INSTABILITIES OF AN INTENSE COASTING BEAM IN THE PRESENCE OF CONDUCTING PLATES
A THEORETICAL INVESTIGATION

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1. INTRODUCTION

Previous studies have shown that conducting plates and their electrical termination can have a considerable effect on the transverse and the longitudinal dynamics of a beam moving in a circular accelerator or storage ring.

Laslett\(^{(1)}\) studied the effect of a centrally loaded clearing electrode plate (c.e.) on the transverse coherent oscillation of a coasting beam in a pipe of rectangular cross section, assuming the charge and the current distribution on the c.e. to be governed by the transmission line equations. The results can be applied also to wavelengths smaller than the length of the c.e.

Sessler and Vaccaro\(^{(2)}\) investigated the longitudinal dynamics of the beam for the case of a pipe of circular cross section. They treated the clearing electrodes as lumped discontinuities of the electrical properties of the vacuum chamber walls. Because of this assumption their results are limited to long wavelengths of the perturbation.

The two different approaches of the above papers result from the fact that for wavelengths comparable or smaller than the length of the c.e. the perturbation sees the details of the clearing electrode structure, whereas for long wavelengths the c.e. are well approximated by a circuit of lumped elements.

Dangerous situations for the beam may occur in two particular cases:

i) for certain impedances of the termination, one can have a resonant behaviour of the beam dynamics, although the termination absorbs current. Only this case has been examined by the above authors.
ii) for certain wavelengths the termination does not absorb current, because it happens to be situated at a node of the standing wave representing the potential induced on the plate by the perturbation. In this case the plate appears as an unloaded transmission line. This case is obviously typical of short wavelengths.

In the present paper we will direct our investigations towards both situations, considering that for the phenomena of type ii) the damping of resonances is given by the power dissipation which occurs on the plate, whereas for the phenomena of type i) the damping is mainly given by the dissipation occurring in the terminating impedance. We will consider a circular geometry in the transverse plane and a periodic distribution of conducting plates, each having one termination, around the machine circumference; no restriction is imposed on the position of the termination. More complicated arrangements of plates can be analyzed using the superposition principle. We will investigate both the longitudinal and the transverse beam dynamics, with no limitations for the wavelength of the perturbation.

We use a circular cylindrical coordinate system $\rho$, $\varphi$, $z$. The $z$-axis coincides with the longitudinal axis of the beam and of the vacuum pipe. The beam has a circular cross-section of radius $a$. The charge density is uniform within this cross-section. The vacuum pipe has a circular cross-section of radius $b$. The plates are sectors of a cylinder having the same radius as the vacuum pipe. They cover the angular range from $-\varphi_0$ to $+\varphi_0$ and have length $l$ along the $z$-axis (Fig. 1).

The notation used in this paper follows as closely as possible Laslett's notation.\(^1\)
2. ELECTRODYNAMICS

2.1 Charges and currents induced on the plates

The perturbed charge per unit surface and linear current density induced at the surfaces by a beam of particles moving with velocity \( v = \beta_p c \) are

\[
\sigma = \sigma_1 e^{-j(kz-\omega t)} \cos m \varphi \quad (2.1a)
\]

\[
I = I_1 e^{-j(kz-\omega t)} \cos m \varphi \quad (2.1b)
\]

We define:

\[
\beta = \frac{I_1}{\sigma_1 c} \quad (2.1c)
\]

The quantity \( k = n/R \) defines the longitudinal mode number \( n \) of the perturbation; \( R \) is the radius of the machine.

The integer \( m \) is the multipole number of the perturbation; \( m = 0 \) for longitudinal (zeropole) motion, and \( m = 1 \) for transverse (dipole) motion. The perturbed charges \( \sigma \) are related to the sources of the perturbation, namely the perturbed linear charge density \( \lambda = \lambda_1 \exp[-j(kz-\omega t)] \) in the longitudinal case and the perturbed electric dipole moment per unit length \( \lambda = \lambda_1 \exp[-j(kz-\omega t)] \) in the transverse case, respectively, by the equations:

\[
\sigma_{1,\text{long}} = -\frac{\lambda_1}{2\pi b} \sigma_{1,\text{long}} \quad (2.2a)
\]

\[
\sigma_{1,\text{trans}} = -\frac{p_1}{\pi b^2} \sigma_{1,\text{trans}} \quad (2.2b)
\]

By a straightforward application of Maxwell's equations and of the charge conservation law (see Ref. 2 and Appendix I), assuming zero resistivity at the boundary, one obtains:

\[
\sigma_{1,\text{long}} = \frac{I_1(qa)}{G_0(qb)} \frac{2}{qa} \quad (2.3a)
\]

\[
\beta_{1,\text{long}} = \beta_w \quad (2.3b)
\]
\[ g_{\text{trans}} = \frac{b}{a} \frac{I_1(qa)}{I_1(qb)} \left[ 1 + 2\beta_w \frac{\beta_w - \beta_P}{1-\beta_w^2} \frac{I_2(qb)}{I_0(qb) + I_2(qb)} \right] \] (2.3c)

\[ \beta_{\text{trans}} = \left[ \beta_P + 2 \frac{\beta_w - \beta_P}{1-\beta_w^2} \frac{I_2(qb)}{I_0(qb) + I_2(qb)} \right]^{-1} \left[ 1 + 2\beta_w \frac{\beta_w - \beta_P}{1-\beta_w^2} \frac{I_2(qb)}{I_0(qb) + I_2(qb)} \right] \] (2.3d)

where the I's are modified Bessel functions and \( q = k \sqrt{1 - \beta_w^2} \.
In the long wavelength limit (\( q_b \ll 1 \)) the \( g \)-factors are unity and
\( \beta_{\text{trans}} = \beta_P \).

