Cosmological consequences of particle creation during inflation

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(September 1996)

Abstract

Particle creation during inflation is considered. It could be important for species whose interaction is of gravitational strength or weaker. A complete but economical formalism is given for spin-zero and spin-half particles, and the particle abundance is estimated on the assumption that the particle mass in the early universe is of order the Hubble parameter \( H \). It is roughly the same for both spins, and it is argued that the same estimate should hold for higher spin particles in particular the gravitino. The abundance is bigger than that from the usual particle collision mechanism if the inflationary energy scale is of order \( 10^{16} \) GeV, but not if it is much lower.

I. INTRODUCTION

According to present ideas, particles species can be created in the very early universe by a number of mechanisms [1,2]. The most familiar, which applies to particles of any spin, is the collision or decay of other particles. A number of other reasonably familiar mechanisms come into play for the case of a spin zero particle, which corresponds to the oscillation of a scalar field. If the field is initially displaced from its vacuum value it will oscillate homogeneously, a familiar example being the oscillation of the inflaton field after inflation. The homogeneous oscillation can in turn cause further particle production, either through the decay of the initial particles as in the original discussion of reheating after inflation, or through collective effects as in the more modern discussions of ‘preheating’. Alternatively, the scalar field can be radiated from topological defects, as in the example of axion production from axionic strings.

The present paper explores a mechanism which is different from any of the above, namely the production of particles from the vacuum fluctuation during inflation. We estimate the abundance of particles arising from the vacuum fluctuation during inflation, and compare it with the abundance from the particle collision mechanism and with the cosmological constraints.

In contrast with earlier work, we take seriously the fact that the effective mass \( m(t) \) in the early universe is not expected to be the same as the true mass \( m \) which is relevant at the present day, because of the effect of interactions. On the usual assumption that we are
dealing with a supersymmetric theory, it has long been recognised that even an interaction of gravitational strength will typically give a spin zero particle a mass of order $H$ during inflation if $H$ is bigger than the true mass [3–5]. More recently it has been emphasised [6] that the same result is to be expected even after inflation. Although it has not so far been stated explicitly in the literature, one may expect this result to apply equally to particles of any spin.

This contribution of order $H$ to the mass can be evaded if the mass is protected by a symmetry. For example gauge bosons cannot acquire mass unless the symmetry is spontaneously broken, and Peccei-Quinn symmetry should keep the axion mass small, but neither of these cases is of interest in the present context (gauge bosons are too short-lived and the axion density will be dominated by other production mechanisms).

During inflation there are more possibilities for evading the mass of order $H$. A number are available for spin zero, as reviewed for example in [7] for the particular case of the inflaton field. Another case is that of the gravitino; its mass is proportional to $|W|$ where $W$ is the superpotential, and in some models of inflation [5,8] it is much less than $H$ during inflation. However, even if the contribution of order $H$ to the mass is absent during inflation it still typically reappears afterwards.

If the interaction is stronger than gravitational strength, one presumably expects a bigger mass [9], but in the present context this case is not of interest because it will typically be associated with production mechanisms that dominate the vacuum fluctuation. In particular, if the interaction is strong enough to maintain thermal equilibrium the mass is of order the temperature $T$, but then the thermal abundance dominates.

We shall assume that $m(t)$ is not varying rapidly on the timescale of the expansion rate $H$. Rapid variation will occur if $m(t)$ is generated by a coupling to a homogeneous oscillating scalar field, because the oscillation frequency is necessarily big compared with $H$ (and so usually is the amplitude). But in that case we are dealing with a separate mechanism of particle creation [10–12], which is decoupled from the vacuum fluctuation that we consider here.

With the above discussion in mind we shall assume that a species with mass $m$ has an effective mass $m(t) = \mu H$ in the early universe, with $\mu$ a constant of order 1. This should give an order of magnitude estimate, though $\eta$ may actually be expected to have different values during and after inflation.

The layout of this paper is as follows. In Section 2 we give the formalism for calculating the occupation number per quantum state, treating separately the cases of spin 0 and spin $1/2$. In Section 3 we estimate the corresponding cosmological abundance and compare it with the abundance coming from particle collisions, on the assumption that these are of only gravitational strength. In Section 4 we summarize the conclusions, and some specific cases are worked out in an Appendix.

II. PARTICLE CREATION

In this section we give the basic formalism for the creation of spin zero and spin half particles. The reader who is interested only in the results may skip to the next section, where the rather simple conclusion is summarized.
In considering particle creation from the vacuum fluctuation, we are interested only in species which have very weak interactions. One reason is that a species with significant interactions will achieve thermal equilibrium, losing all memory of its earlier abundance. The other is that there are strong cosmological constraints on the abundance of a long-lived species, whereas short-lived species with stronger interactions decay before they can have any direct cosmological effect.

In the case of a scalar field the quantum fluctuation is competing with a possible classical displacement of the field from its vacuum value, and this latter effect must be absent if the quantum fluctuation is to be significant. If the moduli fields of superstring theory exist with masses of order 100 GeV, a displacement for them at the classical level leads to a gross over-abundance in the standard cosmology, which is called the ‘moduli problem’ [13–17,6,18,19]. As far as one can tell, the only way of solving the problem seems to be to dilute the moduli abundance with a late bout of ‘thermal’ inflation as discussed in detail in Refs. [19–22] (for earlier incomplete discussions of thermal inflation see the works cited in Ref. [18]). However, this would also dilute the abundance of particles produced by the vacuum fluctuation that we are considering. (The vacuum fluctuation during thermal inflation is likely to be insignificant since the particle masses are likely to be much bigger than $H_*$, which is much less than 100 GeV.) Thus, for our discussion to be relevant in the context of supergravity the moduli problem has to be eliminated by requiring a mass $\gg 100$ GeV, or else no classical displacement, rather than by modifying the standard cosmology.

