DUALITY AND CANonical TRANSFORMATIONS

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Abstract

We present a brief review on the canonical transformation description of some duality symmetries in string and gauge theories. In particular, we consider abelian and non-abelian T-dualities in closed and open string theories as well as S-duality in abelian and non-abelian non-supersymmetric gauge theories.
1 T-Duality in String Theory

Some suggestions had been made in the literature pointing (at least in the simplified situation when all backgrounds are constant or depending only on time) towards an understanding of T-duality as particular instances of canonical transformations [1]. This idea works well when the backgrounds have an abelian isometry [2], laying duality on a simpler setting than before (see for instance [3] for a review on previous formulations). Essentially the canonical transformation provides the non-local change of variables identified as world-sheet T-duality, which reduces to $d \to s d$ for toroidal compactifications. Concerning non-abelian duality it is also possible to give a formulation in terms of canonical transformations when the isometry acts without isotropy [4, 5, 6, 7, 8]. In this case the canonical formulation allows to define the dual theory in arbitrary genus Riemann surfaces, what was not known within the original gauging procedure formulation [9].

There remain however some open problems within this formulation of duality. For the dual theory to be conformally invariant the dilaton must transform as $\tilde{\phi} = \phi - \frac{1}{2} \log R^2$ (for toroidal compactifications, or a straightforward generalization for abelian or non-abelian isometries) [10, 11]. A general argument justifying this transformation in phase space is not available, although we believe it should be along the lines of how the modular anomaly appears in abelian gauge theories, as we will discuss later in the text. Also, the explicit canonical transformation formulation of non-abelian duality for isotropic isometries is not known. There are results in the literature [12, 8, 32] proving that the initial and dual theories are indeed canonically equivalent but the non-local mapping generalizing $d \to s d$ to these transformations has still not been found.

1.1 Abelian duality

It is by now well known that abelian T-duality in sigma models is the result of a simple canonical transformation in the phase space of the theory [2]. Written in configuration space variables this transformation generalizes the duality mapping

$$\partial_+ x = -\frac{1}{R^2} \partial_+ \tilde{x}, \quad \partial_- x = \frac{1}{R^2} \partial_- \tilde{x}$$

(1.1)
of toroidal compactifications to general backgrounds with abelian isometries. Given a general
\(d\) dimensional \(\sigma\)-model with an abelian isometry represented by translations of a \(\theta\)-coordinate:

\[
L = \frac{1}{2} g_{0\alpha} (\dot{\theta}^2 - \theta^{'2}) + (\dot{\theta} + \theta') j_- + (\dot{\theta} - \theta') j_+ + \frac{1}{2} V, \tag{1.2}
\]

where:

\[
j_\pm = \frac{1}{2} k^\pm_\alpha \partial_\pm x^\alpha, \quad \alpha = 1, \ldots, d - 1
\]

\[
k^\pm_\alpha = g_{0\alpha} \pm b_\alpha
\]

\[
V = (g_{\alpha\beta} + b_{\alpha\beta}) \partial_+ x^\alpha \partial_- x^\beta,
\]

the dual with respect to this isometry can be obtained by performing the canonical trans-
formation:

\[
p_\theta = -\dot{\theta} \quad p_\theta = -\theta' \tag{1.4}
\]

in the phase space of the theory or equivalently, the non-local mapping:

\[
\partial_+ \theta = -(g_{00} \partial_+ \tilde{\theta} + \tilde{k}^-_\alpha \partial_+ \tilde{x}^\alpha)
\]

\[
\partial_- \theta = g_{00} \partial_- \tilde{\theta} + \tilde{k}^+_\alpha \partial_- \tilde{x}^\alpha \tag{1.5}
\]

in configuration space. The dual backgrounds are then given by Buscher’s formulas [10]. (1.4) is generated from the type I generating functional:

\[
\mathcal{F} = \int_{D,\partial D=S^1} d\tilde{\theta} \wedge d\theta = \frac{1}{2} \int_{S^1} (\dot{\theta} \tilde{\theta} - \theta \tilde{\theta}^{'}) d\sigma, \tag{1.6}
\]

which can be easily derived from the gauging procedure to abelian T-duality [13]. It is easy
to see that in this approach the initial and dual Lagrangians are equivalent up to a total derivative:

\[
\int_{\Sigma} d\tilde{\theta} \wedge d\theta, \tag{1.7}
\]

usually neglected in closed strings. However under a canonical transformation \(\{q^i, p_i\} \rightarrow \{Q^i, P_i\}\) the initial and canonically transformed Hamiltonians verify:

\[
\dot{q}^i p_i - H(q^i, p_i) = \dot{Q}^i P_i - \tilde{H}(Q^i, P_i) + \frac{d\mathcal{F}}{dt} \tag{1.8}
\]

where \(\mathcal{F}\) is the generating functional, such that \(H = \tilde{H}\) if and only if (assuming \(\mathcal{F}\) is type I
and does not depend explicitly on time):

\[
\frac{\partial \mathcal{F}}{\partial q^i} = p_i, \quad \frac{\partial \mathcal{F}}{\partial Q^i} = -P_i, \tag{1.9}
\]

It is then straightforward to see that the total derivative (1.7) yields the generating func-
tional and corresponding canonical transformation (1.6) and (1.4).

\(\mathcal{F}\) being linear in \(\theta\) and \(\tilde{\theta}\) implies that the classical canonical transformation (1.4) is
also valid quantum mechanically (as explained in [14]) and we can write the relation:

\[
\psi_k[\theta(\sigma)] = N(k) \int \mathcal{D}\theta(\sigma) e^{i\mathcal{F}[\theta(\sigma)]} \phi_k[\theta(\sigma)] \tag{1.10}
\]
between the corresponding Hilbert spaces. Here $\psi_k[\theta]$ and $\phi_k[\theta]$ are chosen as eigenstates corresponding to the same eigenvalue of the respective Hamiltonians and $N(k)$ is a normalization factor introduced to insure the proper normalization of the dual wave functions. From (1.10) we can learn about the multivaluedness and periods of the dual variables. Since $\theta$ is periodic, $\phi_k(\theta + a) = \phi_k(\theta)$ implies for $\theta$: $\theta(\sigma + 2\pi) - \theta(\sigma) = 4\pi/a$, which means that $\theta$ must live in the dual lattice of $\theta$.

In this formulation duality gets very simple conceptually. We can also learn how it can be applied to arbitrary genus Riemann surfaces, because the state $\phi_k[\theta(\sigma)]$ could be the state obtained from integrating the original theory on an arbitrary Riemann surface with boundary. It is also clear that the arguments generalize straightforwardly when we have several commuting isometries. The behavior of currents not commuting with those used to implement duality can also be clarified. In the case of WZW models it becomes rather simple to prove that the full duality group is given by $\text{Aut}(G)_L \times \text{Aut}(G)_R$, both inner and outer, where $L, R$ refer to the left and right currents on the model with group $G$. Due to the chiral conservation of the currents in this case, the canonical transformation leads to a local expression for the dual currents. In the case where the currents are not chirally conserved, then those currents associated to symmetries not commuting with the one used to perform duality become generically non-local in the dual theory and this is why they are not manifest in the dual Lagrangian.

We can say that the canonical transformation is a “minimal” approach in the sense that no extraneous structure (1-forms in Buscher’s approach or fake gauge fields in Rocek and Verlinde’s) has to be introduced, and all standard results in the abelian case (and more) are easily recovered using it. We should mention however that renormalization effects still need to be considered in order to prove the quantum equivalence between the two theories and there are in fact some results in the literature showing that they give corrections to Buscher’s backgrounds [15]. We need to reproduce as well the dilaton shift within the canonical transformation approach. Consider a constant toroidal background of radius $R$. The measure in configuration space is given by $D\theta \det R$. We can regularize this determinant as $R^{B_0}$, where $B_0$ is the dimension of the space of 0-forms in the two dimensional world-sheet (regularized in a lattice, for instance). With this prescription one realizes that the usual measure in phase space: $D\theta Dp_\theta$ gives upon integration on $p_\theta$: $D\theta R^{B_1}$, where $B_1$ is the dimension of the space of 1-forms in the world-sheet and emerges because the momenta are 1-forms. Therefore it differs from our definition of measure in configuration space. In order to reproduce the partition function in configuration space we need to include explicit factors of $R$ in the definition of the measure in phase space. One can check that considering these factors the correct shift of the dilaton is obtained after performing the canonical transformation. These arguments however are only rigorous for constant backgrounds. We believe that a similar reasoning could be applied to the general case.

