Perturbations of spacetime: gauge transformations
and gauge invariance at second order and beyond

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Abstract.

We consider in detail the problem of gauge dependence that exists in relativistic perturbation theory, going beyond the linear approximation and treating second and higher order perturbations. We first derive some mathematical results concerning the Taylor expansion of tensor fields under the action of one-parameter families (not necessarily groups) of diffeomorphisms. Second, we define gauge invariance to an arbitrary order $n$. Finally, we give a generating formula for the gauge transformation to an arbitrary order and explicit rules to second and third order. This formalism can be used in any field of applied general relativity, such as cosmological and black hole perturbations, as well as in other spacetime theories. As a specific example, we consider here second order perturbations in cosmology, assuming a flat Robertson–Walker background, giving explicit second order transformations between the synchronous and the Poisson (generalized longitudinal) gauges.

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1. Introduction

The perturbative approach is a fundamental tool of investigation in general relativity, where exact solutions are most often too idealized to properly represent the realm of natural phenomena. Unfortunately, it has long been known that the invariance of general relativity under diffeomorphisms (two solutions of the Einstein equation are physically equivalent if they are diffeomorphic to each other) makes the very definition of perturbations gauge dependent [1, 2, 3, 4, 5, 6, 7, 8, 9]. A gauge choice is an identification between points of the perturbed and the background spacetimes, and generic perturbations are not invariant under a gauge transformation. This “gauge problem” has been widely treated in linear theory, but what if one wants to consider higher order perturbations? In particular, how do the latter change under a gauge transformation?

Second order treatments have been recently proposed, both in cosmology [10, 11] and compact object theory [12], as a way of obtaining more accurate results to be compared with present and future observations. For example, in view of the increased sensitivity expected from the next generation of detectors, precise computations of microwave background anisotropies or gravitational wave production may require going beyond the linear regime. In addition, second order perturbations provide a reliable measure of the accuracy of the linearized theory (see, e.g., [12]). In cosmology, a second (or higher) order treatment might prove necessary when dealing with scales much smaller than the cosmological horizon, where the linear approximation can either be not accurate enough or just miss some physical effects. Finally, general relativity is an intrinsically non-linear theory, thus it is in principle interesting to look at higher order perturbations as a tool for exploring non-linear features. Unfortunately, already the second order calculations are almost invariably a computational tour de force, and a gauge-invariant treatment is not at hand. As a matter of fact, the computationally more convenient gauge does not necessarily coincide with the most interesting one, and often different authors work in different gauges, but at present a general formalism to deal with second and higher order gauge transformations is not available in the literature, although some partial results can be found in [13, 14, 15] and in [16] (and references therein).

The aim of this paper is to fill this gap. Thus, while we shall consider the problem of gauge transformations from a general geometrical perspective, from the practical point of view our main goal is to derive the effect on a tensor field $T$ of a second order gauge transformation. To this end, we shall show that the latter is necessarily represented in coordinates by

$$\tilde{x}^\mu = x^\mu + \lambda \xi^{(1)}_\mu + \frac{\lambda^2}{2} \left( \xi^{(1),\nu}_\mu \xi^{(1)\nu} + \xi^{(2)}_\mu \right) + \cdots ,$$ (1.1)

where $\xi^{(1)}$ and $\xi^{(2)}$ are two independent vector fields, and that, under (1.1), the first and second order perturbations of $T$ transform as

$$\delta \tilde{T} = \delta T + \mathcal{L}_{\xi^{(1)}} T_0 ,$$ (1.2)

$$\delta^2 \tilde{T} = \delta^2 T + 2 \mathcal{L}_{\xi^{(1)}} \delta T + \mathcal{L}^2_{\xi^{(1)}} T_0 + \mathcal{L}_{\xi^{(2)}} T_0 .$$ (1.3)

Equation (1.2) is the usual first order result in terms of the Lie derivative of the background tensor field $T_0$ along the vector field $\xi^{(1)}$, and the second order perturbation $\delta^2 T$ is defined by (3.3) and (3.4). In fact, we derive a general formula,
equation (4.6), from which the gauge transformations to an arbitrary order \( n \) can be deduced, although we will give the explicit expression only for transformations up to third order. Furthermore, we show that a tensor field \( T \) is gauge-invariant to second order if and only if it is gauge-invariant to first order, i.e.,

\[
\mathcal{L}_\xi T_0 = 0
\]  

for an arbitrary vector field \( \xi \), and, in addition,

\[
\mathcal{L}_\xi \delta T = 0.
\]  

Actually, we generalize the above result giving a condition for gauge invariance to an arbitrary order \( n \).

The plan of the paper is the following. In the next section we give some mathematical results concerning the Taylor expansion of tensor fields under the action of one-parameter families (not necessarily groups) of diffeomorphisms. This material is necessary for the applications to follow; however, the reader interested only in the latter can skip all the proofs, as well as the last two paragraphs of section 2.2; section 2.4 is useful if one wants to translate our results in the language of coordinates. In section 3 we give a general discussion of spacetime perturbations and gauge choices, giving in particular a precise definition of the \( k \)-th order perturbation of a general tensor field. In section 4 we first define total gauge invariance and gauge invariance to order \( n \); then we give the generating formula for a gauge transformation to an arbitrary order, and explicit rules for second and third order transformations. In section 5 we consider perturbations of a flat Robertson–Walker model, and apply our results to the case of the transformations between two specific gauge choices, i.e., the synchronous [18] and the Poisson [19] gauges. Section 6 contains a final discussion.

In general, we shall work on an \( m \)-dimensional Lorentzian manifold of signature \( m-2 \). The abstract mathematical notation is used extensively, but sometimes we shall make reference to charts; in this case, coordinate indices \( \mu, \nu, \ldots \) take values from 0 to \( m-1 \). Units are such that \( c = 1 \). Furthermore, for the sake of simplicity we shall always suppose that manifolds, maps, tensor fields, etc., are as smooth as necessary, or even analytic (see [20] for an extension to \( C^r \) fields).

2. Taylor expansions of tensor fields

Before considering the specific problem of gauge transformations in relativistic perturbation theory we need to establish some general results concerning the Taylor expansion of tensor fields, to be used later. For functions on \( \mathbb{R}^m \), a Taylor expansion is essentially a convenient way to express the value of the function at some point in terms of its value, and the value of all its derivatives, at another point. Of course, this is impossible for a tensor field \( T \) on a manifold \( \mathcal{M} \), simply because \( T(p) \) and \( T(q) \) at different points \( p \) and \( q \) belong to different spaces, and cannot thus be directly compared. A Taylor expansion can therefore be written only if one is given a mapping between tensors at different points of \( \mathcal{M} \). In this section, we study the case in which such a mapping arises from a one-parameter family of diffeomorphisms of \( \mathcal{M} \), starting from the simplest case of a flow (i.e., a one-parameter group of diffeomorphisms) and then proceeding to generalize it. Let us first establish some notation.

\[\text{A short account of the material covered in sections 2–4 is presented in [17].}\]
Let $\varphi : M \to N$ be a diffeomorphism between two manifolds $M$ and $N$. For each $p \in M$, $\varphi$ defines naturally the linear map $\varphi_*|_p : T_pM \to T_{\varphi(p)}N$ between the tangent spaces, called push-forward, and the linear map $\varphi^*|_{\varphi(p)} : T^*_pN \to T^*_{\varphi(p)}M$ between the cotangent spaces, called the pull-back.‡ Using $\varphi^{-1}$, we can define also a push-forward of $T^*_pM$ on $T^*_{\varphi(p)}N$, and a pull-back of $T_{\varphi(p)}N$ on $T_pM$, so that $\varphi_*|_p$ and $\varphi^*|_{\varphi(p)}$ turn out to be well-defined for tensors of arbitrary type. We can also construct maps $\varphi_*$ and $\varphi^*$ for tensor fields, simply requiring that, $\forall \ p \in M$,

$$
(\varphi_*T)(\varphi(p)) := \varphi_*|_p (T(p)) \tag{2.1}
$$
and

$$
(\varphi^*T)(p) := \varphi^*|_{\varphi(p)} (T(\varphi(p))) \tag{2.2}
$$

Hereafter, we drop the suffixes in $\varphi_*|_p$ and $\varphi^*|_{\varphi(p)}$, since there is no real danger of confusion.

2.1. Flows

Let $M$ be a differentiable manifold, and let $\xi$ be a vector field on $M$, generating a flow $\phi : \mathbb{R} \times M \to M$, where $\phi(0, p) = p$, $\forall \ p \in M$.† For any given $\lambda \in \mathbb{R}$, we shall write, following the common usage, $\phi_\lambda(p) := \phi(\lambda, p), \forall \ p \in M$. Let $T$ be a tensor field on $M$. The map $\phi^*_\lambda$ defines a new field $\phi^*_\lambda T$ on $M$, the pull-back of $T$, which is thus a function of $\lambda$.

**Lemma 1:** The field $\phi^*_\lambda T$ admits the following expansion around $\lambda = 0$:

$$
\phi^*_\lambda T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} L^k_\xi T. \tag{2.3}
$$

**Proof:** By analyticity we have

$$
\phi^*_\lambda T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \left. \frac{d^k}{d\lambda^k} \right|_0 \phi^*_\lambda T, \tag{2.4}
$$
where, here and in the following,

$$
\frac{d^k}{d\lambda^k} (\cdots) := \left[ \frac{d^k(\cdots)}{d\lambda^k} \right]_{\lambda=\tau}, \tag{2.5}
$$
and the first derivative is, by definition, just the Lie derivative of $T$ with respect to $\xi$:

$$
\frac{d}{d\lambda} \bigg|_0 \phi^*_\lambda T = \lim_{\lambda \to 0} \frac{1}{\lambda} (\phi^*_\lambda T - T) =: L^\xi T. \tag{2.6}
$$

In order to prove (2.3), it is then sufficient to show that, $\forall \ k$,

$$
\frac{d^k}{d\lambda^k} \bigg|_0 \phi^*_\lambda T = L^k_\xi T, \tag{2.7}
$$

† Here, we are following the most common notation, although some authors (see, e.g., [21]) denote the push-forward and the pull-back exactly in the opposite way.