As we are dealing with the early universe we can assume that the density is critical, so that the line element can be written
\[
 ds^2 = a^2(\eta)[d\eta^2 - dx^2 - dy^2 - dz^2] \tag{1}
\]
where $a$ is the scale factor and $\eta$ is conformal time, related to the proper time of a comoving observer by $d\eta = dt/a$. Using the time coordinate $x^0 \equiv \eta$ and comoving space coordinates $\{x^1, x^2, x^3\} \equiv \{x, y, z\}$, the corresponding metric components are $g_{\mu\nu} = a\delta_{\mu\nu}$. In what follows, a prime will denote $d/d\eta$ and an overdot will denote $d/dt$.

We assume that at early times there is almost exponential inflation, and set the Hubble parameter $H \equiv \dot{a}/a$ equal to a constant $H_*$. Then we can choose $\eta = -(aH_*)^{-1}$. After inflation we typically have either radiation domination when we can choose $\eta = (aH)^{-1}$ or matter domination when we can choose $\eta = 2(aH)^{-1}$.

A. Spin zero particles

The formalism in the case of a spin zero field is well known [23–25], but we reproduce it briefly for comparison with the less well-known spin half case. Then we apply it to the case $m(\eta) \sim H$, which has not been discussed before.

We take as the Lagrangian density of the scalar field $\phi$
\[
 \mathcal{L} = \frac{1}{2}(-\det g)^{1/2}\left[\partial_\mu \phi \partial^\mu \phi - (m^2(\eta) - \xi R)\phi^2\right] \tag{2}
\]
We have allowed a coupling to the curvature scalar $R = -6a''/a^3 = 6[\ddot{a}/a + H^2]$, but will assume that $|\xi| \lesssim 1$. During exponential inflation, $a \propto \exp(Ht)$ with $H$ constant so
\( R = -12H^2 \). During matter domination \( a \propto t^{2/3} \) so \( R = -3H^2 \), and during radiation domination \( a \propto a^{1/2} \) so \( R = 0 \).

With the exception of this coupling, we have assumed that all interactions can be represented by a time dependent mass \( m(\eta) \), and as discussed earlier we set

\[
\begin{align*}
    m(\eta) &= \mu H(\eta) \quad (\mu H > m) \\
    m(\eta) &= m \quad (\mu H < m)
\end{align*}
\]

(3)

where \( m \) is the true mass and \( \mu \) is a constant for order 1.

In order to quantize the theory we may start with annihilation operators \( a_k \), which commute except for \([a_k, a_k^\dagger] = 1 \). Then \( \phi \) is Fourier expanded in a comoving box much bigger than the region of interest in the form

\[
\phi = a^{-1}(\eta) \sum_k \left[ w_k(\eta)a_k + w_k^*(\eta)a_{-k}^\dagger \right] e^{ik \cdot x} \tag{4}
\]

The equation of motion of \( w_k \) is

\[
w_k'' + \Omega_k^2 w_k = 0 \tag{5}
\]

where

\[
\Omega_k^2 \equiv k^2 + a^2 [m^2(\eta) + \left( \frac{1}{6} - \xi \right) R] \tag{6}
\]

In the above expansion, the factor \( 1/a \) was pulled out to make Eq. (5) simple. It has the property that the independent solutions \( w_k \) and \( w_k^* \) have a constant Wronskian, which we set equal to \( i \),

\[
w_k(w_k')^* - w_k^*w_k' = i \tag{7}
\]

If \( \Omega_k \) satisfies the adiabatic condition

\[
\Omega_k'/\Omega_k \ll \Omega_k \tag{8}
\]

then \( w_k \) is a sinusoidal function of \( \eta \) with angular frequency \( \Omega_k \). With our assumptions about \( m(\eta) \) and \( |\xi| \), the adiabatic condition is satisfied whenever the expansion rate \( a'/a = aH \) is negligible compared with \( \Omega_k \). This criterion is satisfied for all all wavenumbers \( k \) in the late-time regime where \( H \) is less than the true mass \( m \), whereas at earlier times it is satisfied only by modes with wavelength inside the horizon, corresponding to \( k \gg aH \),

\[
k \gg aH \quad (H \gtrsim m) \\
\text{all } k \quad (H \ll m) \tag{9}
\]

In this adiabatic regime we recover flat spacetime field theory, with \( \Omega_k/a \) (the physical angular frequency) equal to the particle energy \( E_k = [(k/a)^2 + m^2]^{1/2} \). Keeping the slow variation of the adiabatic quantities, the canonical formulation of flat spacetime field theory corresponds to \( w_k = w_k^{\text{flat}} \), with
where $\chi_k$ is slowly varying and the slowly varying normalization follows from Eq. (7). These expressions lead to the particle concept, the number of particles per quantum state having the expectation value $n_k \equiv \langle a_k^\dagger a_k \rangle$.

