Abstract

The Einstein-Langevin equations take into account the backreaction of quantum matter fields on the background geometry. We present a derivation of these equations to lowest order in a covariant expansion in powers of the curvature. For massless fields, the equations are completely determined by the running coupling constants of the theory.

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The study of the backreaction of quantum matter fields on the spacetime geometry is a very interesting subject. It is relevant to understand the fate of black hole evaporation [1], for the analysis of the eventual smearing of the classical cosmological singularities due to quantum effects [2], for the study of the theoretical possibility of creating closed timelike curves (time machines) [3], etc.

The usual approach to this problem is based on the use of the Semiclassical Einstein Equations (SEE) [4]

$$\frac{1}{8\pi G} \left[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] - \alpha H_{\mu\nu}^{(1)} - \beta H_{\mu\nu}^{(2)} = T_{\mu\nu}^{\text{ch}} + < T_{\mu\nu} >,$$  \hspace{0.5cm} (1)

where the effect of quantum matter fields is taken into account by including as a source the quantum mean value of the energy-momentum tensor. The terms proportional to

$$H_{\mu\nu}^{(1)} = [4R^{\mu\nu} - 4g^{\mu\nu}\Box R] + O(R^2),$$  \hspace{0.5cm} (2)

and

$$H_{\mu\nu}^{(2)} = [4R^{\mu\alpha\nu}_{\alpha\alpha} - 2\Box R^{\mu\nu} - g^{\mu\nu}\Box R] + O(R^2),$$  \hspace{0.5cm} (3)

come from terms quadratic in the curvature in the gravitational action, which are needed to renormalize the theory. The SEE take into account dissipative effects of the quantum fields on the metric, and can be derived from the real part of the Closed Time Path (CTP) effective action [5].

It is clear that these equations cannot provide a full description of the problem, at least when the state of the quantum matter fields is such that the energy momentum tensor has important fluctuations around its mean value [6]. These fluctuations can be taken into account by including an additional stochastic term [7–9] on the right hand side of Eq. 1. This noise-term comes from the imaginary part of the CTP effective action. The SEE thus become “Einstein-Langevin Equations” (ELEs), that include both the dissipative and diffusive effects of the quantum matter on the geometry of spacetime [7].

In previous works, the ELEs were derived for arbitrary small metric perturbations conformally coupled to a massless quantum scalar field in a spatially flat background [9], and,
in a cosmological setting, for a massive field in a spatially flat Friedmann-Robertson-Walker universe [10], and in a Bianchi type-I spacetime [11].

In this paper we will present a derivation of the ELEs for a quantum scalar field, to lowest order in a covariant expansion in powers of the curvature. We will see that, when the scalar field is massless, the ELEs are fully determined by the running couplings of the theory. For massive fields, some additional work is needed in order to obtain the corresponding dissipation and noise kernels.

Our strategy will be as follows. We will first compute the Euclidean effective action up to second order in the curvature. With an adequate replacement of Euclidean propagators by propagators along a closed time path, we will obtain the CTP effective action. The dissipation and noise kernels in the CTP action will allow us to obtain the ELEs.

Consider a massless scalar field on a classical, Euclidean curved background. The classical action is given by

\[ S = S_{\text{grav}} + S_{\text{matter}}, \]

where

\[ S_{\text{grav}} = - \int d^4x \sqrt{g} \left[ \frac{1}{16\pi G_0} (R - 2\Lambda_0) + \alpha_0 R^2 + \beta_0 R_{\mu\nu} R^{\mu\nu} \right], \]

and

\[ S_{\text{matter}} = \frac{1}{2} \int d^4x \sqrt{g} \left[ \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \xi R \phi^2 \right]. \]

Here \( \xi \) is the coupling to the curvature. \( G_0, \Lambda_0 \), and the dimensionless constants \( \alpha_0 \) and \( \beta_0 \) are bare constants.

The effective action for this theory is a complicated, non local object. It is defined by integrating out the quantum scalar field, that is

\[ e^{-S_{\text{eff}}} = N \int \mathcal{D}\phi e^{-S[\phi_\nu, \phi]}, \]

where \( N \) is a normalization constant. It is in general not possible to find a closed form for it. On general grounds, we expect it to be of the form [12]
\[ S_{\text{eff}} = -\int d^4x \sqrt{g} \left[ \frac{1}{16\pi G} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right] + \frac{1}{32\pi^2} \int d^4x \sqrt{g} \left[ F_0 R + R F_1(\Box) R + R_{\mu\nu} F_2(\Box) R^{\mu\nu} + \ldots \right], \]  

where the ellipsis denote terms cubic in the curvature. For simplicity, in the above equation and in what follows we will omit the cosmological constant term. Note that the bare constants in Eq. 5 have been replaced by dressed couplings in Eq. 8.