† In order not to burden the discussion unnecessarily, we suppose that $\phi$ defines global transformations of $M$ [22].
This can be established by induction over \( k \). Suppose that (2.7) is true for some \( k \). Then

\[
\frac{d^{k+1}}{d\lambda^{k+1}} \bigg|_0 \phi^*_\lambda T =
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \frac{d^k}{d\lambda^k} \bigg|_0 \phi^*_\lambda T - \frac{d^k}{d\lambda^k} \bigg|_0 \phi^*_\lambda T \right) =
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \phi^*_{\tau + \varepsilon} T - \phi^*_\tau T \right) =
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \phi^*_{\varepsilon} \mathcal{L}^k T - \mathcal{L}^k T \right) = \mathcal{L}^{k+1} T,
\tag{2.8}
\]

where \( \tau := \lambda - \varepsilon \), and we have used the property that \( \phi_\lambda \) forms a one-parameter group: 

\( \phi_{\tau + \varepsilon} = \phi_\tau \circ \phi_\varepsilon \).

It is worth noticing that (2.3) can also be written \([23]\) in the symbolic form \( \phi^*_\lambda = \exp(\lambda \mathcal{L}_\xi) \).

Equation (2.3) can be applied to the special case in which the tensor \( T \) is just one of the coordinate functions on \( M \), \( x^\mu \). We have then, since \( \phi^*_\lambda x^\mu(p) = x^\mu(\phi_\lambda(p)) \), the usual action of an “infinitesimal point transformation” extended to second order in \( \lambda \):

\[
\tilde{x}^\mu = x^\mu + \lambda \xi^\mu + \frac{\lambda^2}{2} \xi^\mu,\nu \xi^\nu + \cdots;
\tag{2.9}
\]

where we have denoted \( x^\mu(p) \) simply by \( x^\mu \), and \( x^\mu(\phi_\lambda(p)) \) by \( \tilde{x}^\mu \).

2.2. Knight diffeomorphisms

Let us now suppose that there are two vector fields \( \xi^{(1)} \) and \( \xi^{(2)} \) on \( M \). Separately, they generate the flows \( \phi^{(1)} \) and \( \phi^{(2)} \), respectively. We can combine \( \phi^{(1)} \) and \( \phi^{(2)} \) to define a new one-parameter family of diffeomorphisms \( \Psi : \mathbb{R} \times M \to M \), whose action is given by \( \Psi_\lambda := \phi^{(2)}_{\lambda/2} \circ \phi^{(1)}_\lambda \). Thus, \( \Psi_\lambda \) displaces a point of \( M \) a parameter interval \( \lambda \) along the integral curve of \( \xi^{(1)} \), and then an interval \( \lambda^2/2 \) along the integral curve of \( \xi^{(2)} \) (see figure 1). For this reason, we shall call it, with a chess-inspired terminology, a knight diffeomorphism.

This concept can be immediately generalized to the case in which \( n \) vector fields \( \xi^{(1)}, \ldots, \xi^{(n)} \) are defined on \( M \), corresponding to the flows \( \phi^{(1)}, \ldots, \phi^{(n)} \). Then we define a one-parameter family \( \Psi : \mathbb{R} \times M \to M \) of knight diffeomorphisms of rank \( n \) by

\[
\Psi_\lambda := \phi^{(n)}_{\lambda/\sqrt{n}} \circ \cdots \circ \phi^{(2)}_{\lambda/2} \circ \phi^{(1)}_\lambda,
\tag{2.10}
\]

and the vector fields \( \xi^{(1)}, \ldots, \xi^{(n)} \) will be called the generators of \( \Psi \).

Of course, \( \Psi_\sigma \circ \Psi_\lambda \neq \Psi_{\sigma + \lambda} \); consequently, Lemma 1 cannot be applied if we want to expand in \( \lambda \) the pull-back \( \Psi^*_\lambda T \) of a tensor field \( T \) defined on \( M \). However, the result can be easily generalized.

**Lemma 2:** The pull-back \( \Psi^*_\lambda T \) of a tensor field \( T \) by a one-parameter family of knight diffeomorphisms \( \Psi \) with generators \( \xi^{(1)}, \ldots, \xi^{(k)}, \ldots \) can be expanded around \( \lambda = 0 \) as
follows:

$$\Psi_\lambda^\ast T = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \cdots \sum_{l_k=0}^{+\infty} \frac{\lambda^{l_1+2l_2+\cdots+kl_k+\cdots}}{2! \cdots (k!)^l_1 l_2 \cdots l_k \cdots} L_{\xi_1}^{l_1} L_{\xi_2}^{l_2} \cdots L_{\xi_k}^{l_k} \cdots T.$$  

(2.11)

**Proof:**

$$\Psi_\lambda^\ast T = \phi_\lambda^{(1)_\ast} \phi_\lambda^{2/2}_\ast \cdots \phi_\lambda^{k/k!}_\ast \cdots T = \sum_{l_1=0}^{+\infty} \frac{\lambda^{l_1}}{l_1!} L_{\xi_1}^{l_1} \left( \phi_\lambda^{2/2}_\ast \cdots \phi_\lambda^{k/k!}_\ast \cdots T \right)$$

$$= \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \cdots \sum_{l_k=0}^{+\infty} \cdots \frac{\lambda^{l_1+2l_2+\cdots+kl_k+\cdots}}{2! \cdots (k!)^l_1 l_2 \cdots l_k \cdots} L_{\xi_1}^{l_1} L_{\xi_2}^{l_2} \cdots L_{\xi_k}^{l_k} \cdots T,$$

(2.12)

where we have repeatedly used (2.3).

The explicit form of (2.11) up to the third order in $\lambda$ is

$$\Psi_\lambda^\ast T = T + \lambda L_{\xi_1} T + \frac{\lambda^2}{2} \left( L_{\xi_1}^2 + L_{\xi_2} \right) T$$

$$+ \frac{\lambda^3}{3!} \left( L_{\xi_1}^3 + 3 L_{\xi_1} L_{\xi_2} + L_{\xi_3} \right) T + \cdots.$$  

(2.13)

Equations (2.11) and (2.13) apply to a one-parameter family of knight diffeomorphisms of arbitrarily high rank, and can be specialized to the particular case of rank $n$ simply by setting $\xi_k = 0, \forall k > n$. Their meaning is particularly clear in a chart. Denoting, as before, by $x^\mu$ the coordinates of a point $p$, and by $\tilde{x}^\mu$ those of $\Psi_\lambda(p)$, we have, to order $\lambda^2$,

$$\tilde{x}^\mu = x^\mu + \lambda \xi^\mu_1 + \frac{\lambda^2}{2} \left( \xi^\mu_1 \xi^\nu_1 + \xi^\nu_1 \right) + \cdots.$$  

(2.14)

Equation (2.14) is represented pictorially in figure 2.

Since $\Psi_\sigma \circ \Psi_\lambda \neq \Psi_{\sigma + \lambda}$, and $\Psi_{\sigma}^{-1} \neq \Psi_{-\lambda}$, one may reasonably doubt that $\Psi$ forms a group, except under very special conditions. This is confirmed by the following

**Theorem 1:** The only cases in which $\Psi$ forms a group are those for which $\xi_{(k)} = \alpha_k \xi_{(1)}, \forall k \geq 2$, with $\alpha_k$ arbitrary numerical coefficients. Then, under the reparametrization $\lambda := f(\lambda), \text{ with } f(\lambda) := \lambda + \sum_{k=2}^{+\infty} \alpha_k \lambda^k / k!$, $\Psi$ reduces to a flow in the canonical form.

**Proof:** Let us first show that $\xi_{(k)} = \alpha_k \xi_{(1)}, \forall k \geq 2$, is a sufficient condition for $\Psi$ to be a group. Since Lemma 1 implies $\phi_k^{(1)} = \phi_k^{(1)}_{\alpha_k \sigma}, \text{ we have } \Psi_\lambda = \cdots \circ \phi_k^{(1)}_{\alpha_k \lambda^k / k!} \circ \cdots \circ \phi_1^{(1)} = \phi_1^{(1)}$. Thus, (i) $\Psi_\sigma \circ \Psi_\lambda = \phi_\sigma \circ \phi_\lambda^{(1)} = \phi_\sigma^{(1)} = \phi_\sigma^{(1)}_{\sigma + \lambda} = \Psi_\tau$, with $\tau = f^{-1}(\sigma + \lambda)$, and (ii) $\Psi_\lambda^{-1} = \phi_\lambda^{(1)} = \phi_\lambda^{(1)} = \Psi_\rho$, with $\rho = f^{-1}(-\lambda)$.