Modes of interest are well inside the horizon now, and for inflation to do its job they have to be well inside the horizon at the beginning of almost-exponential inflation. (We assume as usual that the present density is critical, $\Omega_0 = 1$.) Thus they start and finish in the adiabatic regime, but if we choose $w_k = w^\text{flat}_k$ at early times we will have at late times some solution $\tilde{w}_k(\eta)$ given by

$$\tilde{w}_k(\eta) \equiv \alpha_k w^\text{flat}_k(\eta) + \beta_k(\omega^\text{flat}_k(\eta))^*$$  \hspace{1cm} (11)

From Eq. (7), $\alpha_k$ and $\beta_k$ are constants which satisfy

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$  \hspace{1cm} (12)

The Fourier expansion Eq. (4) can be written at late times

$$\phi = a^{-1}(\eta) \sum_k \left[ w^\text{flat}_k(\eta) \tilde{a}_k + w^\text{flat}_k(\eta)^\dagger \tilde{a}^\dagger_{-k} \right] e^{ikx}$$  \hspace{1cm} (13)

with where $\tilde{a}_k \equiv \alpha_k a_k + \beta_k^* a^\dagger_{-k}$. Finally, we suppose that no particles exist at early times, $\langle a_k^\dagger a_k \rangle = 0$ for all $k$. Then the number of particles per state at late times is $\langle \tilde{a}_k^\dagger \tilde{a}_k \rangle = |\beta_k|^2$.

By solving Eq. (5) with the initial condition Eq. (10), we can read off $\beta_k$ from Eq. (11). It is negligible, and hence there is no particle creation, if the adiabatic condition Eq. (9) is satisfied at all times. This is in general the case only for modes which are well inside the horizon at the end of inflation, i.e., for modes with $k \gg a_* H_*$ where a star denotes the end of inflation. For longer wavelength modes adiabaticity generally fails leading to significant particle creation. (Adiabaticity holds for all modes if $m^2(\eta)$ is negligible and $\xi = 1/6$, or more generally if $m^2(\eta)$ and $(\xi^2 - \xi)R$ cancel, but there is no reason to expect these special cases to be realized.)

In the absence of any special reason, one expects that $\alpha_k$ and $\beta_k$ will be very roughly of the same order, which implies that very roughly they are both of order 1. One therefore expects that very roughly the occupation number $|\beta_k|^2$ will be of order 1 for modes which are inside the horizon at the end of inflation, and practically zero for shorter wavelength modes. In the appendix some specific cases are examined, and the expectation verified.

It should be emphasized that we are here talking about the case where the mass is of order $H$ during inflation. In the opposite case $m \ll H$ it is well known, from the case of axion physics [2,1,26], that the occupation number $|\beta_k|^2$ becomes huge in the limit of large wavelengths.

**B. Spin half particles**

The spin 1/2 case has been discussed briefly before especially in the approximation of weak particle production [27,28,12], but the following account is more systematic and explicit and we treat for the first time the physically interesting case $m(\eta) \sim H$. 

5
The starting point is the Dirac equation in a Robertson-Walker universe, which can be derived from a Lagrangian [28,12] or from more general arguments [27]. With appropriate normalization as specified in a moment, it is simply the ordinary Dirac equation written in comoving coordinates,

\[ (ia^{-1}\gamma^\mu \partial_\mu - m + i\frac{3}{2} H \gamma^0)\psi = 0 \]  

(14)

To quantize one can start with the usual anticommuting annihilation operators \(a_{r\mathbf{k}}\) and \(b_{r\mathbf{k}}\) (where \(r\) denotes the spin state), and make the Fourier expansion

\[ \psi = a^{-3/2} \sum_{\mathbf{k}} \sum_{r=\pm} \left[ u_{r\mathbf{k}} a_{r\mathbf{k}} + v_{r\mathbf{k}} b_{r\mathbf{k}}^\dagger \right] e^{-i\mathbf{k} \cdot \mathbf{x}} \]  

(15)

(This applies to a Dirac field; for a Majorana field \(a_{r\mathbf{k}} = b_{r\mathbf{k}}\) which will make no difference in what follows.) Substituting this expression into the Dirac equation and choosing for each mode the \(z\) axis along \(\mathbf{k}\) gives

\[ i\gamma^0 u'_{r\mathbf{k}} - [\gamma^z \mathbf{k} + am] u_{r\mathbf{k}} = 0 \]  

(16)

\[ i\gamma^0 v'_{r\mathbf{k}} - [\gamma^z \mathbf{k} + am] v_{r\mathbf{k}} = 0 \]  

(17)

Any solution of this equation can be written as a linear combination of \(u_{1\mathbf{k}}, u_{2\mathbf{k}}, v_{1\mathbf{k}}\) and \(v_{2\mathbf{k}}\).

We impose on these four objects the usual orthonormality condition

\[ u_{r\mathbf{k}}^\dagger u_{s\mathbf{k}} = v_{r\mathbf{k}}^\dagger v_{s\mathbf{k}} = \delta_{rs} \]  

(18)

\[ u_{r\mathbf{k}}^\dagger v_{s\mathbf{k}} = 0 \]  

(19)

This condition is preserved by the evolution equation Eq. (14), because \((\gamma^0)^2\) is the unit matrix and \(\gamma^0 (a^{-1} \gamma \cdot \mathbf{k} + m)\) is Hermitian. Note that we did not pull out a factor \(1/(2E_k)\) before defining \(u_{\mathbf{k}}\) and \(v_{\mathbf{k}}\), as is often done in flat spacetime. Had we done so, \(u_{\mathbf{k}}\) and \(v_{\mathbf{k}}\) would have satisfied a more complicated evolution equation because of the time dependence of \(E_k\). We did however pull out a factor \(a^{-3/2}\), which again is crucial in making the equation simple.