Up to this order, all the information about the effect of the quantum field is encoded in the constant \(F_0\) and in the form factors \(F_1\) and \(F_2\). The form factors are, in general, non-local two point functions constructed with the d’Alambertian and the parameters \(\xi\) and \(m^2\). \(F_0\), \(F_1\), and \(F_2\) also depend on an energy scale \(\mu\), introduced by the regularization method.

The dressed coupling constants depend on the energy scale \(\mu\) according to the renormalization group equations

\begin{align}
\frac{dG}{d\mu} &= \frac{G^2 m^2}{\pi} \left( \xi - \frac{1}{6} \right), \\
\frac{d\alpha}{d\mu} &= -\frac{1}{32\pi^2} \left[ \frac{1}{6} - \xi \right] \left[ \frac{1}{6} - \xi \right] - \frac{1}{90}, \\
\frac{d\beta}{d\mu} &= -\frac{1}{960\pi^2}.
\end{align}

The dependence of \(F_0\), \(F_1\) and \(F_2\) on \(\mu\) is such that the full equation is \(\mu\)-independent. For example, from Eqs. 8 and 9, we see that \(F_0 = m^2 \ln \left( \frac{m^2}{\mu^2} \right) \left( \xi - \frac{1}{6} \right) + \text{const.}\)

When the scalar field is massless, this information is enough to fix completely the form factors. Indeed, as the \(F_i\), \(i = 1, 2\) are dimensionless two point functions, by simple dimensional analysis we obtain \(F_i(\Box, \mu^2, \xi) = F_i(\frac{\Box}{\mu^2}, \xi)\). Inserting this into Eq. 8, using Eqs. 10 and 11, and the fact that \(S_{\text{eff}}\) must be independent of \(\mu\), we obtain

\begin{align}
F_1(\Box) &= \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \ln \left( \frac{-\Box}{\mu^2} \right) + \text{const}, \\
F_2(\Box) &= \frac{1}{60} \ln \left( \frac{-\Box}{\mu^2} \right) + \text{const}.
\end{align}

The final result for the effective action has a clear interpretation: it is just the classical action in which the coupling constants \(\alpha\) and \(\beta\) have been replaced by non-local two point functions that take into account their running in configuration space.
For a massive field, the situation is more complex because there is an additional dimensional parameter. The form factors also depend on $\frac{m^2}{\mu^2}$ and the $\mu$-independence of the effective action is not enough to fix the form of them. Fortunately, they have already been computed in the literature \cite{13,14}

$$F_i(\square) = \int_0^1 d\gamma \chi_i(\xi, \gamma) \ln \left[ \frac{m^2 - \frac{1}{4}(1 - \gamma^2)\square}{\mu^2} \right], \quad (13)$$

where

$$\chi_1(\xi, \gamma) = \frac{1}{2} \left[ \xi^2 - \frac{1}{2} \xi (1 - \gamma^2) + \frac{1}{48} (3 - 6\gamma^2 - \gamma^4) \right],$$

$$\chi_2(\xi, \gamma) = \frac{1}{12} \gamma^4. \quad (14)$$

These equations can be obtained through a covariant perturbation expansion \cite{13}, or by a resummation of the Schwinger DeWitt expansion \cite{14}. Of course these form factors coincide with our previous Eq. 12 in the massless case.

In order to clarify the meaning of the two point functions appearing in Eqs. 12 and 13, it is useful to introduce the following integral representation \cite{15}

$$\ln \left[ \frac{m^2 - \frac{1}{4}(1 - \gamma^2)\square}{\mu^2} \right] = \ln \left( \frac{1 - \gamma^2}{4} \right) + \int_0^\infty dz \left( \frac{1}{z + \mu^2} - G_E^{(z)} \right), \quad (15)$$

so the logarithm of the d’Alambertian is written in terms of the massive Euclidean propagator $G_E^{(z)} = (z + \frac{4m^2}{1-\gamma^2} - \square)^{-1}$. This representation will also be useful to construct the CTP version of the effective action.

Replacing the Euclidean propagator by the Feynman one in the integral representation Eq. 15, one obtains the usual \textit{in-out} effective action. As is very well known, the effective equations derived from this action are neither real nor causal because they are equations for \textit{in-out} matrix elements and not for mean values.