To prove the reverse implication, let us consider first the case of a knight diffeomorphism of rank two. If $\Psi$ has to form a group, then for any $\lambda, \sigma \in \mathbb{R}$, there must exist a $\tau \in \mathbb{R}$ such that $\Psi_\sigma \circ \Psi_\lambda = \Psi_\tau$, i.e.,

$$\phi_{\lambda^{2/2}}^{(2)} \circ \phi_{\sigma}^{(1)} \circ \phi_{\lambda^{2/2}}^{(1)} \circ \phi_\lambda^{(1)} = \phi_{\lambda^{2/2}}^{(2)} \circ \phi_\tau^{(1)}.$$  

(2.15)
Applying Lemma 1 to (2.15) we get, to second order in the parameters and for an arbitrary tensor $T$,

$$
(\tau - \lambda - \sigma) \mathcal{L}_{\xi(\lambda)} T + \frac{1}{2} \left( \tau^2 + \lambda^2 - 2\lambda \tau - \sigma^2 \right) \mathcal{L}^2_{\xi(\lambda)} T + \frac{1}{2} \left( \tau^2 - \lambda^2 - \sigma^2 \right) \mathcal{L}_{\xi(\lambda)} T + \cdots = 0 .
$$

(2.16)

In the limit $\lambda, \sigma \to 0$, one gets from (2.16) that

$$
\tau = \lambda + \sigma - \alpha_2 \lambda \sigma - \beta_2 \lambda^2 - \gamma_2 \sigma^2 + \cdots,
$$

where $\alpha_2, \beta_2, \gamma_2$ are unspecified numerical coefficients. Substituting back into (2.16) we have that

$$
\lambda \sigma \left( \mathcal{L}_{\xi(\lambda)} - \alpha_2 \mathcal{L}_{\xi(\lambda)} \right) T - \beta_2 \lambda^2 \mathcal{L}_{\xi(\lambda)} T - \gamma_2 \sigma^2 \mathcal{L}_{\xi(\lambda)} T + \cdots = 0 .
$$

(2.17)

It is clear that (2.17) can be satisfied only if $\beta_2 = \gamma_2 = 0$ and $\xi(2) = \alpha_2 \xi(1)$. Similarly, considering higher rank knight diffeomorphisms, one can show that $\xi(k) = \alpha_k \xi(1), \forall k$.

Finally, it is perhaps worth pointing out that the result of Lemma 2 cannot be written, even formally, as

$$
\Psi^* T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \mathcal{L}_{\xi(\lambda)}^k T
$$

(2.18)

with

$$
\xi(\lambda) := \sum_{h=0}^{+\infty} \frac{\lambda^h}{h!} \xi(h+1) ,
$$

(2.19)

because this expression fails to agree with (2.11) for $k \geq 3$. One might try to define a vector $\eta(\lambda)$ for which an analog of (2.18) holds, but this does not seem very useful or illuminating.

2.3. General case

Knight diffeomorphisms are of a very peculiar form, and the previous results seem therefore of limited applicability. This is, however, not the case, because any one-parameter family of diffeomorphisms can always be regarded as a one-parameter family of knight diffeomorphisms — of infinite rank, in general — as shown by the following

**Theorem 2:** Let $\Psi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ be a one-parameter family of diffeomorphisms. Then $\exists \phi^{(1)}, \ldots, \phi^{(k)}, \ldots$, one-parameter groups of diffeomorphisms of $\mathcal{M}$, such that

$$
\Psi = \cdots \circ \phi^{(k)}_{\lambda^k} \circ \cdots \circ \phi^{(2)}_{\lambda^2} \circ \phi^{(1)}_{\lambda} .
$$

(2.20)
Proof: Consider the action of $\Psi_\lambda$ on a function $f : \mathcal{M} \to \mathbb{R}$. A Taylor expansion of $\Psi_\lambda^* f$ gives

$$
\Psi_\lambda^* f = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k} \bigg|_0 \Psi_\lambda^* f.
$$

(2.21)

The differential operator $L_{(1)}$ defined by

$$
L_{(1)} f := \frac{d}{d\lambda} \bigg|_0 \Psi_\lambda^* f
$$

(2.22)

is clearly a derivative, so we can define a vector $\xi_{(1)}$ through $\xi_{(1)} f := L_{(1)} f$. Similarly,

$$
L_{(2)} f := \frac{d^2}{d\lambda^2} \bigg|_0 \Psi_\lambda^* f - L^2_{\xi_{(1)}} f
$$

(2.23)

is also a derivative, as one can easily check. Thus, we define the vector $\xi_{(2)}$ such that $L_{\xi_{(1)}} f := L_{(2)} f$, and so on at higher orders. Hence, we recover (2.11) for an arbitrary $f$. But if $\varphi$ and $\psi$ are two diffeomorphisms on $\mathcal{M}$ such that $\varphi^* f = \psi^* f$ for every $f$, it follows that $\varphi \equiv \psi$, as it is easy to see. Thus, we establish (2.20).

It must be noticed that, although we have supposed so far that maps and fields are analytic, it is possible to give versions of Lemmas 1 and 2, and of Theorem 2, that hold in the case of $C^r$ objects [20]. The main change is the substitution of Taylor series like the one in (2.3) by a finite sum of $n - 1$ terms plus a remainder [22]. The meaning of Theorem 2 is then that any one-parameter family of diffeomorphisms can be approximated by a family of knight diffeomorphisms of suitable rank.

2.4. Interpreting the literature: What is what

The abstract mathematical notation that we have used so far is the most appropriate one for the study of gauge transformations in perturbation theory from a general point of view. In order to make explicit calculations in special cases of physical interest, however, one must introduce a chart. Most of the literature on the subject, therefore, is written in the language of coordinates. For this reason, we indicate here how to establish a correspondence between the two formalisms. For the sake of simplicity in the notation, we shall restrict ourselves to consider the action of the pull-back on a vector field, the extension to one-forms and tensors of higher rank being straightforward.

Let therefore $(\mathcal{U}, x)$ be a chart of $\mathcal{M}$, with $\mathcal{U} \subseteq \mathcal{M}$ an open set and $x : \mathcal{U} \to \mathbb{R}^m$ given by $x : p \mapsto (x^0(p), x^1(p), \ldots, x^{m-1}(p)), \forall p \in \mathcal{U}$. Since the function $x^\mu : \mathcal{M} \to \mathbb{R}$ is differentiable, we can define the linear map $x^\mu_*$, that associates to a vector on $\mathcal{M}$ its $\mu$-th component in the coordinate basis defined by the chart $(\mathcal{U}, x)$.

† Notice that, in the language of differential forms, $x^\mu_* = dx^\mu$.

Consider now a vector field $Z$ and its pull-back $\tilde{Z} := \Psi_\lambda^* Z$, which for every value of $\lambda$ is a new vector field on $\mathcal{M}$. For each point $p \in \mathcal{U}$, the components of $\tilde{Z}(p)$ in the chart $(\mathcal{U}, x)$ are

$$
\tilde{Z}^\mu(x(p)) = \left( x^\mu_\lambda \tilde{Z} \right)(x(p)),
$$

(2.24)

† Notice that, in the language of differential forms, $x^\mu_* = dx^\mu$. 

which becomes, using (2.2) and the definition of $\tilde{Z}$,

$$
\tilde{Z}^\mu(x(p)) = x^\mu_\nu \left( \tilde{Z}(p) \right) = x^\mu_\nu (\Psi_\lambda^* Z)(p) = (x^\mu_\nu \circ \Psi_\lambda^{-1})(Z(\Psi_\lambda(p))). \tag{2.25}
$$

Now, let us define a new chart $(\Psi_\lambda(U), y)$, with $y^\mu := x^\mu \circ \Psi_\lambda^{-1}$. In this way, the $y$-coordinates of the point $q := \Psi_\lambda(p)$ coincide with the $x$-coordinates of the point $p$ from which $q$ has come under the action of the diffeomorphism: $y^\mu(q) = x^\mu(p)$. We have then

$$
\tilde{Z}^\mu(x(p)) = y^\mu_\nu (Z(\Psi_\lambda(p))) = (y^\mu_\nu Z)(y(\Psi_\lambda(p))) = (y^\mu_\nu Z)(y(q)). \tag{2.26}
$$

Denoting by $Z^\mu$ and $Z'^\mu$ the components of $Z$ in the charts $(U, x)$ and $(\Psi_\lambda(U), y)$, respectively, we can then write

$$
\tilde{Z}^\mu(x(p)) = Z'^\mu(y(q)) = \left[ \frac{\partial y^\mu}{\partial x^\nu} \right]_x Z^\nu(y(q)). \tag{2.27}
$$

That is, the pull-back of $Z$ is characterized by having, in the chart $(U, x)$ at point $p$, the same components that the original vector field has in the chart $(\Psi_\lambda(U), y)$ at point $q = \Psi_\lambda(p)$. Or, since $y(q) = x(p)$, one can just write $\tilde{Z}^\mu(x) = Z'^\mu(x)$, where $x$ simply stands for a point of $\mathbb{R}^m$. This property can also be used to define $\tilde{Z}$, and corresponds to a passive interpretation of the map $\Psi_\lambda$, regarded as generating a change in the chart on $M$ rather than a transformation of $M$ (active view). Of course, the two viewpoints are equivalent [21], but the active interpretation is much less confusing.

We end this section by writing down the explicit expression, up to second order in $\lambda$, for the coordinate transformation $x \rightarrow y$ associated to a family of diffeomorphisms with generators $\xi_1(\lambda), \xi_2(\lambda), \ldots$. Calling $q := \Psi_\lambda(p)$, we have from (2.14),

$$
x^\mu(q) = x^\mu(p) + \lambda \xi^\mu_1(x(p)) + \frac{\lambda^2}{2} \left( \xi^\mu_1_{,\nu}(x(p)) \xi^\nu_1(x(p)) + \xi^\mu_2(x(p)) \right) + \cdots. \tag{2.28}
$$

By definition, we have also

$$
y^\mu(q) := x^\mu(p) = x^\mu(q) - \lambda \xi^\mu_1(x(p)) - \frac{\lambda^2}{2} \left( \xi^\mu_1_{,\nu}(x(q)) \xi^\nu_1(x(q)) - \xi^\mu_2(x(q)) \right) + \cdots. \tag{2.29}
$$

Expanding the various quantities on the right hand side around $q$, (2.29) becomes finally

$$
y^\mu(q) = x^\mu(q) - \lambda \xi^\mu_1(x(q)) + \frac{\lambda^2}{2} \left( \xi^\mu_1_{,\nu}(x(q)) \xi^\nu_1(x(q)) - \xi^\mu_2(x(q)) \right) + \cdots. \tag{2.30}
$$

Equations (2.28) and (2.30) express the relationship, in the language of coordinates, between the active and the passive views. Whereas (2.28) provides us with the coordinates, in the same chart $(U, x)$, of the different points $p$ and $q = \Psi_\lambda(p)$, equation (2.30) gives the transformation law between the coordinates of the same point $q$ in the two different charts $(U, x)$ and $(\Psi_\lambda(U), y)$. An equivalent form of the transformation (2.30) was already used by Taub in studying the gauge dependence of an approximate stress energy tensor for gravitational fields [13].