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3 This way of regularizing the determinants reproduces the correct modular anomaly under S-duality in abelian gauge theories, as we will see in section 2.
1.2 Non-abelian duality

The conventional gauging approach to non-abelian duality [9] has two important drawbacks. The first one is that the transformation is in general non-invertible (i.e., it is not possible to recover the original theory by repeating the gauging procedure starting from the dual) and the second that it is not valid for non-spherical world-sheets [16, 17]. A canonical transformation description would be in this sense very useful, since it is invertible and the generalization to arbitrary genus Riemann surfaces is straightforward, as we have seen. Such a description is in fact known for those sigma models in which the non-abelian isometry acts without isotropy, i.e., without fixed points. The most general sigma model of this kind is [17]:

\[ S[g, x] = \int d\sigma_+ d\sigma_- [F_{ab}(x)(\partial_+ gg^{-1})^a (\partial_- gg^{-1})^b + F_{aa}^R(x)(\partial_+ gg^{-1})^a \partial_- x^a + F_{aa}^L(x) \partial_+ x^a (\partial_- gg^{-1})^a + F_{aa}(x) \partial_+ x^a \partial_- x^b], \]

where \( g \in G \), a Lie group (which we take to be compact), and \( \partial_{\pm} gg^{-1} = (\partial_{\pm} gg^{-1})^a T_a \) with \( T_a \) the generators of the corresponding Lie algebra\(^4\). This model is invariant under right transformations \( g \to gh \), with \( h \in G \). Let us parametrize the Lie group using the Maurer-Cartan forms \( \Omega_b^k \), such that

\[ (\partial_{\pm} gg^{-1})^a = \Omega_b^k(\theta) \partial_{\pm} \theta^k. \]

The following canonical transformation from \( \{ \theta^i, \Pi_i \} \) to \( \{ \chi^a, \tilde{\Pi}_a \} \):

\[ \Pi_i = -(\Omega_i^a \chi^a + f_{abc} \chi^a \Omega_{i}^b \theta^j) \]

\[ \tilde{\Pi}_a = -\Omega_i^a \theta^i \]

yields the non-abelian dual of (1.11) with respect to its isometry \( g \to gh \):

\[ \tilde{S} = \int d\sigma_+ d\sigma_- [(E + \text{ad}\chi)^{-1}_a(\partial_+ \chi^a + F_{aa}^L(x) \partial_+ x^a)(\partial_- \chi^b - F_{ab}^R(x) \partial_- x^b) + F_{aa}(x) \partial_+ x^a \partial_- x^b] \]

(1.14)

This was first realized in [4] for the case of \( SU(2) \) principal chiral models (where \( F_{ab} = \delta_{ab} \), \( F_{aa}^R = F_{aa}^L = F_{ab} = 0 \)), generalized in [5, 6] to arbitrary group, and shown to apply also to this more general case in [8]. (1.13) reads, in configuration space variables:

\[ \Omega_i^a \partial_+ \theta^i = -M_{ba}(\partial_+ \chi^b + F_{ba}^L \partial_+ x^a) = -(\tilde{g}_{ab} - \tilde{b}_{ab}) \partial_+ \chi^b - (\tilde{g}_{aa} - \tilde{b}_{aa}) \partial_+ x^a \]

\[ \Omega_i^a \partial_- \theta^i = M_{ab}(\partial_- \chi^b - F_{ba}^R \partial_- x^a) = (\tilde{g}_{ab} + \tilde{b}_{ab}) \partial_- \chi^b + (\tilde{g}_{aa} + \tilde{b}_{aa}) \partial_- x^a. \]

(1.15)

These relations generalize (1.5) to non-abelian duality transformations, the main difference being that the components of the torsion in the Lie algebra variables appear explicitly.

(1.13) is generated by:

\[ \mathcal{F}[\chi, \theta] = \oint d\sigma T_r(\chi \partial_+ gg^{-1}) \]

(1.16)

\(^4\{T_a\} \) are normalized such that \( Tr(T_aT_b) = \delta_{ab} \).
which is linear in the dual variables but non-linear in the original ones. This means that in general it will receive quantum corrections when implemented at the level of the Hilbert spaces \[ n5b/1/4/n5d, \] the reason being that we cannot prove a relation like

\[ |\chi^a\rangle = \int \mathcal{D}\theta^i(\sigma)e^{i\mathcal{F}[\chi^a,\theta^i(\sigma)]}\theta^i(\sigma) \]  \hspace{1cm} (1.17)

using the eigenfunctions of the respective Hamiltonians. However, it was shown in [4, 5] that such a relation can in fact be proven using the eigenfunctions of the respective conserved currents in the initial and dual theories. Of course for this to be true we need to have a symmetry in the dual theory, which is not the case for arbitrary backgrounds. When \( E_{[ab]} = F_{oa}^L = F_{oa}^R = 0 \) the original sigma model is invariant under left transformations \( g \rightarrow hg, h \in G \) and we also find a symmetry in the dual theory under \( \chi \) transforming in the adjoint representation\(^5\). Then it is easy to see that the canonical transformation couples the conserved currents associated to the left symmetry of the initial theory:

\[ J_\pm ^{(L)} = \frac{1}{2}E_{(ab)}\Omega_i^a \partial_\pm \theta^i \]  \hspace{1cm} (1.18)

and the ones associated to \( \chi \rightarrow h\chi h^{-1} \) in the dual\(^6\):

\[ \tilde{J}_\pm ^a = \partial_+ \chi^a - \frac{1}{2}E_{(ab)}M_{bc}\partial_+ \chi^c \]

\[ \tilde{J}_- ^a = -\partial_- \chi^a + \frac{1}{2}E_{(ab)}M_{bc}\partial_- \chi^c. \]  \hspace{1cm} (1.19)

We can then show that

\[ \tilde{J}_\pm ^a e^{i\mathcal{F}} = J_\pm ^{(L)} e^{i\mathcal{F}} \]  \hspace{1cm} (1.20)

and prove (1. 17) using the eigenfunctions of the respective conserved currents. From this relation the equivalence between the initial and dual theories for arbitrary genus Riemann surfaces is straightforward and it also allows the derivation of global properties in the dual. As in the abelian case there can still be renormalization effects modifying the classical backgrounds. We should also mention that a dilaton shift is needed in order to preserve conformal invariance \[ n5b/9/n5d, \] exactly as in the abelian case. This remains an open question within the canonical transformation description whose resolution we believe should be along the lines previously mentioned in the abelian case.

1.3 Superstrings

The formulation of abelian \[ n5b/1/8/n5d \] and non-isotropic non-abelian \[ n5b/1/9, n5d T/n5b/1/9, n5d \] T-dualities in \( N = 1 \) superstring theories as canonical transformations is also known. Let us consider first the

\(^5\)This symmetry is the reminiscence of the left symmetry of the original theory since the left and right symmetries commute and we are dualizing with respect to the right action only.