As the evolution of the two spin states is independent we need consider only one of them. This means that we can regard \(\psi\) as only a two component object so we drop the subscript \(r\). The Dirac matrices can be taken to be

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  

(20)

Now drop the subscripts \(\mathbf{k}\) and \(-\mathbf{k}\), and denote the upper and lower components of \(u\) and \(v\) by a subscript \(\pm\). They satisfy uncoupled equations

\[ u''_{\pm} + a^2 [E_k^2 \pm i H m] u_{\pm} = 0 \]

\[ v''_{\pm} + a^2 [E_k^2 \pm i H m] v_{\pm} = 0 \]  

(21)

where as before \(E_k\) is the energy related to the momentum \(p_k = k/a\) by \(E_k^2 = m(\eta)^2 + p_k^2\).
With our assumption about \( m(\eta) \), \( Hm \ll E_k^2 \) during any era in which the adiabatic condition Eq. (9) is satisfied, and we recover flat spacetime field theory with the canonical choice

\[
\begin{align*}
    u^{\text{flat}} &\equiv \left( \frac{E_k + m}{2E_k} \right)^{1/2} \left( 1 \frac{p_k/(E_k + m)}{p_k/(E_k + m)} \right) e^{-i(E_k t + \chi_k)} \\
    v^{\text{flat}} &\equiv \left( \frac{E_k + m}{2E_k} \right)^{1/2} \left( -\frac{p_k/(E_k + m)}{1} \right) e^{i(E_k t + \chi_k)}
\end{align*}
\]

(22)

The ratio of the upper and lower components comes from Eq. (17), \( \chi_k \) is slowly varying, and the slowly varying normalization comes from conditions (18) and (19). As one knows from flat spacetime field theory, these expressions lead to the particle concept, the number of particles per quantum state being \( \langle a^{\dagger}_k a_k \rangle \) and the number of antiparticles being \( \langle b^{\dagger}_k b_k \rangle \).

Just as for a scalar field, we consider solutions \( \tilde{u} \) and \( \tilde{v} \), which become equal to \( u^{\text{flat}} \) and \( v^{\text{flat}} \) well before horizon exit during inflation. Remembering that \( u_+ \) has the same evolution equation as \( u_\ast - \) and similarly for \( v_\pm \), they are of the form

\[
\begin{align*}
    \tilde{u} &= \left( \begin{array}{c} f_1 \\ f_2^* \end{array} \right) \\
    \tilde{v} &= \left( \begin{array}{c} -f_2 \\ f_1^* \end{array} \right)
\end{align*}
\]

(23)

Thus, in the late time adiabatic regime they are of the form

\[
\begin{align*}
    \tilde{u}(\eta) &= \alpha_k u^{\text{flat}}(\eta) + \beta_k v^{\text{flat}}(\eta) \\
    \tilde{v}(\eta) &= \alpha_\ast_k v^{\text{flat}}(\eta) - \beta_\ast_k u^{\text{flat}}(\eta)
\end{align*}
\]

(24) (25)

Here \( \alpha_k \) and \( \beta_k \) are constants, which from the normalization condition (18) satisfy

\[
|\alpha_k|^2 + |\beta_k|^2 = 1
\]

(26)

The contribution of the mode to \( \psi \) is

\[
a^{-3/2} \left( u^{\text{flat}}\tilde{a}_k + v^{\text{flat}}\tilde{b}^{\dagger}_{-k} \right) e^{-ik\cdot x}
\]

(27)

where \( \tilde{a}_k = \alpha_k a_k - \beta_\ast_k b^{\dagger}_{-k} \) and \( \tilde{b}_k = \beta_k a_k + \alpha_\ast_k b^{\dagger}_{-k} \). Finally, we suppose that no particles exist at the beginning of inflation, \( \langle a^{\dagger}_k a_k \rangle = \langle b^{\dagger}_k b_k \rangle = 0 \). Then the number of particles per state at late times is \( \langle \tilde{a}^{\dagger}_k \tilde{a}_k \rangle = |\beta_k|^2 \), and the number of antiparticles \( \langle \tilde{b}^{\dagger}_k \tilde{b}_k \rangle \) is the same.

In order to calculate \( |\beta_k|^2 \) precisely one has to solve the second order differential equation (21) with the initial condition (10). But since it must lie between 0 and 1 one may hope that an adequate order of magnitude estimate will be obtained by setting it equal to 1 if there is an era of non-adiabaticity and equal to 0 otherwise. A couple of cases are worked out in the Appendix, and they support this hope provided that \( \mu \) is somewhat less than 1. (For \( \mu \sim 1 \) one finds \( |\beta_k|^2 \sim e^{-2\pi\mu} \).)