Using (2.30) for the actual computation of the coordinate transformation in (2.27), and expanding every term again at second order around $x(p)$, one can derive the
components in the chart \((U, x)\) of the pull-back \(\Psi^*_\lambda Z\) of \(Z\), given in terms of a second order expansion formula involving \(Z\) and its partial derivatives along \(\xi(1)\) and \(\xi(2)\). Then, properly collecting the various terms, one can check that this leads to the components of the right hand side of (2.13), i.e., to the components of a Taylor expansion of \(\Psi^*_\lambda Z\) in terms of the Lie derivatives along \(\xi(1)\) and \(\xi(2)\) of \(Z\).

3. Perturbations of spacetime and gauge choices

In relativistic perturbation theory one tries to find approximate solutions of the Einstein equation, regarding them as “small” deviations from some known exact solution — the so-called background. The perturbation \(\Delta T\) in any relevant quantity, say represented by a tensor field \(T\), is defined as the difference between the value \(T\) has in the physical spacetime, and the background value \(T_0\). However, it is a basic fact of differential geometry that, in order to make the comparison of tensors meaningful at all, one has to consider them at the same point. Since \(T\) and \(T_0\) are defined in different spacetimes, they can thus be compared only after a prescription for identifying points of these spacetimes is given. A gauge choice is precisely this, i.e., a map between the background and the physical spacetime. Mathematically, any diffeomorphism between the two spacetimes provides one such prescription. A change of this diffeomorphism is then a gauge transformation, and the freedom one has in choosing it corresponds to the arbitrariness in the value of the perturbation of \(T\) at any given spacetime point, unless \(T\) is gauge-invariant. This is the essence of the “gauge problem,” which has been discussed in depth in many papers [1, 2, 3, 5, 6] and review articles [4, 8, 9].

In order to discuss higher order perturbations and gauge transformations, and to define gauge invariance, we must formalize the previous ideas, giving a precise description of what perturbations and gauge choices are. Here we shall mainly follow the approach used in references [2, 6, 7, 21] (cf. also [14, 16]).

Let us thus consider a family of spacetime models \(\{\mathcal{M}(g, \tau)\}\), where the metric \(g\) and the matter fields (here collectively referred to as \(\tau\)) satisfy the field equation
\[
\mathcal{E}[g, \tau] = 0 ,
\]
and \(\lambda \in \mathbb{R}\). We shall assume that \(g\) and \(\tau\) depend smoothly on the parameter \(\lambda\), so that \(\lambda\) itself is a measure of the amount by which a specific \((\mathcal{M}, g, \tau)\) differs from the idealized background solution \((\mathcal{M}, g_0, \tau_0)\), which is supposed to be known. In some applications, \(\lambda\) is a dimensionless parameter naturally arising from the physical problem one is dealing with. In this case one expects the perturbative solution to accurately approximate the exact one for reasonably small \(\lambda\) (see, e.g., [12]). In other problems, \(\lambda\) can be introduced as a purely formal parameter, and in the end, for convenience, one can thus choose \(\lambda = 1\) for the physical spacetime, as we shall do in section 5.

This situation is most naturally described by introducing an \((m + 1)\)-dimensional manifold \(\mathcal{N}\), foliated by submanifolds diffeomorphic to \(\mathcal{M}\), so that \(\mathcal{N} = \mathcal{M} \times \mathbb{R}\). We shall label each copy of \(\mathcal{M}\) by the corresponding value of the parameter \(\lambda\). The manifold \(\mathcal{N}\) has a natural differentiable structure which is the direct product of those of \(\mathcal{M}\) and \(\mathbb{R}\). We can then choose charts in which \(x^\mu\) \((\mu = 0, 1, \ldots, m - 1)\) are coordinates on each leave \(\mathcal{M}_\lambda\), and \(x^m \equiv \lambda\).

Now, if a tensor field \(T_\lambda\) is given on each \(\mathcal{M}_\lambda\), we have that a tensor field \(T\) is automatically defined on \(\mathcal{N}\) by the relation \(T(p, \lambda) := T_\lambda(p)\), with \(p \in \mathcal{M}_\lambda\).† In † It is worth noticing that tensor fields on \(\mathcal{N}\) constructed in this way are “transverse,” in the sense

\[
\mathcal{E}[g, \tau] = 0 ,
\]
particular, on each $M_\lambda$ one has a metric $g_\lambda$ and a set of matter fields $\tau_\lambda$, satisfying the field equation (3.1); correspondingly, the fields $g$ and $\tau$ are defined on $N$.

We want now to define the perturbation in any tensor $T$, therefore we must find a way to compare $T_\lambda$ with $T_0$. As already said, this requires a prescription for identifying points of $M_\lambda$ with those of $M_0$. This is easily accomplished by assigning a diffeomorphism $\varphi_\lambda : N \to N$ such that $\varphi_\lambda|_{M_0} : M_0 \to M_\lambda$. Clearly, $\varphi_\lambda$ can be regarded as the member of a flow $\varphi$ on $N$, corresponding to the value $\lambda$ of the group parameter. Therefore, we could equally well give the vector field $X$ that generates $\varphi$. In the chart introduced above, $X^m = 1$ but, except for this condition, $X$ remains arbitrary. With a slight abuse of terminology, we shall sometimes refer also to such a vector field as a gauge.

The perturbation can now be defined simply as

$$\Delta T_\lambda := \varphi_\lambda^* T|_{M_0} - T_0.$$  \hspace{1cm} (3.2)

The first term on the right hand side of (3.2) can be Taylor-expanded to get

$$\Delta T_\lambda = \sum_{k=1}^{+\infty} \frac{\lambda^k}{k!} \delta^k T,$$  \hspace{1cm} (3.3)

where

$$\delta^k T := \left[ \frac{d^k \varphi_\lambda^* T}{d\lambda^k} \right]_{\lambda=0,M_0}.$$  \hspace{1cm} (3.4)

Equation (3.4) defines then the $k$-th order perturbation of $T$. Notice that $\Delta T_\lambda$ and $\delta^k T$ are defined on $M_0$; this formalizes the statement one commonly finds in the literature, that “perturbations are fields living in the background.” It is important to appreciate that the parameter $\lambda$ labeling the various spacetime models serves also to perform the expansion (3.3), and determines therefore what one means by “perturbations of the $k$-th order.” However, as we have already pointed out, there are applications where $\lambda$ is, to a large extent, arbitrary. In these cases, the split of $\Delta T_\lambda$ into perturbations of first order, second order, and so on, has no absolute meaning, because a change of $\lambda$, i.e., a reparametrization of the family of spacetimes, will mix them up. What is invariantly defined, is only the quantity $\Delta T_\lambda$, whereas the various $\delta^k T$ are meaningful only once a choice of the parameter has been made.

Now, we are interested in those cases in which (3.1) is too difficult to solve exactly, so that one looks for approximate solutions, to some order $n$. In fact, we can now obtain much simpler linear equations from (3.1). At first order, differentiating (3.1) with respect to $\lambda$ and setting $\lambda$ equal to zero, one obtains [21] a linear equation for $\delta g$ and $\delta \tau$. At second order, a second derivative with respect to $\lambda$ of (3.1) at $\lambda=0$ gives an equation of the type

$$L \left[ \delta^2 g, \delta^2 \tau \right] = S \left[ \delta g, \delta \tau \right],$$  \hspace{1cm} (3.5)

which is linear in the second order perturbations $\delta^2 g$ and $\delta^2 \tau$, and where the first order perturbations $\delta g$, $\delta \tau$ now appear as known source terms. This can obviously be extended to higher orders, giving an iterative procedure to calculate $\Delta g_\lambda$ and $\Delta \tau_\lambda$ — hence $g_\lambda$ and $\tau_\lambda$ — to the required accuracy.

that their $m$-th components in the charts we have defined vanish identically.
4. Gauge invariance and gauge transformations

Let us now suppose that two vector fields $X$ and $Y$ are defined on $\mathcal{N}$, such that they have $X^m = Y^m = 1$ everywhere. Correspondingly, their integral curves define two flows $\varphi$ and $\psi$ on $\mathcal{N}$, that connect any two leaves of the foliation. Thus $X$ and $Y$ are everywhere transverse to the $\mathcal{M}_\lambda$, and points lying on the same integral curve of either of the two are to be regarded as the same point within the respective gauge: $\varphi$ and $\psi$ are both point identification maps, i.e., two different gauge choices.

The fields $X$ and $Y$ can both be used to pull back a generic tensor field $T$, and to construct therefore two other tensor fields $\varphi_\lambda^* T$ and $\psi_\lambda^* T$, for any given value of $\lambda$. In particular, on $\mathcal{M}_0$ we now have three tensor fields, i.e., $T_0$, and

$$T_X^\lambda := \varphi_\lambda^* T |_0, \quad T_Y^\lambda := \psi_\lambda^* T |_0,$$

where, for the sake of simplicity, we have denoted the restriction to $\mathcal{N}$ simply by the suffix 0.