\(^6\)Up to a total derivative term which for principal chiral models \( (E_{ab} = \delta_{ab}) \) is the responsible for having curvature-free currents in the dual, that are coupled to the curvature-free currents of the principal chiral model \[ n5b/4/n5d. \]
case of an abelian isometry $\delta x^i = \epsilon k^i$, $i = 1, \ldots, d$. In adapted coordinates to the isometry \{\theta, \psi_\pm^0, x^\alpha, \psi_\pm^\alpha\}, $\alpha = 1, \ldots, d - 1$, we can write the $N = 1$ action as:

$$S = \frac{1}{2} \int d^2 \sigma \{ g_{00}(\dot{\theta}^2 - \theta'^2) + 2(\dot{\theta} + \theta')j_+ + 2(\dot{\theta} - \theta')j_- - i(g_{00}\psi_+^0 + k^\alpha_\alpha)(\dot{\psi}_+^0 - \psi_+^0') - i(g_{00}\psi_-^0 + k^\alpha_\alpha)(\dot{\psi}_-^0 + \psi_-^0') + V \}$$

(1. 21)

where we have defined

$$j_\pm = \frac{1}{2}(k^\alpha_\alpha \partial_x x^\alpha + i k^i_{[i,j]} \psi_+^i \psi_-^j)$$

$$V = (g_{\alpha\beta} + b_{\alpha\beta}) \partial_x x^\alpha \partial_x x^\beta - i\psi_+^i (g_{i\alpha} + b_{i\alpha}) \partial_x \psi_-^\alpha - i\psi_-^i (g_{i\alpha} - b_{i\alpha}) \partial_x \psi_+^\alpha - i\partial_j (g_{ai} - b_{ai}) \psi_+^i \partial_x x^a \psi_-^j - i\partial_j (g_{ai} + b_{ai}) \psi_-^i \partial_x x^a \psi_+^j + \frac{1}{2} R^i_{ijkl} \psi_+^i \psi_-^j \psi_+^k \psi_-^l, \quad (1. 22)$$

and $k^\pm_\alpha = g_{0i} \pm b_{0i}$ as in the previous section.

The canonical momenta associated to the zero coordinates are

$$\Pi_\pm = \frac{i}{2}(g_{00}\psi_\pm^0 + k^\alpha_\alpha \psi_\pm^\alpha) \quad (1. 23)$$

$$p_\theta = g_{00}\dot{\theta} + j_+ + j_-, \quad (1. 24)$$

(1. 23) being two first class constraints.

The generating functional:

$$\mathcal{F} = \frac{1}{2} \int d\sigma \{ \theta'(\dot{\theta} - \theta') - i\psi_+^0 \psi_-^0 + i\psi_-^0 \psi_+^0 \}$$

(1. 25)

induces the change of variables in phase space [18]:

$$\bar{\Pi}_\pm = -\frac{\delta F}{\delta \psi_\pm^0} = \mp \frac{i}{2} \psi_\pm^0, \quad \Pi_\pm = \frac{\delta F}{\delta \psi_\pm^\alpha} = \mp \frac{i}{2} \psi_\pm^\alpha$$

$$p_\theta = -\frac{\delta F}{\delta \dot{\theta}} = -\dot{\theta}', \quad p_\theta = \frac{\delta F}{\delta \dot{\theta}} = -\dot{\theta}, \quad (1. 26)$$

which yields the abelian T-dual with backgrounds given by Buscher’s formulas [21]. In configuration space this corresponds to:

$$\psi_\pm^0 = \mp(g_{00}\psi_\pm^0 + k^\alpha_\alpha \psi_\pm^\alpha)$$

$$\psi_\pm^\alpha = \pm \psi_\pm^\alpha \quad (1. 27)$$

for the fermions, and:

$$\partial_+ \theta = -\bar{g}_{00}\partial_+ \tilde{\theta} - \bar{k}^\alpha_\alpha \partial_+ x^\alpha - i\bar{k}^i_{[i,j]} \psi_+^j \psi_-^i = -\bar{k}^i_\alpha \partial_+ x^i - i\bar{k}^i_{[i,j]} \psi_+^j \psi_-^i$$

$$\partial_- \theta = \bar{g}_{00}\partial_- \tilde{\theta} + \bar{k}^\alpha_\alpha \partial_- x^\alpha + i\bar{k}^i_{[i,j]} \psi_+^i \psi_-^j = \bar{k}^i_\alpha \partial_- x^i + i\bar{k}^i_{[i,j]} \psi_+^i \psi_-^j, \quad (1. 28)$$

for the bosons. (1. 27) and (1. 28) generalize the abelian duality mapping (1, 5) to $N = 1$ sigma models, and can also be obtained from (1. 5) replacing bosonic fields by superfields and derivatives by superderivatives.
\( \mathcal{F} \) being linear in the original and dual variables implies that the original and dual theories are also equivalent quantum mechanically, as in the bosonic case. As in that case its expression can be derived from the total time derivative term that is induced in the gauging procedure.

Let us consider now non-isotropic non-abelian transformations in \( N = 1 \) sigma models. For simplicity we are going to restrict ourselves to the case of principal chiral models: \( g_{ij} = \Omega^a_i \Omega^a_j \) and \( b_{ij} = 0 \). Following \([20]\) we can use tangent space variables for the fermions \( \phi_\pm^a = \Omega^a_i \psi_\pm^i \) and consider an action:

\[
S = \int d\sigma_+ d\sigma_- [ (\partial_+ gg^{-1})^a (\partial_- gg^{-1})^a - i \phi_+^a \partial_- \phi_-^a - i \phi_-^a \partial_+ \phi_+^a + \frac{i}{2} f_{abc} \phi_+^a (\partial_- gg^{-1})^b \phi_-^c + \\
\frac{i}{2} f_{abc} \phi_-^a (\partial_+ gg^{-1})^b \phi_+^c + \frac{1}{8} f_{abc} f_{def} \phi_+^a \phi_+^d \phi_-^e \phi_-^f ] .
\] (1. 29)

Working in phase space variables \( \{ (\theta^i, \Pi_i), (\phi_\pm^a, \Pi_{\phi \pm}^a) \} \):

\[
\Pi_i = \Omega^a_i \Omega^a_j \dot{\theta}^j + \frac{i}{4} f_{abc} \Omega^a_i (\phi_+^b \phi_-^c + \phi_-^b \phi_+^c) \\
(1. 30)
\]

\[
\Pi_{\phi \pm}^a = \frac{i}{2} \phi_\pm^a ,
\] (1. 31)

where (1. 31) are a set of first class constraints, the non-abelian dual of (1. 29) with respect to the right action of the whole symmetry group \( G \) can be obtained through a canonical transformation from \( \{ (\theta^i, \Pi_i), (\phi_\pm^a, \Pi_{\phi \pm}^a) \} \) to \( \{ (\chi^a, \Pi_{\chi}^a), (\phi_\pm^a, \Pi_{\phi \pm}^a) \} \). In particular:

\[
\Pi_i = - (\Omega^a_i \chi^a + f_{abc} \chi^a \Omega^b_j \Omega^c_i \dot{\theta}^j) \\
\Pi_{\phi \pm}^a = - \frac{i}{2} (\phi_\pm^a + f_{abc} \phi_\pm^b \phi_\pm^c) \\
(1. 32)
\]

\[
\Pi_{\phi \pm}^a = \pm \frac{i}{2} \phi_\pm^a
\] (1. 33)

for the fermionic ones. Its generating functional is:

\[
\mathcal{F} = \int d\sigma [ \chi^a \Omega^a_i \theta^i + \frac{i}{4} f_{abc} \chi^a (\phi_+^b \phi_-^c - \phi_-^b \phi_+^c) - \frac{i}{2} (\phi_+^a \phi_-^a - \phi_-^a \phi_+^a) ] .
\] (1. 34)

The dual action is given by \([22]\)\(^7\):

\[
\tilde{S} = \int d\sigma_+ d\sigma_- [ M_{ab} (\partial_+ \chi^a \partial_- \chi^b - i \phi_+^a \partial_- \phi_-^b + i \partial_+ \phi_-^a \phi_-^b + i M_{ac} f_{bde} M_{da} \phi_-^e \phi_-^d \partial_+ \chi^b + \\
i M_{ac} f_{bde} M_{db} \phi_-^a \phi_-^e \partial_- \chi^b + L_{abcd} \phi_-^a \phi_-^b \phi_-^c \phi_-^d ] ,
\] (1. 35)