As in the spin zero case one will have adiabaticity at all times for those modes which are far inside the horizon at the end of inflation. In addition, there is adiabaticity at all times if the mass is negligible in the sense that \( p_k(\eta) \gg m(\eta) \) and \( p(\eta)^2 \gg H m(\eta) \), because then
Eq. (21) reduces to $u''_{\pm} + k^2 u_{\pm} = 0$. However if, as we are supposing, $m(\eta) \sim H$ (until the epoch $H \sim m$, which we are supposing occurs after the end of inflation), these conditions are equivalent to the adiabatic condition $p_k(\eta) \gg H$. The conclusion is therefore that if $m(\eta) \sim H$ (at least during inflation) one may hope for significant particle production in all modes for which the adiabatic condition fails just as in the spin 0 case.

III. COSMOLOGICAL IMPLICATIONS

We have considered a particle, whose true mass $m$ is less than the Hubble parameter $H_*$ during inflation. We have assumed that its mass $m(\eta)$ in the early universe, before the epoch $H \sim m$, is of order $H$ and smoothly varying which is what one expects in general for a particle with an interaction of only gravitational strength. On these assumptions, we have found that of order 1 particle per quantum state is created for modes which are outside the horizon at the end of inflation (corresponding to $k \lesssim a_* H_*$), whereas for smaller scales particle creation is insignificant. Although we have demonstrated this only for spin zero and spin half particles the argument looks quite general and it reasonable so suppose that it holds for any spin.\(^1\)

In this section we see what the estimate implies for cosmology. We assume of course that the species does not thermalize subsequently, since then it loses all memory of its earlier abundance.

A convenient measure of the abundance is provided by the ratio $n/s$, where $n$ is the number density and $s$ is the entropy density. After inflation is over and reheating has occurred, this ratio can be taken to be constant between the epoch when the particle stops interacting and the epoch when it decays. For a species which is initially in thermal equilibrium, $n/s \sim g_*^{-1}$ where $g_* \sim 10$ to $10^3$ is the effective number of particle species at the epoch when thermal equilibrium fails.

Summing over all modes, the vacuum fluctuation gives

$$n = (4\pi^2)^{-1} a^{-3} \int_0^{a_* H_*} |\beta_k|^2 k^2 dk \sim |\beta_{a_* H_*}|^2 (a_* H_*/a)^3$$

(28)

In estimating the integral we have assumed that it is dominated by the upper limit. This is true provided that $|\beta_k|^2 \sim k^\gamma$ with $\gamma > -3$, which holds for the cases we evaluated in the last section. As discussed in the text we expect $|\beta_{a_* H_*}| \sim 1$ and from now on we adopt the value 1.

The energy density just after reheating is of order $g T_R^4$ where $T_R$ is the reheat temperature, and the entropy density is of order $g T_R^3$. Suppose first that reheating is prompt so that $g T_R^4 \sim V_*$ where $V_* = 3H_*^2 M_{Pl}^2$ is the potential at the end of inflation. (Here $M_{Pl} = (8\pi G)^{-1/2} = 2 \times 10^{18}$ GeV is the reduced Planck mass.) Then

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\(^1\)As mentioned earlier, the gravitino is practically massless during inflation according to some models but it will have the canonical mass $\sim H$ afterwards which should be enough to ensure that the stated result holds. If the mass were much less than $H$ even after inflation one would need to ask whether the gravitino field equation is conformally invariant since in that case gravitino production would be negligible. We are not sure of the answer to that question.
\[ n/s \sim (H_*/M_{Pl})^{3/2} \sim (V_*^{1/4}/M_{Pl})^3 \]  

(29)

If reheating is delayed this estimate is reduced by a factor \( T_R/V^{1/4} \) to become

\[ n/s \sim \frac{T_R}{M_{Pl}} \left( \frac{V_*^{1/4}}{M_{Pl}} \right)^2 \]  

(30)

Now let us compare this abundance with that coming from particle collisions, assuming that the interaction of the species under consideration with other particles is of only gravitational strength. Candidates for this case are the gravitino (spin 3/2) [29], the moduli of superstring theory (spin 0) and their superpartners the modulini (spin 1/2) [15,16]. Up to numerical factors and dimensionless couplings, the cross section for producing the species in a typical relativistic particle collision process is of order \( M_{Pl}^{-2} \), and as a result the abundance arising from this mechanism is [30]

\[ n/s = fT_R/M_{Pl} \]  

(31)

where \( f \) comes from the numerical factors and couplings. In the case of a gravitino (with the standard properties), \( f \) can be estimated in a fairly model-independent fashion [30] to be of order \( 10^{-5} \). It is not quite clear what value of \( f \) to expect more generally, for example for moduli and modulini.

Comparing these two estimates, we see that creation from particle collisions dominates creation from the vacuum fluctuation, provided that \( V^{1/4}/M_{Pl} \ll f^{1/2} \). In order not to create too much cmb anisotropy one must have [31] \( V^{1/4} \ll 10^{-2} M_{Pl} \), and we see that if the upper bound is attained the vacuum fluctuation mechanism is marginally stronger, at least for gravitinos.

