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BEAM-BEAM-DRIVEN COUPLING RESONANCE OF FOURTH ORDER

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ABSTRACT

The non-linear force of the beam-beam interaction in storage rings excites coupling resonances between radial and vertical betatron oscillations. In beams of unequal emittances the coupled motion, though stable, can increase the smaller emittance. The behaviour of single-particle motion driven by a periodic beam-beam force near the resonance $2Q_v - 2Q_h = m$ is examined and it is shown that the tune difference between vertical and horizontal planes plays an important rôle. The criteria for avoiding increase of vertical emittance are of particular interest for e^+e^- storage rings and are evaluated for the Large Electron Positron (LEP) machine.

1. INTRODUCTION

The efforts directed towards an understanding of the beam-beam interaction near to the stochastic regime have tended to obscure the importance of single-resonance beam-beam phenomena in storage rings.

One such effect is the coupled motion induced by the non-linear forces of two colliding beams, which can cause an increase in the smaller emittance. The LEP design requires a rather small vertical beam size, obtained by careful compensation of the linear coupling¹⁾, and vertical emittance growth due to non-linear coupling must be avoided. It will be shown that this imposes certain constraints on the choice of vertical and horizontal betatron tunes.

The coupled motion examined here occurs in the vicinity of the difference resonance $2Q_v - 2Q_h = m$ and is excited by the octupole component of the beam-beam space-charge force. This is the lowest order non-linear resonance of this type which can be excited with the assumed beam symmetry. Such difference resonances are known to be stable and involve a interchange of energy between horizontal and vertical motion.

The resonance $2Q_v - 2Q_h = 0$ was studied in detail some years ago in connection with the single-beam space forces in a synchrotron²⁾. This work can be applied to the case of beam-beam excitation with a few modifications to take account of the periodic perturbation, the distinction between beam size and emittance, and the sign of the space-charge force for colliding beams of opposite charge.

2. SUMMARY OF THEORY

The beam-beam potential can be treated as a delta-function at each of S identical interaction points and expanded as a Fourier series :

$$V(x,y,\theta) = V_0 I(x,y) \sum_{n=-\infty}^{\infty} e^{-inS\theta} \quad (1)$$

where

$$I(x,y) = \frac{2}{a(a+b)} \cdot x^2 + \frac{2}{b(a+b)} y^2 - \frac{(2a+b)}{3a^3(a+b)^2} x^4 - \frac{(2b+a)}{3b^3(a+b)^2} y^4 - \frac{2}{ab(a+b)^2} x^2 y^2 \quad (2)$$

is the development of the potential coefficients of the Gaussian distribution up to fourth order. The beam size parameters can be related to the standard deviations of the Gaussian by :

$$a = \sqrt{2} \sigma_x, \quad b = \sqrt{2} \sigma_y \quad (3)$$

Following the procedures of Refs. 2 and 3 with (1) as the perturbation potential, results in the Hamiltonian with resonant behaviour in the vicinity of :

$$2 Q_y - 2 Q_x = \pm 2 q_0 \quad (4)$$

where

$$q_0 + \Delta q = Q_y - Q_x, \quad (5)$$

q_0 and Δq being the integer and fractional parts respectively of the tune difference. Retaining only the zero-frequency and low-frequency terms appropriate to the resonance (4) yields the Hamiltonian, up to fourth order in particle amplitudes :

$$H_2(A_x, \phi_x, A_y, \phi_y, \theta) = K_{20} A_x + K_{02} A_y + K_{40} A_x^2 + K_{04} A_y^2 + K_{22} A_x A_y \left[1 + \frac{1}{2} \cos \{ 2(\phi_x - \phi_y) - 2\Delta q \theta \} \right] \quad (6)$$

where the action variables conjugate to the phase variables ϕ_x, ϕ_y are scaled into single-particle emittances :

$$A_x = \frac{\hat{x}^2}{\beta_x^*}, \quad A_y = \frac{\hat{y}^2}{\beta_y^*} \quad (7)$$

The coefficients in (6) can be expressed in the form :

$$\left. \begin{aligned} K_{20} &= \Delta Q_y \frac{\sqrt{k}}{h}; & K_{02} &= \Delta Q_y; & K_{40} &= -\Delta Q_y \cdot \frac{1}{\epsilon_x} \cdot \frac{\sqrt{k}}{h} \cdot \frac{1}{16} \cdot \frac{(2+\sqrt{k})}{(1+\sqrt{k})}; \\ K_{04} &= -\Delta Q_y \cdot \frac{1}{\epsilon_x} \cdot \frac{h}{k} \cdot \frac{1}{16} \cdot \frac{(2\sqrt{k}+1)}{(1+\sqrt{k})}; & K_{22} &= -\Delta Q_y \cdot \frac{1}{\epsilon_x} \cdot \frac{1}{16} \cdot \frac{4}{(1+\sqrt{k})} \end{aligned} \right\} \quad (8)$$