\(^7\)In this reference this dual action is derived following the gauging procedure.
manifestly supersymmetric and where \( M = (1 + \text{ad} \chi)^{-1} \) and \( L_{abcd} = -(f_{aef} f_{ebc} + f_{abe} f_{fd}) M_{ci} M_{eg} M_{fd} \). The dual momenta are:

\[
\Pi_a = \frac{1}{2} (M_{ab}) \chi^b - M_{[ab]} \chi^b - i (M \text{ad} \phi_- M)_{ab} \tilde{\phi}_a^b - i (M \text{ad} \phi_+ M)_{ba} \tilde{\phi}_b^a \quad (1.36)
\]

\[
\tilde{\Pi}_{\phi+}^a = \frac{i}{2} M_{ba} \tilde{\phi}_a^b \]

\[
\tilde{\Pi}_{\phi-}^a = \frac{i}{2} M_{ab} \tilde{\phi}_a^b \quad (1.37)
\]

where \( (\text{ad} \phi_{\pm})_{ab} = f_{abc} \tilde{\phi}_c^a \). From (1.37) and (1.33) we see that the fermions simply transform with the change of scale:

\[
\phi^a_+ = -M_{ba} \tilde{\phi}_a^b \\
\phi^a_- = M_{ab} \tilde{\phi}_a^b. \quad (1.38)
\]

The corresponding non-local transformation for the bosonic part is given in terms of dual backgrounds by:

\[
(\partial_+ gg^{-1})^a = -(\tilde{g}_{ab} - \tilde{b}_{ab}) \partial_+ \chi^b - i \partial_c (\tilde{g}_{ab} - \tilde{b}_{ab}) \tilde{\phi}_+^c \tilde{\phi}_+^d - i (\phi_+^2)^a \\
(\partial_- gg^{-1})^a = (\tilde{g}_{ab} + \tilde{b}_{ab}) \partial_- \chi^b + i \partial_c (\tilde{g}_{ab} + \tilde{b}_{ab}) \tilde{\phi}_-^c \tilde{\phi}_-^d - i (\phi_-^2)^a, \quad (1.39)
\]

where the last terms need still be written in terms of the dual fermions. In this form we see that they generalize (1.28) by means of the last quadratic terms in the fermions, which are zero in the abelian case. (1.38) and (1.39) can also be obtained from the corresponding (1.15) in superspace by introducing chiral superfields [20]. As in the pure bosonic non-abelian case the canonical transformation couples the conserved currents associated to the left symmetry \( g \to hg \) of the initial theory and the ones associated to transformations in the adjoint in the dual. Namely:

\[
J^a_{\pm} = (\partial_\pm gg^{-1})^a + i (\phi_\pm^2)^a = \Omega^a_i \partial_\pm \xi^i + i f_{abc} \phi_\pm^b \phi_\pm^c \quad (1.40)
\]

with

\[
J^a_+ = \partial_+ \chi^a - M_{ba} \partial_+ \chi^b + i (M \text{ad} \phi_+ M)_{ba} \tilde{\phi}_b^a \\
J^a_- = -\partial_- \chi^a + M_{ab} \partial_- \chi^b - i (M \text{ad} \phi_- M)_{ab} \tilde{\phi}_b^a. \quad (1.41)
\]

From here we can also establish the quantum equivalence between the two theories.

### 1.4 Open Strings

Recently there has been renewed interest in the study of open string theories with the last developments in string dualities (see for instance [23] and references therein). In [24] Polchinski showed that open strings with certain exotic boundary conditions (D-branes) were the carriers of the RR charges required by string duality. This identification allowed for many new tests of string duality. D-branes first arised as particular features under T-duality in
theories of open strings [25, 26]. The duality transformation (1.1) maps Neumann boundary conditions: $\partial_\nu x = 0$, to Dirichlet boundary conditions: $\partial_k \bar{x} = 0$, where $\partial_\nu$ and $\partial_k$ are the normal and tangent derivatives to the boundary. The ends of the strings are then confined to the $\bar{x}$ plane, which is itself dynamical. These particular objects with mixed Neumann and Dirichlet boundary conditions are the D-branes [25]. For type I superstrings crosscap boundary conditions for the unoriented topologies are mapped to orientifold conditions [25, 27] and the dual D-brane is hidden in the orientifold plane. These dual theories may seem rather exotic, but they are just a more suitable description at small distances of the same original open string theory.

The open string-D-brane dualities of toroidal compactifications have been extended to more general backgrounds following the gauging procedure to T-duality\(^8\). Namely, to backgrounds with abelian [28, 29] and non-abelian isometries [30]. Certain backgrounds without isometries and more recently WZW models have also been studied in [31] and [32] within the Poisson-Lie T-duality. We are going to discuss T-duality for open strings in various backgrounds within the canonical transformation approach [33]. This approach is particularly useful in obtaining information about the boundary conditions, since it provides an explicit mapping between initial and dual variables.

Let us consider open and closed strings propagating in a $d$ dimensional background of metric, antisymmetric tensor and abelian gauge field\(^9\). In the neutral case the action can be written:

$$S = \int d\Sigma d\sigma_+ d\sigma_- (g_{ij} + b_{ij}) \partial_+ x^i \partial_- x^j + \int \partial \Sigma V_i \partial_k x^i$$

(1.42)

where $V_i$ denotes the abelian background gauge field and $\partial_k$ is the tangent derivative to the boundary\(^10\). The boundary term can be absorbed in the action by just considering:

$$S = \int d\Sigma d\sigma_+ d\sigma_- (g_{ij} + B_{ij}) \partial_+ x^i \partial_- x^j$$

(1.43)

with $B_{ij} = b_{ij} + F_{ij} = b_{ij} + \partial V_j - \partial_j V_i$. The torsion term is absent for the unoriented topologies. Let us assume that there exists a Killing vector $k^i$ such that $\mathcal{L}_k g_{ij} = 0$ and $\mathcal{L}_k B_{ij} = 0$ (this means we can have: $\mathcal{L}_k b_{ij} = \partial_k v_j - \partial_j v_i$ and $\mathcal{L}_k V_i = -v_i + \partial_k \varphi$, for some $v_i$, $\varphi$). The dual with respect to this isometry can be constructed as in the closed string case by performing the canonical transformation:

$$p_\theta = -\theta'$$

$$p_\theta = -\theta'$$

(1.44)

the only difference being that $b$ is replaced by $B$ in order to absorb the background gauge field. In this case we also need to care about the boundary conditions. The canonical transformation approach is particularly adequate to deal with boundary conditions since it provides the explicit relation between the target space coordinates of the original and dual theories. From (1.44) we get\(^11\):

$$\bar{\theta} = -(g_{\alpha \alpha} \theta' + g_{\alpha \alpha} \theta'^\alpha - B_{\alpha \alpha} \bar{x}^\alpha)$$

(1.45)

---

\(^8\)In the first of [29] there is also a brief study with canonical transformations.

\(^9\)We will only consider abelian background gauge fields. For a non-abelian treatment see the first of [29].

\(^10\)We consider $\sigma = \text{constant boundaries}$ but in certain, specified, cases.

\(^11\)We use capital letters for the dual backgrounds to account for its dependence on the abelian background gauge field.
and
\[
g_{\alpha\beta}\dot{\theta} + g_{\alpha\beta}x^{\beta} + B_{\alpha\beta}\dot{x}^{\beta} = \bar{G}_{\alpha\beta}\dot{\theta} + \bar{G}_{\alpha\beta}x^{\beta} + \bar{B}_{\alpha\beta}\dot{x}^{\beta}. \tag{1.46}
\]

Then, Neumann boundary conditions for the original theory \([34]\):
\[
g_{ij}x^{j} - B_{j}\dot{x}^{j} = 0 \tag{1.47}
\]
imply in the dual:
\[
\dot{\theta} = 0 \\
\bar{G}_{\alpha\beta}\dot{\theta} + \bar{G}_{\alpha\beta}x^{\beta} + \bar{B}_{\alpha\beta}\dot{x}^{\beta} = 0. \tag{1.48}
\]

These mixed boundary conditions represent a flat Dirichlet \((d - 2)\)-brane in the dual background \([28]\). Also from here we can deduce the collective motion of the brane. Decomposing \(B_{\alpha\alpha} = b_{\alpha\alpha} - \partial_{\alpha}V_{0}\) we realize that the usual Buscher’s backgrounds for closed strings (with the torsion \(b\)) are gotten provided we redefine \(\dot{\theta} \equiv \theta + V_{0}(x^{\alpha})\). Therefore \(V_{0}(x^{\alpha})\) gives the transverse position of the brane in the dual theory\(^{12}\). If we dualize \(n\) commuting isometries it is straightforward to check that a Dirichlet \((d - n - 1)\)-brane is obtained in the dual.