Since $X$ and $Y$ represent gauge choices for mapping a perturbed manifold $\mathcal{M}_\lambda$ into the unperturbed one $\mathcal{M}_0$, $T_X^\lambda$ and $T_Y^\lambda$ are the representations, in $\mathcal{M}_0$, of the perturbed tensor according to the two gauges. We can write, using (3.2)–(3.4) and Lemma 1,

$$T_X^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta^k T_X = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \xi^k T |_0 = T_0 + \Delta^\varphi T_\lambda,$$

(4.2)

$$T_Y^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta^k T_Y = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \xi^k T |_0 = T_0 + \Delta^\psi T_\lambda.$$

(4.3)

4.1. Gauge invariance

If $T_X^\lambda = T_Y^\lambda$, for any pair of gauges $X$ and $Y$, we say that $T$ is totally gauge-invariant. This is a very strong condition, because then (4.2) and (4.3) imply that $\delta^k T_X = \delta^k T_Y$, for all gauges $X$ and $Y$ and for any $k$. In any practical case one is however interested in perturbations to a fixed order $n$; it is thus convenient to weaken the definition above, saying that $T$ is gauge-invariant to order $n$ iff $\delta^k T_X = \delta^k T_Y$ for any two gauges $X$ and $Y$, and $\forall k \leq n$. We have then the following ($\delta^0 T := T_0$, $\delta T := \delta^1 T$)

**Proposition 1:** A tensor field $T$ is gauge-invariant to order $n \geq 1$ iff $\xi^k T = 0$, for any vector field $\xi$ on $\mathcal{M}$ and $\forall k < n$.

**Proof:** Let us first show that the statement is true for $n = 1$. In fact, if $\delta T_X = \delta T_Y$, we have $\xi^k T |_0 = 0$. But since $X$ and $Y$ define arbitrary gauges, it follows that $X - Y$ is an arbitrary vector field $\xi$ with $\xi^m = 0$, i.e., tangent to $\mathcal{M}$. Let us now suppose that the statement is true for some $n$. Then, if one has also $\delta^{n+1} T_X |_0 = \delta^{n+1} T_Y |_0$, it follows that $\xi^{n} T_X = 0$, and we establish the result by induction over $n$. \qed

As a consequence, $T$ is gauge-invariant to order $n$ iff $T_0$ and all its perturbations of order lower than $n$ are, in any gauge, either vanishing, or constant scalars, or a combination of Kronecker deltas with constant coefficients. Thus, this generalizes to an arbitrary order $n$ the results of references [1, 2, 6, 7]. Further, it then follows that $T$ is totally gauge-invariant iff it is a combination of Kronecker deltas with coefficients depending only on $\lambda$. 

4.2. Gauge transformations

If a tensor $T$ is not gauge-invariant, it is important to know how its representation on $\mathcal{M}_0$ changes under a gauge transformation. To this purpose, it is useful to define, for each value of $\lambda \in \mathbb{R}$, the diffeomorphism $\Phi_{\lambda} : \mathcal{M}_0 \to \mathcal{M}_0$ given by

$$\Phi_{\lambda} := \varphi_{-\lambda} \circ \psi_{\lambda} .$$

(4.4)

The action of $\Phi_{\lambda}$ is illustrated in figure 3. We must stress that $\Phi : \mathbb{R} \times \mathcal{M}_0 \to \mathcal{M}_0$ so defined, is not a one-parameter group of diffeomorphisms on $\mathcal{M}_0$. In fact, $\Phi_{-\lambda} \neq \Phi_{\lambda}^{-1}$, and $\Phi_{\lambda+\sigma} \neq \Phi_{\sigma} \circ \Phi_{\lambda}$, essentially because the fields $X$ and $Y$ have, in general, a non vanishing commutator, as depicted in figure 4. However, Theorem 2 guarantees that, to order $n$ in $\lambda$, the one-parameter family of diffeomorphisms $\Phi$ can always be approximated by a one-parameter family of knight diffeomorphisms of rank $n$† (see figure 3 for the action of $\Phi_{\lambda}$ to second order).

It is very easy to see that the tensor fields $T^X_\lambda$ and $T^Y_\lambda$ defined by the gauges $\varphi$ and $\psi$ are connected by the linear map $\Phi^*_\lambda$:

$$T^Y_\lambda = \psi^*_\lambda T|_0 = (\psi^*_\lambda \varphi_{-\lambda} \varphi^*_\lambda T)|_0 = \Phi^*_\lambda (\varphi^*_\lambda T)|_0 = \Phi^*_\lambda T^X_\lambda .$$

(4.5)

Thus, Theorem 2 allows us to use (2.11) as a generating formula for a gauge transformation to an arbitrary order $n$:

$$T^Y_\lambda = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \ldots \sum_{l_k=0}^{\infty} \lambda^{l_1+2l_2+\cdots+kl_k+\cdots} \frac{1}{2!} \ldots (k!)^l \ldots l_1! l_2! \ldots l_k! \ldots \left( \mathcal{L}_{\xi(1)} T^Y + \mathcal{L}_{\xi(2)} T^Y + \mathcal{L}_{\xi(n)} T^Y \right) \cdot \mathcal{T}^X_\lambda .$$

(4.6)

To third order, we have explicitly

$$T^Y_\lambda = T^X_\lambda + \lambda \mathcal{L}_{\xi(1)} T^X_\lambda + \frac{\lambda^2}{2} \left( \mathcal{L}_{\xi(1)}^2 + \mathcal{L}_{\xi(2)} T^X \right) T^X_\lambda$$

$$+ \frac{\lambda^3}{3!} \left( \mathcal{L}_{\xi(1)}^3 + 3 \mathcal{L}_{\xi(1)} \mathcal{L}_{\xi(2)} T^X + \mathcal{L}_{\xi(3)} T^X \right) T^X_\lambda + \ldots ,$$

(4.7)

where $\xi(1)$ and $\xi(2)$ are now the first two generators of $\Phi_{\lambda}$, or of the gauge transformation, if one prefers.

We can now relate the perturbations in the two gauges. To the lowest orders, this is easy to do explicitly:

**Proposition 2**: Given a tensor field $T$, the relations between the first, second, and third order perturbations of $T$ in two different gauges are‡:

$$\delta T^Y - \delta T^X = \mathcal{L}_{\xi(1)} T_0 ;$$

(4.8)

$$\delta^2 T^Y - \delta^2 T^X = \left( \mathcal{L}_{\xi(1)} + \mathcal{L}_{\xi(1)}^2 \right) T_0 + 2 \mathcal{L}_{\xi(1)} \delta T^X ;$$

(4.9)

$$\delta^3 T^Y - \delta^3 T^X = \left( \mathcal{L}_{\xi(1)} + 3 \mathcal{L}_{\xi(1)} \mathcal{L}_{\xi(2)} + \mathcal{L}_{\xi(1)}^2 \right) T_0$$

$$+ 3 \left( \mathcal{L}_{\xi(1)} + \mathcal{L}_{\xi(1)}^2 \right) \delta T + 3 \mathcal{L}_{\xi(1)} \delta^2 T^X .$$

(4.10)

† This result confirms a claim in [14].

‡ A second order gauge transformation equivalent to (4.9) has recently been given in [15], see their section III C.
Proof: Substitute (4.2) and (4.3) into (4.7).

This result is consistent with Proposition 1, of course. Equation (4.8) implies that $T_\lambda$ is gauge-invariant to the first order iff $\mathcal{L}_\xi T_0 = 0$, for any vector field $\xi$ on $\mathcal{M}$. In particular, one must have $\mathcal{L}_{\xi(\lambda)} T_0 = 0$, and therefore (4.9) leads to $\mathcal{L}_\xi \delta T = 0$. Similarly, one has then $\mathcal{L}_\xi \delta^2 T = 0$ from (4.10), and so on recursively at higher orders.

It is also possible to find the explicit expressions, in terms of $X$ and $Y$, for the generators $\xi_{(k)}$ of a gauge transformation:

**Proposition 3:** The first three generators of the one-parameter family of diffeomorphisms $\Phi$ are:

1. $\xi_{(1)} = Y - X$; (4.11)
2. $\xi_{(2)} = [X,Y]$; (4.12)
3. $\xi_{(3)} = [2X - Y, [X,Y]]$. (4.13)

**Proof:** Substituting (4.2) and (4.3) into (4.7), using the fact that $\mathcal{L}_{\xi(\lambda)} \lambda = 0$, and identifying terms of first order in $\lambda$, we find

$$\mathcal{L}_{\xi_{(1)}} T_0 = \mathcal{L}_{Y-X} T_0|_0.$$ (4.14)

Since $Y^m - X^m = 0$ and $T$ is arbitrary, we have (4.11). Substituting back, and identifying terms of order $\lambda^2$, we have now, similarly,

$$\mathcal{L}_{\xi_{(2)}} T_0 = \mathcal{L}_{[X,Y]} T_0|_0.$$ (4.15)

But $[X,Y]^m = 0$, so we obtain (4.12). Analogously, one finds

$$\mathcal{L}_{\xi_{(3)}} T_0 = \mathcal{L}_{[2X - Y, [X,Y]]} T_0|_0.$$ (4.16)

Since $[2X - Y, [X,Y]]^m = 0$, we get (4.13).

5. **An example from cosmology**

As an example of the applications of the gauge transformation obtained, we now show how the perturbations on a spatially flat Robertson–Walker background in two different gauges are related, up to second order. We shall first consider the metric perturbations, then those in the energy density and 4-velocity of matter. Thus in this section we choose $m = 4$, so that the Greek indices $\mu, \nu, \ldots$ take values from 0 to 3, and the Latin ones $i, j, \ldots$ from 1 to 3.