Now let us consider the cosmological constraints. First suppose that the species is unstable. Then it should decay with only gravitational strength or as we have just discussed the abundance from the vacuum fluctuation cannot be significant. The decay rate is then \( \Gamma = 10^{-2}\gamma m^3/M_{Pl}^2 \) where the coefficient \( \gamma \) is expected [19,21] to be \( \lesssim 1 \). The corresponding decay time \( \Gamma^{-1} \) is typically after the beginning of nucleosynthesis, and as a result the abundance during nucleosynthesis is constrained to be [30]

\[ n/s < 10^{-12} \text{ to } 10^{-15} \]  

(32)

Taking \( n/s \ll 10^{-15} \), the constraint on the reheat temperature following from the abundance due to particle collisions is

\[ T_R \lesssim 10^{-15} f^{-1} M_{Pl} \]  

(33)

Assuming that (standard) gravitinos exist they correspond to \( f \sim 10^{-5} \) which implies the well known constraint [30]

\[ T_R \lesssim 10^8 \text{ GeV} \]  

(34)

Using the vacuum fluctuation abundance instead multiplies this result by a factor \( f(M_{Pl}/V^{1/4})^2 \), which as we just noted might by marginally bigger than 1.
Now consider a stable particle species. The requirement $\Omega_0 < 1$, where $\Omega_0$ is the present density in units of critical density, gives \[ n/s < (1 \text{ eV}/m) \] (35)
where $m$ is the mass of the species. Using the abundance from particle collisions, this gives
\[
\left( \frac{m}{10 \text{ MeV}} \right) \leq \left( \frac{10^{-5}}{f} \right) \left( \frac{10^{16} \text{ GeV}}{T_R} \right)
\] (36)
Using instead the abundance Eq. (30) coming from the vacuum fluctuation it gives
\[
\left( \frac{m}{10^5 \text{ GeV}} \right) \leq \left( \frac{10^{16} \text{ GeV}}{V^{1/4}} \right)^2 \left( \frac{10^8 \text{ GeV}}{T_R} \right)
\] (37)
If reheating is instantaneous this becomes
\[
\left( \frac{m}{1 \text{ MeV}} \right) \leq \left( \frac{10^{16} \text{ GeV}}{V^{1/4}} \right)^3
\] (38)

IV. CONCLUSION

We have presented a complete and unified formalism for the cases spin zero and spin half. It invokes minimal physical assumptions, so that for example canonical quantization is required only in the adiabatic regime, with the classical evolution of the (Heisenberg representation) operators being the only quantum physics that is used in the intervening regime. The formalism is more complete than earlier discussions for the spin half case, and in both cases it is here applied for the first time to the case of an effective mass $\simeq H$.

The bottom line consists of the bounds on the parameters presented in the previous section, associated with the existence of particles having gravitational strength interactions. Though they are are at best only comparable with similar ones derived on the assumption that the particles are created through collisions after reheating, they are of interest in principle, and might be relevant if the inflation scale is high and production through collisions is more suppressed than for the gravitino.

APPENDIX

In this appendix some specific cases are worked out.

Spin zero

For modes which are well inside the horizon at the epoch $H = m$ (equivalently, for modes which are relativistic at that epoch), it is enough to solve Eq. (5) with $m(\eta) = \mu H$. It then has the form
\[ w_k'' + \left[ k^2 + \mu^2 \left( \frac{a'}{a} \right)^2 + (6\xi - 1) \frac{a''}{a} \right] w_k = 0 \]  
(39)

During exponential inflation, radiation domination and matter domination, this is equivalent to a Bessel equation for \( w_k/\eta^{1/2} \), with the order \( \nu \) given in Table 1, and assuming that the universe makes abrupt transitions between these regimes we can calculate \( \beta_k \) by matching \( w_k \) and its first derivative at each transition.

Let us drop the subscript \( k \), and let \( w_{\text{inf}} \) denote \( \tilde{w} \) during inflation. For argument \( x \gg 1 \), the Bessel function becomes

\[ H^{(1)}_\nu \simeq \sqrt{2\pi} x e^{i\left(z + \frac{1}{4} \nu \pi - \frac{1}{4} \pi \right)} \]  
(40)

By virtue of our assumptions, \( \nu^2_{\text{inf}} = \frac{9}{4} - \mu^2 - 12\xi \) is not much bigger than 1 so the regime of large argument corresponds precisely to the early-time adiabatic regime \( k\eta \gg 1 \). In this same regime \( E_k = k/a \), so one sees that the solution with the required early-time behaviour Eq. (10) is

\[ w_{\text{inf}} = \frac{\sqrt{\pi}}{2} \left(-\eta \right)^{1/2} H^{(1)}_{\nu_{\text{inf}}} (-k\eta) \exp[(2\nu_{\text{inf}} + 1) \frac{\pi}{4} i] \]  
(41)

We shall assume that inflation gives way quickly to an era of either radiation or matter domination, which persists until the adiabatic regime is encountered. Then, the solution \( w_{\text{era}} \) of Eq. (5) which reduces to Eq. (10) at late times is the solution \( w_{\text{rad}} \) which has the required form Eq. (10) at late times is

\[ w_{\text{era}} = \frac{\sqrt{\pi}}{2} \left(-\eta \right)^{1/2} H^{(1)}_{\nu_{\text{era}}} (-k\eta) \exp[-(2\nu_{\text{era}} + 1) \frac{\pi}{4} i] \]  
(42)

We are looking for a solution of the form

\[ \tilde{w} = \alpha w_{\text{era}} + \beta w_{\text{era}}^* \]  
(43)

whose value and first derivative matches the inflationary solution. Remembering the normalization Eq. (7) one has one finds

\[ \alpha = i(w_{\text{inf}} w_{\text{era}}' - w_{\text{inf}}' w_{\text{era}}) \]

\[ \beta = i(w_{\text{inf}} w_{\text{era}}' - w_{\text{inf}}' w_{\text{era}}) \]  
(44)