It is perhaps worth mentioning that there are some particular backgrounds (those whose conserved currents associated to the isometry are chiral \([13]\)) which are at the same time backgrounds of open strings and D-branes depending on the boundary conditions, which are in turn related by a T-duality transformation.

Let us now analyze the unoriented topologies. Invariance under world-sheet parity implies that the antisymmetric tensor and the abelian gauge field are projected out of the spectrum. We can still have non-abelian gauge fields in \(SO(N)\) and \(USp(N)\) but they must be treated differently (see for instance the first in \([29]\)). Unoriented topologies can be obtained from oriented ones by identifications of points on the boundary \([35]\). For instance the projective plane is obtained from the disk\(^{13}\) identifying opposite points. The topology thus obtained is a crosscap. Under abelian T-duality we should get the mapping from crosscap to orientifold conditions \([28]\). Crosscap boundary conditions for the coordinate adapted to the isometry:
\[
\dot{\theta}(\sigma + \pi) = -\dot{\theta}(\sigma) \quad \theta'(\sigma + \pi) = \theta'(\sigma), \tag{1.49}
\]
where we are parametrizing the boundary of the disk by \((0, 2\pi)\) and identifying opposite points: \(\theta(\sigma + \pi) = \theta(\sigma)\), translate to:
\[
p_{\theta}(\sigma + \pi) = -p_{\theta}(\sigma) \quad \theta'(\sigma + \pi) = \theta'(\sigma) \tag{1.50}
\]
in phase space. Then \((1.44)\) implies:
\[
\tilde{\theta}'(\sigma + \pi) = -\tilde{\theta}'(\sigma) \quad p_{\tilde{\theta}}(\sigma + \pi) = p_{\tilde{\theta}}(\sigma), \tag{1.51}
\]

\(^{12}\)A particular case is when \(V_{0}\) is taken pure gauge locally breaking \(U(N)\) to \(U(1)^{N}\), i.e. when a Wilson line \(V_{0} = \text{diag}(\theta_{1}, \ldots, \theta_{N})\) is included. In this case we get a maximum of \(N\) D-branes in the dual theory with fixed positions at \(\theta_{i}, i = 1, \ldots, N\) \([23]\).

\(^{13}\)We have to make first a Wick rotation to imaginary time.
which are orientifold conditions in phase space since \( p_\theta(\sigma + \pi) = p_\theta(\sigma) \) implies \( \dot{\theta}(\sigma + \pi) = \dot{\theta}(\sigma) \). The orientifold plane is at \( \theta = 0 \) and it’s non-dynamical, because the abelian gauge field is zero in the unoriented case. The rest of the coordinates still satisfy crosscap boundary conditions.

We can make an analogous analysis for the non-abelian backgrounds (1. 11) [33]. The abelian gauge fields that are compatible with the non-abelian isometry \( g \to gh \) have the form \( V_i = \frac{1}{2} \Omega^a_i C^a(x) \), with \( C \) arbitrary, and \( V_\alpha \theta^\beta \)-independent. Then \( E_{[\alpha\beta]} = b_{\alpha\beta} + f_{abc}C^c(x) \), where \( b_{\alpha\beta} \) is the closed strings antisymmetric tensor and \( F_{\alpha\alpha} = f_{\alpha\alpha} - \frac{1}{2} \partial_\alpha C^a(x) \), \( F_{\alpha\alpha} = f_{\alpha\alpha}^L + \frac{1}{2} \partial_\alpha C^a(x) \) (with \( f^R, f^L \) the corresponding closed strings backgrounds). The canonical transformation (1, 13) implies that the dual of Neumann boundary conditions are mixed Dirichlet and Neumann conditions weighted with the initial metric and torsion respectively, for the coordinates transforming under the isometry group. The rest of the coordinates still satisfy Neumann boundary conditions in the dual. However in those cases in which we can establish a correspondence between the Hilbert spaces of the initial and dual theories (when the dual theory admits an isometry) the dual boundary conditions for the coordinates transforming under the isometry reduce to generalized Dirichlet conditions:

\[
\tilde{g}_{ab} \chi^b - \tilde{b}_{ab} \chi^\alpha = 0, \tag{1. 52}
\]

generalized in the sense that it is the momentum associated to a non-flat background which vanishes at the ends of the string. We can then conclude that for certain kinds of sigma models with non-abelian isometries a curved \((d-\dim G-1)\) D-brane with metric \( g_{ab} \) and torsion \( \tilde{b}_{ab} \) is obtained in the dual. The backgrounds of unoriented strings are among the ones for which we get Dirichlet boundary conditions. In these cases we can also study the mapping of crosscap boundary conditions. The result is that in the dual, generalized orientifold conditions:

\[
\tilde{g}_{ab} \chi^\alpha(\sigma + \pi) - \tilde{b}_{ab} \chi^b(\sigma + \pi) = - (\tilde{g}_{ab} \chi^\alpha(\sigma) - \tilde{b}_{ab} \chi^b(\sigma)) \]
\[
\tilde{g}_{ab} \chi^b(\sigma + \pi) - \tilde{b}_{ab} \chi^\alpha(\sigma + \pi) = \tilde{g}_{ab} \chi^b(\sigma) - \tilde{b}_{ab} \chi^\alpha(\sigma) \tag{1. 53}
\]

are satisfied, meaning that the momentum must be equal at the identification points but the momentum flows out of them must have opposite sign.

1.4.1 Superstrings

In a similar way we can study the mapping of the boundary conditions in open superstring theory [33]. We consider abelian background gauge fields, which are absorbed in a torsion term \( B_{ij} = b_{ij} + F_{ij} \). Starting with the usual Neumann R-NS boundary conditions\(^\text{15}\) for the

\(^{14}\)Integration on the first equation implies \( \dot{\theta}(\sigma + \pi) = - \dot{\theta}(\sigma) \) so that the world-sheet parity reversal is accompanied by a \( Z_2 \) transformation in space-time. These kinds of constructions are the orientifolds [25, 27].

\(^{15}\)R-NS are minima of the bulk action and therefore are classical boundary conditions only if \( B_{ij} = 0 \). However the simplest minima of the full action give trivial dynamics for the fermions at the boundary, being in this sense too restrictive. One can study if this is also the case for arbitrary minima and find that in general the fermionic contribution at the boundary is not zero [33]. Here we are still going to consider R-NS for simplicity.
\[ g_{ij} x'^j - B_{ij} \dot{x}^i = 0 \]
\[ \psi_+^i = \eta \psi_-^i; \quad \eta = \pm 1 \]  
(1.54)

we get after an abelian duality transformation (1. 27), (1. 28):
\[ \tilde{\psi}_+^0 = \eta \tilde{\psi}_-^0 \]
\[ \tilde{\psi}_+^0 + \eta \tilde{\psi}_-^0 = 2\eta k^*_\alpha \tilde{\psi}_-^\alpha \]
\[ \dot{\theta} = i \partial_\alpha k^*_\alpha \tilde{\psi}_-^\alpha \tilde{\psi}_+^\beta \]
\[ \tilde{G}_{\alpha \beta} \tilde{x}^\alpha - B_{\alpha \beta} \dot{x}^\beta = -ik^*_\alpha \partial_\beta k^2 \tilde{\psi}_-^\alpha \tilde{\psi}_+^0 + i(k^*_\alpha \partial_\beta k^*_\sigma - k^*_\alpha \partial_\beta \tilde{k}^*_\sigma) \tilde{\psi}_+^\beta \tilde{\psi}_-^\sigma \]  
(1.55)