The components of a perturbed spatially flat Robertson–Walker metric can be written as

$$g_{00} = -a(\tau)^2 \left( 1 + 2 \sum_{r=1}^{+\infty} \frac{1}{r!} \psi^{(r)}(r) \right),$$ (5.1)

$$g_{0i} = a(\tau)^2 \sum_{r=1}^{+\infty} \frac{1}{r!} \omega^{(r)}_i,$$ (5.2)
\[ g_{ij} = a(\tau)^2 \left( 1 - 2 \sum_{r=1}^{+\infty} \frac{1}{r!} \phi^{(r)} \right) \delta_{ij} + \sum_{r=1}^{+\infty} \frac{1}{r!} \chi_{ij}^{(r)} \right], \quad (5.3) \]

where \( \chi_{ij}^{(r)} = 0 \), and \( \tau \) is the conformal time. The functions \( \psi^{(r)} \), \( \omega_i^{(r)} \), \( \phi^{(r)} \), and \( \chi_{ij}^{(r)} \) represent the \( r \)-th order perturbation of the metric.

It is standard to use a non-local splitting of perturbations into the so-called scalar, vector and tensor parts, where scalar (or longitudinal) parts are those related to a scalar potential, vector parts are those related to transverse (divergence-free, or solenoidal) vector fields, and tensor parts to transverse trace-free tensors. Such a splitting generalizes the Helmholtz theorem of standard vector calculus (see, e.g., [24]), and can be performed on any spacetime (see, e.g., [6] and references therein) imposing suitable boundary conditions. In our case, the shift \( \omega_i^{(r)} \) can be decomposed as

\[ \omega_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)} \perp, \quad (5.4) \]

where \( \omega_i^{(r)} \perp \) is a solenoidal vector, i.e., \( \partial^i \omega_i^{(r)} \perp = 0 \). Similarly, the traceless part of the spatial metric can be decomposed at any order as

\[ \chi_{ij}^{(r)} = D_{ij} \chi^{(r)} + \partial_i \chi_i^{(r)} \perp + \partial_j \chi_j^{(r)} \perp + \chi_{ij}^{(r)\perp}, \quad (5.5) \]

where \( \chi^{(r)} \) is a suitable function, \( \chi_i^{(r)} \perp \) is a solenoidal vector field, and \( \partial^i \chi_{ij}^{(r)\perp} = 0 \); hereafter,

\[ D_{ij} := \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2. \quad (5.6) \]

Now, consider the energy density \( \mu \), or any other scalar that depends only on \( \tau \) at zero order: this can be written as

\[ \mu = \mu(0) + \sum_{r=1}^{+\infty} \frac{1}{r!} \delta^r \mu. \quad (5.7) \]

For the 4-velocity \( u^\mu \) of matter we can write

\[ u^\mu = \frac{1}{a} \left( \delta^\mu_0 + \sum_{r=1}^{+\infty} \frac{1}{r!} v_i^{(r)} \right). \quad (5.8) \]

In addition, \( u^\mu \) is subject to the normalization condition \( u^\mu u_\mu = -1 \); therefore at any order the time component \( v_0^{(r)} \) is related to the lapse perturbation, \( \psi^{(r)} \). For the first and second order perturbations we obtain, in any gauge:

\[ v_0^{(1)} = - \psi^{(1)}; \quad (5.9) \]
\[ v_0^{(2)} = - \psi^{(2)} + 3 \psi^{2} + 2 \omega_i^{(1)} v_i^{(1)} + v_i^{(1)} v_i^{(1)} \]. \quad (5.10) \]

The velocity perturbation \( v_i^{(r)} \) can also be split into a scalar and vector (solenoidal) part:

\[ v_i^{(r)} = \partial^i v_i^{(r)} + v_i^{(r)} \perp. \quad (5.11) \]

\(^\dagger\) Indices are raised and lowered using \( \delta^{ij} \) and \( \delta_{ij} \), respectively.
As we have seen in the last section, the gauge transformation is determined by the vectors \( \xi^{(r)} \). Splitting their time and space parts, one can write
\[
\xi_0^{(r)} = \alpha^{(r)},
\]
and
\[
\xi_{i}^{(r)} = \partial^i \beta^{(r)} + d^{(r)i},
\]
with \( \partial_i d^{(r)i} = 0 \).

5.1. First order

We begin by reviewing briefly some well-known results about first order gauge transformations, as we shall need them in the following. From now on, we will drop the suffixes \( X \) and \( Y \) used previously to denote the “old” and “new” gauge choices, simply using a tilde to denote quantities in the new gauge.

For the sake of completeness, we recall here the basic coordinate expressions of the Lie derivative along a vector field \( \xi \). For a scalar \( f \), a vector \( Z \) and a covariant tensor \( T \) of rank two, these are, respectively:
\[
\mathcal{L}_\xi f = f_{,\mu} \xi^\mu ;
\]
\[
\mathcal{L}_\xi Z^\mu = Z^\mu_{,\nu} \xi^\nu - \xi^\mu_{,\nu} Z^\nu ;
\]
\[
\mathcal{L}_\xi T_{\mu\nu} = T_{\mu\nu,\sigma} \xi^\sigma + \xi^\sigma_{,\mu} T_{\sigma\nu} + \xi^\sigma_{,\nu} T_{\mu\sigma} .
\]

Expressions for any other tensor can easily be derived from these.

5.1.1. General transformation

From (4.8), it follows that the first order perturbations of the metric transform as
\[
\delta \tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_{\xi^{(1)}} g^{(0)}_{\mu\nu} ,
\]
where \( g^{(0)}_{\mu\nu} \) is the background metric. Therefore, using (5.16), we obtain the following transformations for the first order quantities appearing in (5.1)–(5.3):
\[
\tilde{\psi}^{(1)} = \psi^{(1)} + \alpha_{(1)} + \frac{a'}{a} \alpha_{(1)} ;
\]
\[
\tilde{\omega}^{(1)}_i = \omega^{(1)}_i - \alpha_{(1)} + \beta_{(1)i}^{(1)r} + d_{(1)i}^{(1)r} ;
\]
\[
\tilde{\phi}^{(1)} = \phi^{(1)} - \frac{1}{3} \nabla^2 \beta_{(1)} + \frac{a'}{a} \alpha_{(1)} ;
\]
\[
\tilde{\chi}_{ij}^{(1)} = \chi_{ij}^{(1)} + 2D_{ij} \beta^{(1)} + d_{ij}^{(1)} + d_{ij}^{(1)} ;
\]
where a prime denotes the derivative with respect to \( \tau \).

For a scalar \( \mu \), from (4.8), (5.7), and (5.14) we have
\[
\delta \tilde{\mu} = \delta \mu + \mu^{(1)}_{(0)} \alpha_{(1)} .
\]
For the 4-velocity \( u^{\mu} \), we have from (4.8)
\[
\delta \tilde{u}^{\mu} = \delta u^{\mu} + \mathcal{L}_{\xi^{(1)}} u^{\mu}_{(0)} .
\]
Using (5.15) and (5.8) this gives:

\[ \hat{v}^0_0^{(1)} = v^0_0^{(1)} - \frac{a'}{a} \alpha^{(1)}_0 - \alpha'^{(1)}_0 ; \]  
\[ \hat{v}^i_0^{(1)} = v^i_0^{(1)} - \beta'^{(1)}_i - d'^{(1)}_i . \]  

(5.24)

(5.25)

The 4-velocity is however subject to the constraint (5.9), therefore (5.24) reduces to (5.18).

5.1.2. Transforming from the synchronous to the Poisson gauge

Let us now consider the particular case of the transformation from the synchronous to the Poisson gauge. The synchronous gauge has been the one most frequently used in cosmological perturbation theory; it is defined by the conditions \( g_{00} = -a(\tau)^2 \) and \( g_{0i} = 0 \) [18]. In this way the four degrees of freedom associated with the coordinate (or diffeomorphism) invariance of the theory are fixed. The Poisson gauge, recently discussed by Bertschinger [19], is instead defined by \( \omega^{(r),i} = \chi^{(r),i} = 0 \). Then, one scalar degree of freedom is eliminated from \( g_{0i} \) \( (\omega^{(r)} = 0) \), and one scalar and two vector degrees of freedom from \( g_{ij} \) \( (\chi^{(r)} = 0) \). This gauge generalizes the well-known longitudinal gauge to include vector and tensor modes. This gauge, in which \( \omega^{(r)} = \chi^{(r)} = 0 \), has been widely used in the literature to investigate the evolution of scalar perturbations [8]. Since the vector and tensor modes are set to zero by hand, the longitudinal gauge cannot be used to study perturbations beyond the linear regime, because in the nonlinear case the scalar, vector, and tensor modes are dynamically coupled.†

Given the perturbation of the metric in one gauge, it is easy to obtain, from (5.18)–(5.21), the gauge transformation to the other one, hence the perturbations in the new gauge. In the particular case of the synchronous and Poisson gauges, we have:

\[ \psi^{(1)} = \alpha^{(1)} + \frac{a'}{a} \alpha^{(1)} ; \]  
\[ \alpha^{(1)} = \beta^{(1)} ; \]  
\[ \omega^{(1)} = d^{(1)} ; \]  
\[ \phi^{(1)} = \phi^{(1)}_b - \frac{1}{3} \nabla^2 \beta^{(1)} - \frac{a'}{a} \alpha^{(1)} ; \]  
\[ D_{ij} \left( \chi^{(1)}_{s} + 2 \beta^{(1)} \right) = 0 ; \]  
\[ \chi^{(1)}_{s} (i,j) + d^{(1)}_{i,j} = 0 ; \]  
\[ \chi^{(1)}_{s} (i,j) = \chi^{(1)}_{s} (i,j) . \]  

(5.26)

(5.27)

(5.28)

(5.29)

(5.30)

(5.31)

(5.32)

The parameters \( \alpha^{(1)} \), \( \beta^{(1)} \), and \( d^{(1)}_{i,j} \) of the gauge transformation can be obtained from (5.27), (5.30), and (5.31) respectively, while the transformed metric perturbations follow from (5.26), (5.28), (5.29), and (5.32).