where the linear beam-beam tune shift per interaction region in the vertical plane

$$\Delta Q_y = \frac{r_0 N}{2 \pi \gamma} \cdot \frac{\beta_y^*}{\sigma_y^* (\sigma_x^* + \sigma_y^*)} \quad (9)$$

quantifies the strength of the perturbation, and the parameters for beam size and emittance are defined by

$$\sqrt{k} = \frac{\sigma_y^*}{\sigma_x^*}; \quad h = \frac{\beta_y^*}{\beta_x^*}; \quad \epsilon_x = \frac{\sigma_x^{*2}}{\beta_x^*}; \quad \epsilon_y = \frac{\sigma_y^{*2}}{\beta_y^*}; \quad \frac{\epsilon_y}{\epsilon_x} = \frac{k}{h} \quad (10)$$

The Hamiltonian H_2 of Eqn.(6) is of identical form to that of Eqn.(4.1) in Ref.2 and the further results of that paper may be taken over as they stand. The two invariants of the coupled motion

$$A_x + A_y = C_1 \quad (11)$$

and

$$H_2 + Q_x A_x + Q_y A_y = C_2 \quad (12)$$

are used, through a succession of canonical transformations, to reduce the problem to a one-dimensional Hamiltonian :

$$H_s(\alpha, \psi) = \Delta Q_e \left[(\eta - \cos \psi) \alpha^2 + \chi \alpha + (C_3 + \frac{1}{4} \cos \psi) \right] \quad (13)$$

with canonical variables :

$$\alpha = \frac{A_y}{C_1} - \frac{1}{2}, \left(-\frac{1}{2} \leq \alpha \leq +\frac{1}{2}\right); \quad \psi = 2 \left[(\phi_y - \phi_x) + \Delta q \theta\right] \quad (14)$$

The parameters of H_5 are given as follows ;

$$\eta = \frac{2(K_{04} + K_{40})}{K_{22}} - 2 = \frac{1}{2Kh} \left[h^2 (2\sqrt{K} + 1) + k\sqrt{K}(2 + \sqrt{K}) \right] - 2, \quad (\eta \geq 1) \quad (15)$$

is a beam shape parameter,

$$\Delta Q_e = K_{22} C_1 \quad (16)$$

is the excitation width of the coupling resonance and

$$\chi = \frac{2}{K_{22} C_1} \left[\Delta q + (K_{02} - K_{20}) + C_1 (K_{04} - K_{40}) \right] = 2 \left[\frac{\Delta q}{\Delta Q_e} + \frac{\Delta Q_l}{\Delta Q_e} + \frac{\Delta Q_u}{\Delta Q_e} \right] \quad (17)$$

describes the distance from the resonant condition of a particle with invariant emittance C_1 . Here Δq is the unperturbed fractional tune difference between vertical and horizontal planes respectively, $\Delta Q_l = K_{02} - K_{20}$ is the difference between the two beam-beam linear tune shifts and $\Delta Q_u = C_1 (K_{04} - K_{40})$ is the non-linear amplitude-dependent tune difference.

With eqns.(8) and (10) we can express (17) as :

$$\chi = \left[\frac{1}{2Kh} \left\{ h^2 (2\sqrt{K}+1) - k\sqrt{K} (2+\sqrt{K}) \right\} - \frac{8}{\sigma} (1+\sqrt{K}) \left\{ \frac{\Delta q}{8\Delta Q_y} + 1 - \frac{\sqrt{K}}{h} \right\} \right] \quad (18)$$

where we have put :

$$\sigma = C_1 / \epsilon_x \quad (19)$$

3. RANGE OF α

The beating between horizontal and vertical motion is characterised by the action α and phase ψ variables. The criterion used in Ref.2 for avoiding undue increase of vertical emittance is that no particle in the range $-\frac{1}{2} < \alpha < \frac{1}{2}$ cross the resonance. Since this range of α is bounded by the definition of the variables it is not immediately obvious that it is initially fully populated and that, therefore, the condition is a necessary one.