where \( k^*_\alpha = B_{\alpha \alpha} \). These results are in agreement with (3.12) in [28]. The non-trivial terms (those that spoil Dirichlet NS-R boundary conditions in the dual) are all proportional to \( B_{\alpha \alpha} \). Therefore a super D-brane is obtained in the dual only if the original background is such that \( B_{\alpha \alpha} = 0 \). If this occurs (1. 55) turns into: \( \tilde{\psi}_+^0 = -\eta \tilde{\psi}_-^0 \), accounting for the reversal of space-time chirality under T-duality [36], \( \dot{\theta} = 0 \), and Neumann R-NS boundary conditions for the rest of the coordinates. This is the case, in particular, for the type I superstring where the D-brane is actually an orientifold. In this theory consistency conditions restrict the possible D-manifolds to one, five and nine-branes [23]. Since the only consistent open superstring theory is the type I superstring, which contains unoriented topologies, it is interesting to analyze in some more detail the unoriented world-sheets. As in the previous section we consider the projective plane, obtained from the disk by identifying opposite points. Crosscap boundary conditions for the fermions contain an \( i \) factor due to the fact that we are taking a constant time boundary [34]:
\[ \psi_+^i (\sigma + \pi) = i\eta \psi_-^i (\sigma), \quad \eta = \pm 1 \]  
(1.56)
\[ x^\beta (\sigma + \pi) = x^\beta (\sigma), \quad \dot{x}^\beta (\sigma + \pi) = -\dot{x}^\beta (\sigma). \]  
(1.57)

These conditions are mapped under (1. 27) and (1. 28) into:
\[ \tilde{\psi}_+^\alpha (\sigma + \pi) = i\eta \tilde{\psi}_-^\alpha (\sigma) \quad \tilde{\psi}_+^0 (\sigma + \pi) = -i\eta \tilde{\psi}_-^0 (\sigma) \]  
(1.58)

for the fermions, giving the usual change of sector for the 0-component, and to:
\[ \dot{\theta}(\sigma + \pi) = \dot{\theta}(\sigma) \quad \dot{x}^\alpha (\sigma + \pi) = -\dot{x}^\alpha (\sigma) \]
\[ \ddot{\theta}(\sigma + \pi) = -\ddot{\theta}(\sigma) \quad \ddot{x}^\alpha (\sigma + \pi) = \ddot{x}^\alpha (\sigma) \]  
(1.59)

for the bosons, i.e. orientifold conditions for the \( \dot{\theta} \) coordinate and crosscap for the rest. Therefore the dual theory is an orientifold, static since the abelian electromagnetic field is absent for unoriented strings.

Let us now concentrate on the non-abelian models (1. 29) [33]. Restricting to the case of R-NS boundary conditions for the fermions we get in the dual:
\[ \tilde{\phi}_+^a = -\eta M_{ba}^{-1} M_{bc} \tilde{\phi}_-^c, \]  
(1.60)

12
which are not NS-R boundary conditions. We can just point out that they could be interpreted as NS-R plus corrections in $\text{ad}\chi$.

Concerning the bosons, if we start with Neumann boundary conditions: $\Omega^a_i \theta^i = 0$, we obtain in terms of the dual backgrounds:

$$\tilde{g}_{ab} \chi^b - \tilde{b}_{ab} \chi^b + \frac{i}{2} \partial_e (\tilde{g}_{ab} + \tilde{b}_{ab}) \tilde{\phi}_e^a \tilde{\phi}_e^b + \frac{i}{2} \partial_e (\tilde{g}_{ab} - \tilde{b}_{ab}) \tilde{\phi}_e^a \tilde{\phi}_e^b = 0. \quad (1.61)$$

This equation represents the vanishing of $\Pi_a$ (given by (1.36)) at the ends of the string. Therefore we find a curved $N = 1$ supersymmetric D-brane (in this particular example (-1)-brane, since we haven’t allowed for inert coordinates) with metric and torsion given by $\tilde{g}_{ab}$ and $\tilde{b}_{ab}$. However since the only consistent open superstring theory contains unoriented topologies the D-brane is an orientifold, as happened in the abelian case. In particular, one can see that crosscap boundary conditions are mapped to:

$$\Pi_a (\sigma + \pi) = \Pi_a (\sigma)$$

$$((g_{ab} \chi^b - b_{ab} \chi^b + \frac{i}{2} \partial_e (g_{ab} - b_{ab}) \phi^a_e \phi^b_e - \frac{i}{2} \partial_e (g_{ab} + b_{ab}) \phi^a_e \phi^b_e) |_{\sigma + \pi} = -((g_{ab} \chi^b - b_{ab} \chi^b + \frac{i}{2} \partial_e (g_{ab} - b_{ab}) \phi^a_e \phi^b_e - \frac{i}{2} \partial_e (g_{ab} + b_{ab}) \phi^a_e \phi^b_e) |_{\sigma}, \quad (1.62)$$

where the second equation represents that the momenta flowing out of the identification points must have opposite signs, as in the bosonic non-abelian case. The dual fermions satisfy:

$$\tilde{\phi}^a_+ (\sigma + \pi) = -i \eta M^a_{bc} \phi^c_- (\sigma). \quad (1.63)$$

## 2 S-duality in Gauge Theories

### 2.1 Abelian gauge theories

Four dimensional abelian gauge theories are invariant under strong-weak coupling duality\textsuperscript{16}, generated by the interchange of the electric and magnetic degrees of freedom of the theory: $dA \to *dA$, with $A$ the abelian gauge field. Both T-duality in two dimensional sigma models\textsuperscript{17} and S-duality in four dimensional abelian gauge theories are generated by the mapping $d \to *d$ in the corresponding two dimensional world-sheet or four dimensional space-time. In fact, they are particular cases of a more general duality present in $d$ dimensional theories of $p$ forms [38, 39]:

$$S \sim \int \frac{1}{g^2} d^d x \; dA_p \wedge *dA_p \quad (2.1)$$

(in 2 dim $g$ is the inverse of the compactification radius) where the mapping $d \to *d$ yields a dual theory formulated in terms of $(d - p - 2)$ forms. This duality can also be described as a canonical transformation in the corresponding phase space, as one would expect. In this

\textsuperscript{16}For reviews on S-duality in gauge theories see for instance [37].

\textsuperscript{17}What follows holds for toroidal compactifications. We have seen in the previous sections the generalization to other backgrounds.
section we study in detail the canonical transformation description of S-duality in Maxwell theory \((d = 4, p = 1)\) [40]. The generalization to arbitrary \(d, p\) will also be described at the end.

Let us consider the Maxwell Lagrangian with \(\theta\)-term:

\[
L = \frac{1}{8\pi} \left( \frac{4\pi}{g^2} F_{mn} F^{mn} + \frac{i\theta}{4\pi} \epsilon_{mnpq} F^{mn} F^{pq} \right) = \frac{i}{8\pi} (\tau F_{mn}^+ F^{-mn} - \tau F_{mn}^- F^{mn})
\]

(2. 2)

defined on a Euclidean four-manifold \(M_4\). Here \(\tau = \theta/2\pi + 4\pi i/g^2\), \(F_{mn} = \partial_m A_n - \partial_n A_m\), \(F_{mn}^\pm = \frac{1}{2} \epsilon_{mnpq} F^{pq}\) and \(F_{mn}^\pm = \frac{1}{2} (F_{mn} \pm * F_{mn})\).

The canonical momenta are given by\(^{19}\):

\[
\Pi_0 = 0 \\
\Pi^\alpha = 4\tau F_{+0\alpha} - 4\tau F_{-0\alpha},
\]

(2. 3)

where \(\alpha\) runs over spatial indices, and the Hamiltonian:

\[
H = \frac{1}{4(\bar{\tau} - \tau)} \Pi_\alpha \Pi^\alpha + \partial_\alpha A_\beta \Pi^\beta - \frac{\bar{\tau} + \tau}{\bar{\tau} - \tau} \Pi_\alpha * F^{0\alpha} + \frac{4\tau}{\bar{\tau} - \tau} * F^{0\alpha} * F_\alpha.
\]

(2. 4)

The primary constraint \(\Pi_0 = 0\) implies the secondary \(\partial_\alpha \Pi^\alpha = 0\), therefore we can drop the \(\partial_\alpha A_\beta \Pi^\beta\) term keeping in mind that the Hamiltonian is defined in this restricted phase space.