The solution to this problem is also well known. Using the CTP formalism one can construct an \textit{in-in} effective action that produces real and causal field equations for \textit{in-in} expectation values \cite{5}. The effective action can be written as

$$e^{i S_{eff}[g^+, g^-]} = Ne^{i [S_{\text{grav}}[g^+] - S_{\text{grav}}[g^-]]} \int \mathcal{D}\phi^+ \mathcal{D}\phi^- e^{i [S_{\text{matter}}[g^+, \phi^+] - S_{\text{matter}}[g^-, \phi^-]]}, \quad (16)$$
and the field equations are obtained taking the variation of this action with respect to the \( g_{\mu\nu}^+ \) metric, and then setting \( g_{\mu\nu}^+ = g_{\mu\nu}^- \).

In an alternative, and more concise notation, we can write this effective action as [16]

\[
e^{iS_{eff}/\hbar} = N e^{iS_{matter}/\hbar} \int D\phi e^{iS_{matter}[\phi]},
\]

where we have introduced the CTP complex temporal path \( \mathcal{C} \), going from minus to plus infinity \( \mathcal{C}_+ \) and backwards \( \mathcal{C}_- \), with a decreasing (infinitesimal) imaginary part. Time integration over the contour \( \mathcal{C} \) is defined by \( \int_{\mathcal{C}} dt = \int_{\mathcal{C}_+} dt - \int_{\mathcal{C}_-} dt \). The field \( \phi \) appearing in Eq. 17 is related to those in Eq. 16 by \( \phi(t, x) = \phi_\pm(t, x) \) if \( t \in \mathcal{C}_\pm \). The same applies to \( g_{\mu\nu} \).

This equation is useful because it has the structure of the usual \textit{in-out} or the Euclidean effective action. Feynman rules are therefore the ordinary ones, replacing Euclidean propagator by

\[
G(x, y) = \begin{cases} 
G_F(x, y) = i \langle 0, in|T\phi(x)\phi(y)|0, in \rangle, & t, t' \text{ both on } \mathcal{C}_+ \\
G_D(x, y) = -i \langle 0, in|\bar{T}\phi(x)\phi(y)|0, in \rangle, & t, t' \text{ both on } \mathcal{C}_- \\
G_+(x, y) = -i \langle 0, in|\phi(x)\phi(y)|0, in \rangle, & t \text{ on } \mathcal{C}_-, t' \text{ on } \mathcal{C}_+ \\
G_-(x, y) = i \langle 0, in|\phi(y)\phi(x)|0, in \rangle, & t \text{ on } \mathcal{C}_+, t' \text{ on } \mathcal{C}_-
\end{cases}
\]

Introducing Riemann normal coordinates, we can write, up to lowest order in the curvature

\[
G_F(x, y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + m^2 - i\epsilon} = G_D^*(x, y),
\]

\[
G_\pm(x, y) = \mp \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} 2\pi i \delta(p^2 - m^2) \theta(\pm p^0).
\]

All of the preceding formulation of the effective action is valid for any field theory. In our particular case, we must replace the Euclidean propagator \( g_E^{(z)} \) in Eq. 15 by the propagator \( G(x, y) \) of Eq. 18 with a mass given by \( \frac{4m^2}{1-\gamma^2} + z \). After integration in \( z \) we obtain

\[
\ln \left[ \frac{4m^2}{1-\gamma^2} \frac{\Box}{\mu^2} \right]_{\text{CTP}} = \begin{cases} 
\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \ln \left( \frac{(1-\gamma^2)(p^2 - i\epsilon) + 4m^2}{\mu^2} \right) & t, t' \text{ both on } \mathcal{C}_+ \\
\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \ln \left( \frac{(1-\gamma^2)(p^2 + i\epsilon) + 4m^2}{\mu^2} \right) & t, t' \text{ both on } \mathcal{C}_- \\
\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} 2\pi i \theta(p^0) \theta(-p^2 - \frac{4m^2}{1-\gamma^2}) & t \text{ on } \mathcal{C}_-, t' \text{ on } \mathcal{C}_+ \\
-\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} 2\pi i \theta(-p^0) \theta(-p^2 - \frac{4m^2}{1-\gamma^2}) & t \text{ on } \mathcal{C}_+, t' \text{ on } \mathcal{C}_-
\end{cases}
\]
With the expression for the CTP logarithm of the d’Alambertian we can calculate explicitly the CTP effective action. Using the previous notation with $g_{\mu\nu}^+$ and $g_{\mu\nu}^-$, the CTP effective action reads

$$S_{\text{eff}}[g^+, g^-] = S_{\text{grav}}^+[g^+] - S_{\text{grav}}^-[g^-]$$

$$+ \frac{i}{8\pi^2} \int d^4x \int d^4y \Delta(x) \Delta(y) N_1(x, y) - \frac{1}{8\pi^2} \int d^4x \int d^4y \Delta(x) \Sigma(y) D_1(x, y)$$