Once these parameters are known, the transformation rules for the energy density \( \mu \) or any other scalar, and those for the 4-velocity \( u^\mu \), follow trivially from (5.22), (5.24), and (5.25).

† In other words, even if one starts with purely scalar linear perturbations as initial conditions for the second order theory, vector and tensor modes are dynamically generated [10].
5.2. Second order

We now extend these well-known transformation rules of linear metric perturbations to the second order.

5.2.1. General transformation  Second order perturbations of the metric transform, according to (4.9), as

\[
\delta^2 \tilde{g}_{\mu\nu} = \delta^2 g_{\mu\nu} + 2 \mathcal{L}_{\xi(1)} \delta g_{\mu\nu} + \mathcal{L}_{\xi(1)}^2 g_{\mu\nu} + \mathcal{L}_{\xi(2)} g_{\mu\nu}.
\]  

(5.33)

This leads to the following transformations in the second order quantities appearing in (5.1)–(5.3):

\textbf{lapse perturbation}

\[
\tilde{\psi}^{(2)} = \psi^{(2)} + \alpha^{(1)} \left[ 2 \left( \psi^{(1)} + 2 \frac{\alpha'}{a} \psi^{(1)} \right) + \alpha''^{(1)} + \frac{5}{a} \alpha'_{(1)} + \left( \frac{\alpha''}{a} + \frac{\alpha'^2}{a^2} \right) \alpha^{(1)} \right] \\
+ \xi^{i}_{(1)} \left( 2 \psi^{(1)} + \alpha'_{(1)} + \frac{\alpha^{(1)}}{a} \right) + 2 \alpha'_{(1)} \left( 2 \psi^{(1)} + \alpha'_{(1)} \right) \\
+ \xi^{\mu}_{(1)} \left( \alpha^{(1)} - \xi^{(1)} - 2 \omega^{(1)} \right) + \alpha^{(2)} + \frac{\alpha}{\alpha} \alpha^{(2)}. 
\]  

(5.34)

\textbf{shift perturbation}

\[
\tilde{\omega}^{(2)} = \omega^{(2)} - 4 \psi^{(1)} \alpha^{(1)} + \alpha^{(1)} \left[ 2 \left( \omega^{(1)} + 2 \frac{\alpha'}{a} \omega^{(1)} \right) - \alpha^{(1)} + \xi^{(1)} \right] \\
- \frac{\alpha' a}{a} \left( \alpha^{(1)} - \xi^{(1)} \right) \right] + \xi^{i}_{(1)} \left( 2 \omega^{(1)} - \alpha^{(1)} + \xi^{(1)} \right) \\
+ \alpha'_{(1)} \left( 2 \omega^{(1)} - 3 \alpha^{(1)} + \xi^{(1)} \right) + \xi^{i}_{(1)} \left( -4 \phi^{(1)} \delta_{ij} + 2 \chi_{ij}^{(1)} + 2 \chi_{ij}^{(1)} + \chi^{(1)} \right) \\
+ \xi^{j}_{(1)} \left( 2 \omega^{(1)} - \alpha^{(1)} \right) - \alpha^{(2)} + \xi^{(2)}; 
\]  

(5.35)

\textbf{spatial metric, trace}

\[
\tilde{\phi}^{(2)} = \phi^{(2)} + \alpha^{(1)} \left[ 2 \left( \phi^{(1)} + 2 \frac{\alpha'}{a} \phi^{(1)} \right) - \frac{\alpha''}{a} + \frac{\alpha'^2}{a^2} \right] \alpha^{(1)} - \frac{\alpha'}{a} \alpha^{(1)} \right] \\
+ \xi^{i}_{(1)} \left( 2 \phi^{(1)} - \frac{\alpha' a}{a} \alpha^{(1)} \right) - \frac{1}{3} \left( -4 \phi^{(1)} + \alpha^{(1)} \partial_{(i} + \xi^{i}_{(1)} \partial_{i} + 4 \frac{\alpha'}{a} \alpha^{(1)} \right) \nabla^2 \beta^{(1)} \\
- \frac{1}{3} \left( 2 \omega^{(1)} - \alpha^{(1)} + \xi^{(1)} \right) \alpha^{(1)} - \frac{1}{3} \left( 2 \chi^{(1)} + \chi^{(1)} + \chi^{(1)} \right) \xi^{i}_{(1)} \\
- \frac{\alpha}{a} \alpha^{(2)} - \frac{1}{3} \nabla^2 \beta^{(2)}; 
\]  

(5.36)

\textbf{spatial metric, traceless part}

\[
\tilde{\chi}^{(2)}_{ij} = \chi^{(2)}_{ij} + 2 \left( \chi^{(1)}_{ij} + \frac{\alpha'}{a} \chi^{(1)}_{ij} \right) \alpha^{(1)} + \chi^{(1)}_{ij} \xi^{k}_{(1)} \\
+ 2 \left( -4 \phi^{(1)} + \alpha^{(1)} \partial_{(i} + \xi^{k}_{(1)} \partial_{k} + 4 \frac{\alpha'}{a} \alpha^{(1)} \right) \left( \delta^{(1)}_{ij} + D_{ij} \beta^{(1)} \right) \\
+ 2 \left( 2 \omega^{(1)} - \alpha^{(1)} + \xi^{(1)} \right) \alpha'_{(1)} - \frac{1}{3} \delta_{ij} \left( 2 \omega^{(1)} - \alpha^{(1)} + \xi^{k}_{(1)} \right) \alpha^{(1)} 
\]  

(5.37)
For the first order quantities in terms of $\beta$, we can easily obtain from (5.34), using (5.27) and the condition (5.37), (5.39), and (5.42).

The more general transformation expressions follow straightforwardly from (5.34)–(5.37).

5.2.2. Transforming from the synchronous to the Poisson gauge

For this example, let us consider the simplified case in which only scalar first order perturbations are present as initial conditions for the second order problem. In the first order analysis presented above, this corresponds to having $\chi_{(1)} = \chi_{(1)}^\top = \nabla_{(1)} = 0$, and thus $d_{(1)}^i = \nabla_{(1)}^i = \lambda_{(1)}^i = 0$. The second order vector and tensor perturbations are however non-vanishing as the dynamical coupling of the modes makes them grow when non-linear terms are considered in the evolution equation. We consider this restriction just for the sake of simplicity and because it describes a physically interesting situation. The more general transformation expressions follow straightforwardly from (5.34)–(5.37), (5.39), and (5.42).

Transforming from the synchronous to the Poisson gauge, the expression for $\psi_{(2)}$ can be easily obtained from (5.34), using (5.27) and the condition $d_{(1)}^i = 0$ to express all the first order quantities in terms of $\beta_{(1)}$:

$$
\psi_{(2)} = \beta_{(1)}\left[\beta_{(1)}'' + 5\frac{a'}{a}\beta_{(1)}'' + \left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\beta_{(1)}'ight]
$$
\[ + \beta_i^{(1)} \left( \beta_i^{(1)\prime} + \frac{a'}{a} \beta_i^{(1)\prime} \right) + 2 \beta_i^{(2)\prime} + \alpha^{(2)} + \frac{a'}{a} \alpha^{(2)}. \quad (5.43) \]

For \(\omega_i^{(2)} \) and \(\phi_i^{(2)}\) we get:
\[ \omega_i^{(2)} = -2 \left( 2 \phi_i^{(1)} + \beta_i^{(1)\prime} - \frac{2}{3} \nabla^2 \beta_i^{(1)} \right) \beta_i^{(1)\prime} - 2 \beta_i^{(1)\prime} \beta_i^{(1)\prime} - \alpha_i^{(2)} + \beta_i^{(2)\prime} + d_i^{(2)\prime}; \quad (5.44) \]
\[ \phi_i^{(2)} = \phi_i^{(2)} + \beta_i^{(1)} \left[ 2 \left( \phi_i^{(1)} + \frac{a'}{a} \phi_i^{(1)} \right) - \left( \frac{a'}{a} + \frac{a'}{a^2} \right) \beta_i^{(1)\prime} + \frac{a'}{a} \beta_i^{(1)\prime} \right] \]
\[ - \frac{1}{3} \left( -4 \phi_i^{(1)} + \beta_i^{(1)} \partial_0 + \beta_i^{(1)} \partial_i + 4 \frac{a'}{a} \beta_i^{(1)} + \frac{4}{3} \nabla^2 \beta_i^{(1)} \right) \nabla^2 \beta_i^{(1)} \]
\[ + \beta_i^{(1)} \left( 2 \phi_i^{(1)} - \frac{a'}{a} \beta_i^{(1)\prime} \right) + \frac{2}{3} \beta_i^{(1)} \beta_i^{(1)} - \frac{a'}{a} \alpha_i^{(2)} - \frac{1}{3} \nabla^2 \beta_i^{(2)}. \quad (5.45) \]

For \(\chi_i^{(2)}\) we obtain:
\[ \chi_i^{(2)} = \chi_i^{(2)} + 2 \left( 4 \frac{\nabla^2 \beta_i^{(1)} - 4 \phi_i^{(1)} - \beta_i^{(1)} \partial_0 - \beta_i^{(1)} \partial_i}{D_{ij} \beta_i^{(1)}} \right) \]nabla^2 \beta_i^{(1)} \]
\[ - 2 \left( \beta_i^{(1)} \beta_i^{(1)} - \frac{1}{3} \delta_{ij} \beta_i^{(1)} \beta_i^{(1)} \right) \]nabla^2 \beta_i^{(1)} \]
\[ + 2 \left( \beta_i^{(1)} \nabla^2 \beta_i^{(1)} + \beta_i^{(1)} \nabla^2 \beta_i^{(1)} + \beta_i^{(1)} \nabla^2 \beta_i^{(1)} \right). \quad (5.46) \]