The interchange between electric and magnetic degrees of freedom can be written as the following canonical transformation in the phase space of the theory\(^{19}\):

\[
\Pi^\alpha = -4 * \vec{F}^{0\alpha} \\
\vec{F}^\alpha = 4 * F^{0\alpha},
\]

(2. 5)

where \(\vec{F} = d \vec{A}\), with generating functional:

\[
\mathcal{F} = -2 \int_{M_3} d^3x (\partial_\alpha * F^{0\alpha} + A_\alpha * \vec{F}^{0\alpha}) = - \int_{D_4/\partial D_4 = M_3} d^4x dA \wedge d\vec{A}.
\]

(2. 6)

(2. 5) yields the following Hamiltonian:

\[
\tilde{H} = \frac{1}{4(\bar{\tau} - \tau)} \Pi_\alpha \Pi^\alpha + \partial_\alpha \vec{F}^{0\alpha} + \frac{4}{\bar{\tau} - \tau} * \vec{F}^{0\alpha} * \vec{F}^{0\alpha},
\]

(2. 7)

in which \(\bar{\tau} = -1/\tau\). Since the original Hamiltonian is defined in the restricted phase space given by \(\Pi_0 = 0\), \(\partial_\alpha \Pi^\alpha = 0\) we need to analyse as well the mapping of the constraints and check that the dual Hamiltonian is defined in the same restricted phase space. The constraint \(\Pi_0 = 0\) is straightforwardly obtained from the generating functional, since there is no dependence on \(A_0\). The secondary constraint \(\partial_\alpha \Pi^\alpha = 0\) is obtained from the Bianchi identity \(\partial_\alpha * F^{0\alpha}\) of the original theory and finally the constraint \(\partial_\alpha \Pi^\alpha = 0\) implies that \(F\) is

\(^{18}\)We have dropped the global \(i/8\pi\) factor. It will then appear when exponentiating these quantities.

\(^{19}\)Note that in the definition of \(\Pi^\alpha\) there is also a contribution from \(* F^{0\alpha}\) when \(\theta \neq 0\) [41].
derived from a vector potential $\bar{A}$ as a consequence of Poincaré’s lemma. We can introduce a $\partial_\alpha A_0 \bar{\Pi}^\alpha$ term in the Hamiltonian imposing the constraint $\partial_\alpha \bar{\Pi}^\alpha = 0$ as in (2.4) and finally read a dual Lagrangian:

$$\bar{L} = \frac{i}{8\pi} \left( -\frac{1}{\tau} F^+_{mn} \bar{F}^{+mn} + \frac{1}{\tau} F^-_{mn} \bar{F}^{-mn} \right).$$

(2.8)

Some useful information can be obtained within this approach. The generating functional is linear in both the original and dual variables. We can then write:

$$\psi_k [\bar{A}] = N(k) \int \mathcal{D}A(x^\alpha) e^{\frac{i}{\hbar} \mathcal{F}[\bar{A}, A(x^\alpha)]} \phi_k [A(x^\alpha)]$$

(2.9)

with $\phi_k [A]$ and $\psi_k [\bar{A}]$ eigenfunctions of the initial and dual Hamiltonians respectively. From this relation global properties can be easily worked out. The Dirac quantization condition:

$$\int_{\Sigma} F = 2\pi n, \quad n \in \mathbb{Z},$$

(2.10)

for $\Sigma$ homologically non-trivial two-cycles in the manifold, implies the same quantization in the dual:

$$\int_{\Sigma} \bar{F} = 2\pi m, \quad m \in \mathbb{Z}.$$

(2.11)

We can also analyze the transformation of the partition function in phase space:

$$Z_{ps} = \int \mathcal{D}A_m \mathcal{D}\Pi^\alpha e^{-\frac{i}{\hbar} \int d^4x (\bar{A}_m \Pi^\alpha - H)}.$$  

(2.12)

Under (2.5) $\mathcal{D}A_0 \mathcal{D}\Pi^\alpha = \mathcal{D}\bar{A}_0 \mathcal{D}\bar{\Pi}^\alpha$ and therefore $\bar{Z}_{ps} = Z_{ps}$ (we have previously seen that both theories are defined in the same restricted phase space), which implies that in phase space the partition function is invariant under duality. However we know that it should transform as a modular form with a given modular weight $[42]$. This modular factor appears when going to configuration space, as we are going to show.

Integrating the momenta in (2.12) gives:

$$Z_{ps} = \int \mathcal{D}A_m (\text{Im}\tau)^{B_2/2} e^{-\int d^4x L}$$

(2.13)

with $L$ given by (2.2). The factor $(\text{Im}\tau)^{B_2/2}$ in the measure is the regularized $(\det \text{Im}\tau)^{1/2}$ coming from the gaussian integration over the momenta. $B_2$ is the dimension of the space of 2-forms in the four dimensional manifold $M_4$ (regularized on a lattice) and emerges because the momenta are 2-forms. The same calculation in the dual phase space partition function gives:

$$\bar{Z}_{ps} = \int \mathcal{D}\bar{A}_m (\text{Im}\tau)^{B_2/2} e^{-\int d^4x \bar{L}}$$

(2.14)

with $\bar{L}$ given by (2.8), where we have regularized

$$(\det (\text{Im} - \frac{1}{\tau}))^{1/2} = (\text{Im}\tau)^{B_2/2} e^{-B_2/2}$$

(2.15)

The integration over $\Pi_0$ is canceled by the gauge group volume and integration on $A_0$ yields the constraint $\partial_\alpha \Pi^\alpha = 0$, which in the dual theory implies that $\bar{F}$ is derived from a vector potential.
and \( B_2^\pm (B_2^-) \) is the dimension of the space of self-dual (anti-self-dual) 2-forms. In configuration space the partition function is defined by [42]:

\[
Z = (\text{Im} \tau)^{(B_1 - B_0)/2} \int \mathcal{D} \mathcal{A}_m e^{-S} = (\text{Im} \tau)^{(B_1 - B_0 - B_2)/2} Z_{ps} \tag{2.16}
\]

and in the dual model

\[
\tilde{Z} = \left( \frac{\text{Im} \tau}{\tau^2} \right)^{(B_1 - B_0)/2} \int \mathcal{D} \tilde{\mathcal{A}}_m e^{-\tilde{S}} = (\text{Im} \tau)^{(B_1 - B_0 - B_2)/2} \tau^{(\chi - \sigma)/4} \tilde{\tau}^{(\chi + \sigma)/4} \tilde{Z}_{ps}, \tag{2.17}
\]

where \( \chi = 2(B_0 - B_1) + B_2 \) is the Euler number (the regularization is such that \( B_p = B_{4-p} \)) and \( \sigma = B_2^+ - B_2^- \) is the signature of the manifold. From \( Z_{ps} = \tilde{Z}_{ps} \) we get

\[
Z = \tau^{-(\chi - \sigma)/4} \tilde{\tau}^{-(\chi + \sigma)/4} \tilde{Z}. \tag{2.18}
\]

Therefore in configuration space the partition function transforms as a modular form [42]. It is clear that the solution to this puzzle is that the regularization prescription for the determinants is such that \( Z \) is not obtained from \( Z_{ps} \) after integrating out the momenta, as it is clearly shown in (2.16) and (2.17). If we impose this requirement to the phase space partition function the corresponding factors in the measure need to be introduced and we also obtain that it transforms as (2.18).

