1) was found via a Feynman diagram construction that leads to the same $O(g^4)$ kernel. The diagrams involved do not contribute at the level of perturbation theory associated with the $t$-channel construction. Nevertheless the Feynman diagram construction may prove useful in understanding the general significance of higher-order interactions such as (7.6).
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References