$$+ \frac{i}{8\pi^2} \int d^4x \int d^4y \Delta_{\mu\nu}(x) \Delta^{\mu\nu}(y) N_2(x, y) - \frac{1}{8\pi^2} \int d^4x \int d^4y \Delta_{\mu\nu}(x) \Sigma_{\mu\nu}(y) D_2(x, y),$$

where $\Delta = R^+ \mp R^-$, $\Sigma = R^+ \mp R^-$, $\Delta_{\mu\nu} = \frac{R_{\mu\nu}^+ - R_{\mu\nu}^-}{2}$, $\Sigma_{\mu\nu} = \frac{R_{\mu\nu}^+ + R_{\mu\nu}^-}{2}$. The classical gravitational action $S_{\text{grav}}^\alpha$ contains the dressed, $\mu$-dependent coupling constants and we absorbed $F_0$ into the gravitational constant $G$.

The real and imaginary parts of $S_{\text{eff}}$ can be associated with the dissipation and noise, respectively. The dissipation $D_i$ and noise $N_i$ kernels are given by

$$D_i(x, y) = \int_0^1 d\gamma \gamma_i(\xi, \gamma) \int \frac{d^4p}{(2\pi)^4} \cos[p(x - y)] \ln \left| \frac{(1 - \gamma^2)p^2 + 4m^2}{\mu^2} \right|, \quad (23)$$

$$N_i(x, y) = \int_0^1 d\gamma \gamma_i(\xi, \gamma) \int \frac{d^4p}{(2\pi)^4} \cos[p(x - y)] \theta \left( -p^2 - 4m^2 \right). \quad (24)$$

It is important to note that the imaginary part of this effective action must be positive definite. To make this point explicit, one can write the imaginary part in terms of the Weyl tensor $C_{\mu\nu\alpha\beta}$ and the scalar curvature $R$ by means of the following relation: $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = 2R_{\mu\nu}R^{\mu\nu} - 2/3R^2$. It is not difficult to show that the scalar and tensor contributions to the imaginary part of the effective action are both positive.

One can regard the imaginary part of the closed-time-path-effective-action (CTPEA) as coming from two classical stochastic sources $\eta(x)$ and $\eta^{\mu\nu\alpha\beta}(x)$, where the last tensor has the symmetries of the Weyl tensor. In fact, as usually done in statistical physics, we can write the imaginary part of the CTPEA as

$$\int \mathcal{D}\eta(x) \mathcal{D}\eta^{\mu\nu\alpha\beta}(x) P[\eta, \eta^{\mu\nu\alpha\beta}] \exp \left\{ i \left\{ \Delta(x) \eta(x) + \Delta_{\mu\nu\alpha\beta} \eta^{\mu\nu\alpha\beta} \right\} \right\}$$

$$= \exp \left\{ - \int d^4x \int d^4y \left[ \Delta(x) \tilde{N}(x - y) \Delta(y) + \Delta_{\mu\nu\alpha\beta}(x) N_2(x - y) \Delta^{\mu\nu\alpha\beta}(y) \right] \right\}, \quad (25)$$

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where $\tilde{N}(x, y) = N_1(x, y) + 1/3N_2(x, y)$, and $\Delta_{\mu\nu\alpha\beta} = \frac{c^+_{\mu\nu\alpha\beta} - c^-_{\mu\nu\alpha\beta}}{2}$. $P[\eta, \eta^{\mu\nu\alpha\beta}]$ is a Gaussian functional probability distribution given by

$$P[\eta, \eta^{\mu\nu\alpha\beta}] = A \exp \left\{ -\frac{1}{2} \int d^4x \int d^4y \eta(x) \left[ \tilde{N}(x, y) \right]^{-1} \eta(y) \right\} \times \exp \left\{ -\frac{1}{2} \int d^4x \int d^4y \eta_{\mu\nu\alpha\beta}(x) \left[ N_2(x, y) \right]^{-1} \eta^{\mu\nu\alpha\beta}(y) \right\},$$

(26)

with $A$ a normalization factor.