For modes with \( k \gg a_\star H_\star \) the evolution is adiabatic and \( |\beta| \) is negligible. Considering for simplicity only the opposite case \( k \ll a_\star H_\star \), we can use the small small-argument limits of the Bessel functions,

\[ H^{(1)}_\nu (x) = \frac{-i}{\sin \nu \pi} \left( \frac{1}{\Gamma(1 - \nu)} \left( \frac{x}{2} \right)^{-\nu} - \frac{e^{-i\nu \pi}}{\Gamma(1 + \nu)} \left( \frac{x}{2} \right)^{\nu} \right) \]  
(45)

\[ H^{(2)}_\nu (x) = \frac{i}{\sin \nu \pi} \left( \frac{1}{\Gamma(1 - \nu)} \left( \frac{x}{2} \right)^{-\nu} - \frac{e^{+i\nu \pi}}{\Gamma(1 + \nu)} \left( \frac{x}{2} \right)^{\nu} \right) \]  
(46)

Using them one finds

\[ \beta = \frac{i}{4\pi} \left(X(\nu_{\text{inf}}, \nu_{\text{era}}) + X(-\nu_{\text{inf}}, -\nu_{\text{era}}) + X(\nu_{\text{inf}}, -\nu_{\text{era}}) + X(-\nu_{\text{inf}}, \nu_{\text{era}})\right) \]  
(47)

where

\[ X(a, b) = (1 + a + b) e^{i(b-a)\frac{\pi}{2}} \Gamma(-a) \Gamma(-b) \left( \frac{pm^2}{2} \right)^{a+b} \]  
(48)

11
Radiation domination after inflation

For the case of a scalar field with conformal coupling ($\xi = \frac{1}{6}$) the equations of motion during radiation domination are identical to those during inflation (ie $\nu_{\text{inf}} = \nu_{\text{rad}}$). For this situation the equation for $\beta$ simplifies significantly

$$\beta = \frac{i}{4\pi} \left( (1 + 2\nu)\Gamma(-\nu)^2 \left( \frac{1}{2a_*H_*} \right)^{2\nu} + (1 - 2\nu)\Gamma(\nu)^2 \left( \frac{1}{2a_*H_*} \right)^{-2\nu} - \frac{2\pi \cos \pi \nu}{\nu \sin \pi \nu} \right)$$

(49)

For the case where $c = \mu$ we have $\nu = i\sqrt{\frac{3}{4}}$ and hence the third term dominates

$$\beta \simeq \frac{i \cosh \sqrt{\frac{3}{4}}}{\sqrt{3} \sinh \sqrt{\frac{3}{4}}}$$

(50)

$$|\beta|^2 \simeq \frac{1}{3}$$

(51)

In general the other two (neglected) terms in the formula for $\beta$ will produce oscillatory terms in the particle numbers, but do not affect the total number significantly.

For the case of a scalar field without coupling to the Ricci scalar $\nu$ will in general be different during the different era’s under consideration. During inflation $\nu = \sqrt{\frac{9}{4} - \mu^2}$ and during radiation domination $\nu = \sqrt{\frac{1}{4} - \mu^2}$. For $\frac{1}{2} < \mu < \frac{3}{2}$ then $\nu_{\text{inf}}$ will be real whilst $\nu_{\text{rad}}$ will be imaginary. In this case one of the terms in the expansion will dominate. If we set $\nu_{\text{inf}} = a$ and $\nu_{\text{rad}} = ib$ for convenience, where both $a$ and $b$ are real, then taking the leading term in our expression gives

$$\beta \simeq \frac{i}{4\pi} (1 - a - ib)e^{\frac{\pi a}{4}}e^{\frac{\pi b}{4}} \Gamma(a)\Gamma(ib) \left( \frac{1}{2a_*H_*} \right)^{-a-ib}$$

(52)

$$|\beta|^2 \simeq \frac{e^{\beta \pi}}{16\pi b \sinh \pi b} ((1 - a)^2 + b^2)\Gamma(a)^2 \left( \frac{1}{2a_*H_*} \right)^{-2a}$$

(53)

In the case where $\mu = 1$ this works out as

$$|\beta|^2 \approx 0.03 \left( \frac{1}{2a_*H_*} \right)^{-\sqrt{5}}$$

(54)

Matter domination after inflation

If a conformally coupled scalar field re-enters the particle horizon during during matter domination as opposed to radiation domination as discussed above then the equations describing its evolution will no longer be the same during the two epochs and hence the full general solution must be used. Taking a particular choice of mass such that $\mu = 1$ then the
orders of the two Bessel functions during the subsequent epochs will be \( \nu_{\text{inf}} = i \sqrt{3/4} \) during inflation and \( \nu_{\text{mat}} = 1 \sqrt{15/4} \) during matter domination. For these values it is clear that the third term in the general expression will dominate, and hence our approximate solution is

\[
\beta \approx \frac{i}{4\pi} (1 - 1.07i) e^{1.4\pi} \Gamma \left( i \sqrt{3/4} \right) \Gamma \left( i \sqrt{15/4} \right) \left( \frac{k}{a_{\ast}H_{\ast}} \right)^{-i 0.07} \\
|\beta|^2 \approx \frac{2.145}{(4\pi)^2} e^{2.8\pi} \left| \Gamma \left( i \sqrt{3/4} \right) \right|^2 \left| \Gamma \left( i \sqrt{15/4} \right) \right|^2 (55)
\]

\[
\approx 0.3 (56)
\]