The situation can be examined with the Hardt diagram of Fig.1, which shows the single particle emittance A_y plotted against the horizontal A_x . The hatched area of the unperturbed distribution inside the nominal beam emittance is bounded by the axes and the line 1. The emittance ratio is designated

$$\kappa = \frac{\epsilon_y}{\epsilon_x} = \frac{k}{h}.$$

Line 2 is a typical trajectory in the diagram corresponding to the invariant $C_1 = \sigma \epsilon_x$ of (11) and (19), and has the equation $A_y = \sigma \epsilon_x - A_x$. Its intersection with boundary line 1, whose equation is $A_y = \kappa(\epsilon_x - A_x)$, lies at

FIG.1

$$A_y = \frac{\kappa \epsilon_x (1-\sigma)}{(1-\kappa)}$$

The maximum value of α present in the initial distribution is therefore, from (14) :

$$\hat{\alpha}_0 = \frac{\kappa(1-\sigma)}{\sigma(1-\kappa)} - \frac{1}{2} \quad (20)$$

For $\sigma < \kappa$, the beating motion does not give rise to any increase in vertical emittance; only particles for which $\kappa < \sigma < 1$ can contribute to this. The corresponding range of $\hat{\alpha}_0$ in (20) is therefore

$$+\frac{1}{2} > \hat{\alpha}_0 > -\frac{1}{2}.$$

Thus the whole dynamically-possible range of α is present in the initially unperturbed beam and the condition that no particle in the initial distribution cross the resonance is, as stated in Ref.2 :

$$|\chi| > (n+1) \quad (21)$$

4. EQUAL LINEAR TUNE SHIFTS

Parameters of e^+e^- storage rings are usually chosen to give equal linear beam-beam tune shifts in the two planes. This condition, from (8), corresponds to $h = \sqrt{K}$ which, from (15) and (18) results in :

$$n+1 = \frac{(1+\sqrt{K})^2}{2\sqrt{K}} \quad \text{and} \quad \chi = (1+\sqrt{K}) \left[\frac{(1-\sqrt{K})}{2\sqrt{K}} - \frac{8}{\sigma} \cdot \frac{\Delta q}{S\Delta Q_y} \right]$$

In order to apply the criterion of (21) for no resonance crossing we have to distinguish between two cases

$$(a) \quad \chi > 0 \quad \text{leads to} \quad -\frac{8}{\sigma} \cdot \frac{\Delta q}{S\Delta Q_y} > 1 \quad (22)$$

For colliding beams of opposite charges $\Delta Q_y > 0$ (focussing) and (22) can only be satisfied for $\Delta q < 0$. Then, since the maximum value of σ is 1, the criterion for $\chi > 0$ becomes :

$$\frac{8(-\Delta q)}{S\Delta Q_y} > 1 \quad (23)$$

$$(b) \quad \chi < 0 \text{ with } \sigma = 1, \text{ leads to } \frac{8\sqrt{K} \cdot \Delta q}{S\Delta Q_y} > 1 \quad (24)$$

This requires $\Delta q > 0$ and, since $\sqrt{K} < 1$, imposes a stronger constraint on the fractional part Δq of the tune split than case (a).

The general conclusion is then that, in e^+e^- storage rings operating near the resonant condition of (4), the fractional part of the vertical tune should be less than that of the horizontal tune, and that the fractional tune difference should satisfy (23).

5. IMPLICATIONS FOR LEP

Several factors which govern the choice of horizontal and vertical tunes in LEP, notably the coherent beam-beam effect⁴⁾ and the reduction of β_x^*, β_y^* by the beam-beam focussing force⁵⁾, favour tunes which lead to operation near the resonant condition of (4).

With a linear tune shift $\Delta Q_y = 0.06$ in each of 8 interaction regions, the requirement (a) of (23), $\Delta q < 0$ becomes $(-\Delta q) > 0.06$ which is not difficult to satisfy.

If however $\Delta q > 0$, then (24), with $\sqrt{k} = 1/16$ would impose $\Delta q > 0.96$ which is not acceptable.

The consequences of failing to satisfy the criteria may be roughly estimated from the beating-range diagram of Fig.2, in which particles move on vertical lines of constant C inside the regions bounded by the parabolae corresponding to $\cos \psi = +1$ or -1 . Lines of constant C joining two points on a $-$ curve without intersecting the corresponding $+$ curve represent resonance crossing motion.

One sees that the maximum beating range in α is normally less than 0.5 even for $\chi = 0$. A typical value of $\hat{\alpha}_0 = 0$ yields from (20), $\sigma \approx 2\kappa$ ($\kappa \ll 1$) and sets an upper limit of beating range. For LEP, therefore, the emittance growth does not exceed a factor of two, even with unfavourably chosen parameters. However, in the upper energy range where very small vertical beam size is required to obtain the required luminosity¹⁾, the constraint of (23) is necessary. The increase in vertical beam size observed in PETRA⁶⁾ might be partly due to non-linear beam-beam coupling since, in the experiment described, $\Delta q > 0$ and (24) was not fulfilled.

Finally, as discussed in Ref.2, the coupling term K_{22} could be compensated by a suitable arrangement of magnetic octupoles. However, it is not likely that such compensation would be needed for LEP.

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