Therefore, the CTPEA can be written as

$$\exp \{i S_{eff} \} = \int D\eta D\eta_{\mu\nu\alpha\beta} P[\eta, \eta_{\mu\nu\alpha\beta}] \exp \{ iA_{eff} [\Delta, \Delta_{\mu\nu\alpha\beta}, \Sigma, \Sigma_{\mu\nu}, \eta, \eta_{\mu\nu\alpha\beta}] \},$$

(27)

where

$$A_{eff} = Re S_{eff} + \int d^4x [\Delta(x)\eta(x) + \Delta_{\mu\nu\alpha\beta}(x)\eta^{\mu\nu\alpha\beta}].$$

(28)

The field equations $\frac{\delta A_{eff}}{\delta g_{\mu\nu}} |_{g^{++}_{\mu\nu}=g^{--}_{\mu\nu}} = 0$, the Einstein-Langevin equations, are

$$\frac{1}{8\pi G} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = \bar{\alpha} H^{(1)}_{\mu\nu} - \bar{\beta} H^{(2)}_{\mu\nu}$$

$$= -\frac{1}{32\pi^2} \int d^4y D_1(x, y) H^{(1)}_{\mu\nu}(y) - \frac{1}{32\pi^2} \int d^4y D_2(x, y) H^{(2)}_{\mu\nu}(y)$$

$$+ g^{\mu\nu} \Box \eta - \eta^{\mu\nu} + 2\eta^{\mu\nu\alpha\beta} \tilde{\alpha}_{\alpha\beta},$$

(29)

where $\bar{\alpha}$ and $\bar{\beta}$ differ from $\alpha$ and $\beta$ by $\xi$-dependent finite constants. The Eq. 29 is our main result. The r.h.s. consists of the mean value of the energy-momentum tensor for the scalar field plus a stochastic correction characterized by the two point correlation functions

$$<\eta(x)\eta(y)> = \tilde{N}(x, y)$$

$$<\eta_{\mu\nu\alpha\beta}(x)\eta_{\rho\sigma\lambda\tau}(y)> = T_{\mu\nu\alpha\beta\rho\sigma\lambda\tau} N_2(x, y),$$

(30)

where the tensor $T_{\mu\nu\alpha\beta\rho\sigma\lambda\tau}$ is a linear combination of four-metric products in such a way that the r.h.s of Eq. 30 keeps the Weyl's symmetries (it is explicitly given in the Appendix of Ref. [9]). The scalar-noise kernel is given by

$$\tilde{N}(x, y) = \frac{1}{2} \int_0^1 d\gamma \left[ \left( \xi - \frac{(1 - \gamma^2)}{4} \right)^2 - \frac{\gamma^4}{36} \right] \int \frac{d^4p}{(2\pi)^4} \cos[p(x - y)] \theta \left( -p^2 - \frac{4m^2}{1 - \gamma^2} \right).$$

(31)
In the massless case $\bar{N}$ is proportional to $(\xi - 1/6)^2$, and vanishes for conformal coupling. Therefore this term is present when the quantum fields are massive and/or when the coupling is not conformal. This is to be expected, since the imaginary part of the CTPEA is directly associated to gravitational particle creation. For massless, conformally coupled quantum fields, particle creation takes place only when spacetime is not conformally flat. Therefore in this case the only contribution to the imaginary part of the CTPEA is proportional to the square of the Weyl tensor. When the fields are massive and/or non-conformally coupled, particle creation takes place even when the Weyl tensor vanishes. This is why an additional contribution proportional to $R^2$ appears in the imaginary part of the effective action.

From Eq. 29 we can define the effective energy-momentum tensor

$$T_{\mu\nu}^{\text{eff}} = <T_{\mu\nu}> + T_{\mu\nu}^{\text{stoch}} = <T_{\mu\nu}> + g^\mu\nu \Box \eta - \eta^{\mu\nu} + 2\eta^{\mu\nu\alpha\beta} \alpha\beta,$$  \hspace{1cm} (32)

where $<T_{\mu\nu}>$ is the quantum expectation value of the energy-momentum tensor of the quantum field and $T_{\mu\nu}^{\text{stoch}}$ is the contribution of the stochastic force, which in turn has contributions from the scalar and tensor noises. In the massless-conformal case the scalar-noise kernel vanishes, and $(T_{\mu\nu})^{\text{stoch}} = 0$, because the noise-source $\eta^{\mu\nu\alpha\beta}$ has vanishing trace. This means that there is no stochastic correction to the trace anomaly [9].

To summarize, we have obtained the ELEs using a covariant expansion in powers of the curvature. Our results are valid for quantum scalar fields with arbitrary mass and coupling to the curvature $\xi$, thus generalizing previous results [9]. In the massless case, still for arbitrary $\xi$, we have shown that it is possible to obtain the noise and dissipation kernels using only dimensional analysis and the running of the coupling constants.

We are currently computing the CTPEA and the ELEs in models of dilaton gravity in two dimensions, in order to discuss the relevance of the stochastic effects in the quantum to classical transition of the background metric.

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