The same analysis made for Maxwell’s theory can be straightforwardly generalized to \( p \)-forms abelian gauge theories in \( d \) dimensions [40]. In this case the generating functional and corresponding canonical transformation are:

\[
\mathcal{F} = -\int_{D_0/D_d = M_{d-1}} d^d x \, dA \wedge d \tilde{A}. \tag{2.19}
\]

\[
\Pi^{\alpha_1 \ldots \alpha_p} = \frac{\delta \mathcal{F}}{\delta A_{\alpha_1 \ldots \alpha_p}} = -(p + 1)!(d - p - 1)! \ast F^{\alpha_1 \ldots \alpha_p}
\]

\[
\Pi^{\alpha_1 \ldots \alpha_d-p-2} = -\frac{\delta F}{\delta A_{\alpha_1 \ldots \alpha_{d-p-2}}} = (p + 1)!(d - p - 1)! \ast F^{\alpha_1 \ldots \alpha_{d-p-2}}. \tag{2.20}
\]

The phase space partition function is invariant under these transformations. However upon integration on the momenta we get for the partition function in configuration space [38]:

\[
Z = (4\pi/g^2)^{(-1)^p \chi/2} \tilde{Z} \text{ in general, and } Z = \tau^{-(\chi - \sigma)/4} \tilde{\tau}^{-(\chi + \sigma)/4} \tilde{Z} \text{ for } d = 2(p + 1), \text{ } p \text{ odd, i.e. when a } \theta \text{-term is allowed in the theory.}
\]

### 2.2 Non-abelian gauge theories

Non-supersymmetric Yang-Mills theories are not invariant under the interchange of the electric and magnetic degrees of freedom. However following a first order formalism or a gauging-type procedure [43] it is possible to construct the dual of the theory, which turns out to be a Freedman-Townsend’s type of theory [44], depending on 2-forms that are not derivable from a vector potential. A generalized duality transformation relating it to the initial Yang-Mills theory has been given in [45] in terms of loop variables.
It is also possible to construct the dual theory by performing a simple canonical transformation in phase space, as we are now going to show. Written in configuration space variables this transformation provides the generalization to the non-abelian case of the electric-magnetic mapping of abelian gauge theories.

Starting with the Yang-Mills Lagrangian for a compact group $G$ on a Euclidean manifold:

\[
L = \frac{1}{8\pi} \frac{4\pi}{g^2} F_{mn}^{(a)} F^{(a)mn} + i \theta \frac{1}{4\pi} \varepsilon^{mn} F_{mn}^{(a)} F^{(a)}
\]

\[
= \frac{i}{8\pi} (\tau F_{mn}^{(a)} + F^{(a)+mn} - \tau F_{mn}^{(a)-mn})
\]

(2. 21)

where $F = dA - A \wedge A$ and we have chosen $Tr(T^a T^b) = \delta^{ab}$ ($T^a$ are the generators of the Lie algebra) we can construct the Hamiltonian:

\[
H = \frac{1}{4} \frac{1}{\tau - \tau} \Pi^a \Pi^{a0} + (\partial_\alpha A^a_0 + f_{abc} A^b_0 A^c_0) \Pi^{a0} - \frac{\tau + \tau}{\tau - \tau} \Pi^{a0} + \frac{4\tau}{\tau - \tau} F_{00}^{(a)} F^{(a)00},
\]

(2. 22)

where $f_{abc}$ are the structure constants of the Lie algebra and

\[
\Pi^{a0} = 2(\tau - \tau) F^{(a)00} + 2(\tau + \tau) F^{(a)00}
\]

\[
\Pi^{a0} = 0.
\]

(2. 23)

The interchange between electric and magnetic degrees of freedom:

\[
\Pi^{a0} = -4 * \tilde{F}^{(a)00}
\]

\[
\tilde{\Pi}^{a0} = 4 * F^{(a)00},
\]

(2. 24)

where $\tilde{F} = d\tilde{A} - \tilde{A} \wedge \tilde{A}$, yields a Hamiltonian in which the coupling has transformed as $\tau \rightarrow -1/\tau$. However in the non-abelian case (2. 24) is not a canonical transformation since the Poisson brackets are not left invariant. Moreover the secondary constraints:

\[
\partial_\alpha \Pi^{a0} - f_{abc} A^b_0 \Pi^{a0} = 0
\]

(2. 25)

of Yang-Mills, imply in the dual theory:

\[
\partial_\alpha * \tilde{F}^{(a)00} - f_{abc} A^b_0 (\tilde{F}) * \tilde{F}^{(c)00} = 0
\]

(2. 26)

from where in the absence of a non-abelian analogue of Poincaré’s lemma we cannot conclude that $\tilde{F}$ is derived from a vector potential.

In fact we know from previous calculations [43] that the dual theory is given by:

\[
\tilde{L} = \frac{i}{8\pi} (\frac{1}{\tau} \tilde{F}_{mn}^{(a)} + \tilde{F}^{(a)+mn} - \tau \tilde{F}_{mn}^{(a)-mn} + 2 R^{ab}_{mn} (\tilde{F}) \partial_q * \tilde{F}^{(a)qn} \partial_p * \tilde{F}^{(b)np}),
\]

(2. 27)

where $\tilde{F}$ are arbitrary two forms in the manifold and $R$ is the inverse of $\text{ad} * \tilde{F}$ and it is a well defined matrix for arbitrary $\tilde{F}$ in four dimensions.
It can be seen that (2.27) is generated from the original theory by the following canonical transformation:

\[
\Pi^{\alpha a} = -4 \bar{F}^{(a)}_{\alpha 0}
\]

\[
\Lambda_{\alpha a} = \frac{1}{4} \bar{\Pi}_{\alpha a}
\]

(2.28)

where:

\[
\bar{\Pi}_{\alpha a} \equiv \frac{\delta \tilde{L}}{\delta \bar{F}_{\alpha a}} = 4 R_{\alpha 0 n} \partial_m * \bar{F}_{nm}^{(b)}.
\]

(2.29)

The generating functional is given by:

\[
\mathcal{F} = -4 \int_{M_3} \text{Tr}(\bar{F}_{\alpha 0} A_{\alpha}),
\]

(2.30)

which has the same form than (2.6) but now \(\tilde{F}\) is not derived from a vector potential. Written in configuration space (2.28) amounts to:

\[
(\bar{\tau} - \tau) F_{\alpha 0}^{(a)} + (\bar{\tau} + \tau) F_{\alpha 0}^{(a)} = -2 \bar{F}_{\alpha 0}^{(a)}
\]

\[
A_{\alpha} = R_{\alpha 0 n} \partial_m * \bar{F}_{nm}^{(b)}.
\]

(2.31)

Compatibility of these equations is guaranteed on-shell. The equations of motion in the initial theory are mapped to the identities:

\[
\partial_n * \bar{F}_{mn}^{(a)} - f_{abc} * \bar{F}_{mn}^{(c)} R_{\alpha 0 q} \partial_p * \bar{F}_{qp}^{(d)} = 0
\]

(2.32)

in the dual, which are verified straightforwardly from the second equation in (2.31). The complete correspondence between equations of motion and Bianchi identities requires also \(A_{0}^{\alpha} = R_{0 n}^{ab} \partial_m * \bar{F}_{nm}^{(b)}\). The Bianchi identities in the dual theory are then satisfied with a “vector potential” \(V_{n}^{\alpha} \equiv R_{nm}^{ab} \partial_{p} * \bar{F}_{mp}^{(b)}\) defined from the fundamental 2-forms of the dual theory, and which transforms as a connexion under the local gauge symmetry of the dual Lagrangian\(^{21}\): \(\tilde{F} \rightarrow h \tilde{F} h^{-1}\). Inversely, the dual equations of motion are mapped to identities in the original theory.

The initial and dual theories are proved to be equivalent under (2.28) on-shell. It would be very interesting to extend the previous formulation to four dimensional supersymmetric non-abelian gauge theories, in particular to \(N = 4\) and those \(N = 2\) for which S-duality is known to be an exact symmetry \([46, 47]\). For these theories a path integral or a canonical transformation description is not known and it could be interesting to see if such a description exists already at the classical level. For \(N = 2\) Yang-Mills theories without matter the low energy effective action is invariant under S-duality \([48]\). In the low energy limit the whole non-abelian symmetry group is broken to its maximal abelian subgroup and then the remaining gauge symmetry is abelian. The canonical transformation description can then also be applied straightforwardly to this case.

\(^{21}\) \(\tilde{L}\) has this symmetry up to a total derivative.
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