Given the metric perturbations in the synchronous gauge, these constitute a set of coupled equations for the second order parameters of the transformation, \(\alpha_i^{(2)}, \beta_i^{(2)}, \) and \(d_i^{(2)}, \) and the second order metric perturbations in the Poisson gauge, \(\psi_i^{(2)}, \omega_i^{(2)}, \) \(\phi_i^{(2)}, \) and \(\chi_i^{(2)}\). This system can be solved in the following way. Since in the Poisson gauge \(\partial^2 \chi_i^{(2)} = 0,\) we can use the fact that \(\partial^2 \partial^2 \chi_i^{(2)} = 0\) and the property \(\partial^2 d_i^{(1)} = 0,\) together with (5.46), to obtain an expression for \(\nabla^2 \nabla^2 \beta_i^{(2)},\) from which \(\beta_i^{(2)}\) can be computed:
\[ \nabla^2 \nabla^2 \beta_i^{(2)} = - \frac{3}{4} \chi_i^{(2),ij} + 6 \phi_i^{(1),ij} \beta_i^{(1)} - 2 \nabla^2 \phi_i^{(1)} \nabla^2 \beta_i^{(1)} + 8 \phi_i^{(1)},i \beta_i^{(1)} \]
\[ + 4 \phi_i^{(1)} \nabla^2 \beta_i^{(1)} + 4 \nabla^2 \beta_i^{(1)} \beta_i^{(1)} - \frac{5}{2} \partial_i \beta_i^{(1)} \beta_i^{(1)} \]
\[ + 2 \beta_i^{(1)} \nabla^2 \beta_i^{(1)} + \beta_i^{(1)} \nabla^2 \beta_i^{(1)} + \beta_i^{(1)} \nabla^2 \beta_i^{(1)}. \quad (5.47) \]

Then, using the condition \(\partial^2 \chi_i^{(2)} = 0\) and substituting \(\beta_i^{(2)}\) we obtain an equation for \(d_i^{(2)}:\)
\[ \nabla^2 d_i^{(2)} = \frac{4}{3} \beta_i^{(2)} - \chi_i^{(2),ij} + 8 \phi_i^{(1),ij} D_{ij} \beta_i^{(1)} + \frac{16}{3} \phi_i^{(1)} \nabla^2 \beta_i^{(1)} \]
\[ + \frac{2}{3} \nabla^2 \beta_i^{(1)} \beta_i^{(1)} + \frac{10}{3} \beta_i^{(1)} \beta_i^{(1)} - \frac{8}{3} \nabla^2 \beta_i^{(1)} \nabla^2 \beta_i^{(1)} \]
\[ + 2 \beta_i^{(1)} D_{ij} \beta_i^{(1)} + \frac{4}{3} \beta_i^{(1)} \nabla^2 \beta_i^{(1)} + \frac{4}{3} \beta_i^{(1)} \nabla^2 \beta_i^{(1)}. \quad (5.48) \]
Finally, using $\partial^i \omega_i^{(2)} = 0$ and substituting $\beta^{(2)}$, we get an equation for $\alpha^{(2)}$:

$$\nabla^2 \alpha^{(2)} = \nabla^2 \beta^{(2)} - 2 \left( 2 \phi^{(1),i} + \beta^{(1),i} + \frac{1}{3} \nabla^2 \beta^{(1)} \right) \beta^{(1),i} - 2 \left( \phi^{(1)} + \beta''_{(1)} - \frac{2}{3} \nabla^2 \beta^{(1)} \right) \nabla^2 \beta^{(1)} - 2 \beta^{(1),ij} \beta^{(1),ij}. \quad (5.49)$$

Having obtained, at least implicitly, all the parameters of the gauge transformation to second order, one can in principle compute the metric perturbations in the Poisson gauge from (5.43)–(5.46).

Similarly, once the parameters are known, the perturbations in any scalar and 4-vector, and in particular those in the energy density and in the 4-velocity of matter, follow trivially from (5.39)–(5.42).

6. Conclusions

In this paper we have studied the problem of gauge dependence in relativistic perturbation theory, considering perturbations of arbitrary order in a geometrical perspective. In fact, the problem itself is of a purely geometrical nature, dealing with the arbitrariness in the mapping between the physical spacetime and the background unperturbed one. Since no dynamics is involved, the formalism developed here can actually find application not only in general relativity, but in any spacetime theory. In considering a specific example, we have assumed a flat Robertson–Walker background, and derived the second order transformation between the well-known synchronous gauge [18], and the Poisson (generalized longitudinal) gauge discussed in [19].

In linearized perturbation theory a gauge transformation is generated by an arbitrary vector field $\xi^{(1)}$, defined on the background spacetime, and associated with a one-parameter group of diffeomorphisms (a flow): the gauge transformation of the perturbation $\delta T$ of a tensor field $T$ is then given by the Lie derivative $L_{\xi^{(1)}} T_0$ of the background field $T_0$. However, in considering a gauge transformation from an exact point of view, we have found that it is not represented by a flow, but rather by a more general one-parameter family of diffeomorphisms. The question then was, how can we approximate the latter to a given order $n$? To this end, we have developed in section 2 the necessary mathematical formalism. First, we have introduced certain families of mappings, dubbed knight diffeomorphisms of rank $n$, defined by (2.10). Then, in Theorem 2, we have proved that any one-parameter family of diffeomorphisms may always be approximated, to order $n$, by a knight diffeomorphism of rank $n$. This result (which confirms a claim in [14]) is fundamental for gauge transformations of order $n$, as it guarantees that they are correctly represented by knight diffeomorphisms of the same rank. From the applicative point of view, Lemma 2 is thus all we need to have a generating formula for the gauge transformation to an arbitrary order, (4.6). Since a knight diffeomorphism of rank $n$ is basically the composition of $n$ flows, and is thus generated by $n$ vector fields $\xi^{(1)}, \ldots, \xi^{(n)}$, a gauge transformation of order $n$ for the $n$-th order perturbation $\delta^n T$ of a tensor field $T$ involves an appropriate combination of the Lie derivatives along $\xi^{(1)}, \ldots, \xi^{(n)}$ of $T_0, \ldots, \delta^{n-1} T$.

Gauge transformations found their main application in considering the time evolution of perturbations of a given background spacetime: in a subsequent paper[25] we shall look at second order perturbations of an Einstein de Sitter universe, comparing results in the synchronous and the Poisson gauges, thus applying the results presented
in section 5. Beyond these applications, there are many topics that we have not
touched upon here which, in a way or another, are related to gauge transformations,
and become even more cumbersome in the non-linear case. We mention only a couple
of them. We have implicitly assumed the applicability of the perturbative method;
also, we have not considered the problem of eliminating spurious gauge modes. In
particular, in our cosmological example, the synchronous gauge as defined in section
5 should actually be regarded as a class of point identification maps [4].

A final issue we would like to mention is that of gauge-invariant quantities. In
relation to this, we have first defined gauge invariance in an exact sense, and then given
the conditions for the gauge invariance of a tensor field $T$ to an arbitrary perturbative
order $n$. However, we have not faced the problem of finding or constructing such
quantities. In particular, in considering cosmological perturbations, it would be
useful to have at hand a set of second order gauge-invariant variables defined à la
Bardeen [3]. However, given the gauge-dependent metric perturbations and their
transformation rules presented in section 5, the construction of such variables seems
impractical. Moreover, it is far from obvious that a complete set giving a full second
order description exists at all, as is the case at first order [3, 26]. Another possibility
is to look for covariant quantities [5, 27]: in order to be second order gauge-invariant
these should vanish in the background and at first order. Assuming purely scalar first
order perturbations, two examples are the magnetic part of the Weyl tensor and the
vorticity of the 4-velocity of matter. Other second order gauge-invariant quantities
can be defined by taking products of first order gauge-invariant tensors that vanish in
the background. In particular, this is the case for scalars such as $E_{\mu\nu}E^{\mu\nu}$, where $E$
is the electric part of the Weyl tensor. Once again, it seems difficult that a complete
set of such variables could even exist. Nevertheless, it is worth pointing out that
quantities which are quadratic in first order gauge-invariant variables are useful in
specific problems; for example they may intervene in the construction of effective
energy momentum tensors of perturbations, which are important for the study of back
reaction problems [13, 28]. Another possible application of the formalism developed
here can be the study of the gauge dependence of these quantities.

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Figure 1. The action of a knight diffeomorphism $\Psi_\lambda$ generated by $\xi^{(1)}$ and $\xi^{(2)}$. Solid lines: integral curves of $\xi^{(1)}$. Dashed lines: integral curves of $\xi^{(2)}$. The parameter lapse between $p$ and $\phi^{(1)}_\lambda(p)$ is $\lambda$, and that from $\phi^{(1)}_\lambda(p)$ to $\phi^{(2)}_{\lambda^2/2} \circ \phi^{(1)}_\lambda(p)$ is $\lambda^2/2$. 
Figure 2. The action of a knight diffeomorphism of rank two, represented in a chart to second order.
Figure 3. The action of a gauge transformation $\Phi_\lambda$, represented on the background spacetime $\mathcal{M}_0$ by its second order approximation, generated by the two vector fields $\xi_{(1)}$ and $\xi_{(2)}$. 

$q = \Phi_\lambda(p) \\
\psi_\lambda(p) = \varphi_\lambda(q)$
Figure 4. If $X$ and $Y$ do not commute, $\Phi$ is not a flow on $\mathcal{M}_0$. 