For a minimally coupled field, \( \xi = 0 \), the equation for \( \beta \) is the same as that for re-entry during radiation domination with the exception that the constant \( b \) is replaced by the corresponding value for matter domination (ie \( \nu_{\text{mat}} = ib \)). Calculating the result for the case where \( \mu = 1 \) gives

\[
|\beta|^2 \approx 0.03 \left( \frac{k}{a_{\ast}H_{\ast}} \right)^{-\sqrt{5}} (58)
\]

### Spin half

In contrast with the spin zero case, we definitely need to consider only modes which are well inside the horizon at the epoch \( H = m \), corresponding to \( k \gg a_{\ast}H_{\ast} \). The reason is that modes with smaller wavenumber are killed by the phase space factor \( k^2 \) in Eq. (28), owing to the fact that the occupation number can \( |\beta| \) can never exceed 1. Thus it is enough to set \( m(\eta) = \mu H \) at all times. Then, during exponential inflation, matter domination and radiation domination Eq. (21) is equivalent to a Bessels equation for \( u_{\pm}/\eta^{1/2} \) and \( v_{\pm}/\eta^{1/2} \) with the index given in Table 1. During inflation, the solutions reducing to Eq. (22) at early times are

\[
\begin{align*}
    f_{\text{inf}(1)} &= \frac{i}{2} \sqrt{\pi k|\eta|} e^{\mu \frac{\eta}{2} H_{\ast}(1)_{\frac{1}{2}-i\mu}} (k|\eta|) \\
    f_{\text{inf}(2)} &= -\frac{i}{2} \sqrt{\pi k|\eta|} e^{-\mu \frac{\eta}{2} H_{\ast}(2)_{\frac{1}{2}-i\mu}} (k|\eta|)
\end{align*}
\]

(59)

(60)

We have used this formalism to estimate the occupation numbers on the assumption that inflation is promptly followed by either radiation or matter domination. The solutions during radiation domination which reduce to Eq. (22) at late times are

\[
\begin{align*}
    f_{\text{rad}(1)} &= -\frac{i}{2} \sqrt{\pi k\eta} e^{-\mu \frac{\eta}{2} H_{\ast}(2)_{\frac{1}{2}-i\mu}} (k\eta) \\
    f_{\text{rad}(2)} &= \frac{i}{2} \sqrt{\pi k\eta} e^{\mu \frac{\eta}{2} H_{\ast}(1)_{\frac{1}{2}-i\mu}} (k\eta)
\end{align*}
\]

(61)

(62)

The solutions for matter domination are the same with \( \mu \) replaced by \( 2\mu \). Matching either set of solutions gives
\[ f_{\text{inf}(1)} = \alpha f_{\text{era}(1)} - \beta f_{\text{era}(2)} \quad (63) \]
\[ f_{\text{inf}(2)} = \alpha f_{\text{era}(2)} + \beta f_{\text{era}(1)} \quad (64) \]

And hence

\[ \beta = (f^*_{\text{inf}(2)} f_{\text{era}(1)} - f_{\text{inf}(1)} f^*_{\text{era}(2)}) \quad (65) \]

where the right hand side is evaluated at the end of inflation.

For radiation domination this gives

\[ \beta = \frac{1}{2\pi} \left[ \Gamma \left( \frac{1}{2} - i\mu \right)^2 \left( \frac{k}{2 a_s H_*} \right)^2 - \text{c.c.} \right] \quad (66) \]

and for matter domination it gives

\[ \beta = \frac{1}{2\pi} (Be^{\frac{7}{2}i\mu} - B^*e^{-\frac{7}{2}i\mu}) \quad (67) \]

where

\[ B = \Gamma \left( \frac{1}{2} + i\mu \right) \Gamma \left( \frac{1}{2} + 2i\mu \right) \left( \frac{k}{a_s H_*} \right)^{-3i\mu} \quad (68) \]

Since \(|\Gamma(\frac{1}{2} \pm iy)|^2 = \pi / \cosh(\pi y)\) these expressions give \(|\beta|^2 < 1\) as required. In the adiabatic regimes \(k \gg a_s H_*\) and \(\mu \gg 1\) they give the expected result \(|\beta|^2 \ll 1\). In the regime \(k \ll a_s H_*\) and \(\mu \sim 1\), they make \(|\beta|^2\) a rapidly oscillating function of \(k\) with a mean value roughly of order \(e^{-2\pi\mu}\), which is roughly of order 1 with our assumed value \(\mu \sim 1\).

**TABLE 1. The order of the Bessel function**

<table>
<thead>
<tr>
<th>Scalars</th>
<th>Spinors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation</td>
<td>(\nu^2 = \frac{9}{4} - 12\xi - \mu^2)</td>
</tr>
<tr>
<td>Radiation dom.</td>
<td>(\nu^2 = \frac{1}{4} - \mu^2)</td>
</tr>
<tr>
<td>Matter dom.</td>
<td>(\nu^2 = \frac{9}{4} - 12\xi - 4\mu^2)</td>
</tr>
</tbody>
</table>
Acknowledgements

We are indebted to Ewan Stewart for many helpful discussions about supergravity. The work is partially supported by grants from PPARC and from the European Commission under the Human Capital and Mobility programme, contract No. CHRX-CT94-0423.
REFERENCES