2.2 Solution of the transmission line equations for perfectly conducting plates

The Haslett's transmission line equations (1) for the supplemental scalar potential and longitudinal vector potential produced by the plate, in the case of circular geometry of the pipe are:

\[ \frac{1}{c} \frac{\partial V_1}{\partial t} + \frac{\partial A_1}{\partial z} = j2Z_0 k c \sigma (\beta - \beta_w) \varphi_o b \] (2.4a)

\[ \frac{\partial V_1}{\partial z} + \frac{1}{c} \frac{\partial A_1}{\partial t} = 0 \] (2.4b)

The characteristic impedance of the line is \( Z_0 = \sqrt{L/C} \); \( C \) and \( L = 1/c^2C \) are the capacitance and inductance per unit length of the plate, respectively. Equation (2.4a) expresses the charge conservation law. The r.h.s. term in Eq. (2.4b) is zero because of the assumption that the material is perfectly conductive.

At the two open ends of the line (\( z = z_s \), and \( z = z_s + l \) for the \( s \)-th electrode)
\[ A_1(z_s, t) = -2Z_0 I(z_s, t) \varphi_o b \]  
(2.5a)

\[ A_1(z_s + \ell, t) = -2Z_0 I(z_s + \ell, t) \varphi_o b \]  
(2.5b)

At the feed point (\( z = z_{os} = z_s + d \))*, the termination impedance \( Z_T \) imposes the condition:

\[ A_1(z_{os}^+, t) - A_1(z_{os}^-, t) = -\frac{Z_0}{Z_T} v_l \]  
(2.6a)

\[ v_l(z_{os}^+, t) - v_l(z_{os}^-, t) = 0 \]  
(2.6b)

where \( z_{os}^+ \) and \( z_{os}^- \) mean \( z \to z_{os} \) from the right and from the left side.

A solution to Eqs. (2.4a and b) can be obtained in the form

\[ v_l = \left[ a^e^{-j \frac{w}{c} (z-z_{os})} + b^e^{-j \frac{w}{c} (z-z_{os})} \right] e^{jwt} + \frac{2\beta w (\beta w - \beta) \varphi_o b}{1-\beta_w^2}, \]  
(2.7a)

and

\[ A_1 = \left[ a^e^{-j \frac{w}{c} (z-z_{os})} - b^e^{-j \frac{w}{c} (z-z_{os})} \right] e^{jwt} + \frac{2(\beta w - \beta) \varphi_o b}{1-\beta_w^2}, \]  
(2.7b)

where the coefficients \( a^+, b^+ \) apply for \( z > z_{os} \) and the coefficients \( a^-, b^- \) for \( z < z_{os} \). Introducing Eqs. (2.7a,b) into Eqs. (2.5a,b) and (2.6a,b) one obtains a system of four linear equations for the

* Although \( d \) does not carry the subscript \( s \), one should remark that the expressions worked out in this section can also be applied to the case where the clearing electrodes have different \( d \)’s.
four unknowns $a^\pm, b^\pm$. Introducing the parameter $r = z_\infty/z_\infty = cCz_\infty$ one finds:

\[
a^- = -\frac{2\sigma_{1\beta_w}}{D} \left\{ e^{-j kz s (1 - \beta_w \beta)} \left[ \text{re} \frac{j w}{c} (l - d) + \cos \frac{w}{c} (l - d) \right]
\right.
\]
\[\vspace{0.5cm}\]
\[-jk (z_\infty + l) (1 - \beta_w \beta) e^{j \frac{w}{c} d}
\right.
\[\vspace{0.5cm}\]
\[\left. + e^{-j k z_\infty (\beta_w - \beta)} e^{j \frac{w}{c} d} \cos \frac{w}{c} (l - d) \right\} \frac{\varphi_0 b \cos m \varphi}{1 - \beta_w^2}, \tag{2.8a}
\]

\[
b^- = -\frac{2\sigma_{1\beta_w}}{c D} \left\{ e^{-j kz s (1 - \beta_w \beta)} \left[ \text{re} -j \frac{w}{c} (l - d) - \cos \frac{w}{c} (l - d) \right]
\right.
\]
\[\vspace{0.5cm}\]
\[-jk (z_\infty + l) e^{j \frac{w}{c} d}
\right.
\[\vspace{0.5cm}\]
\[\left. + e^{-j k z_\infty (\beta_w - \beta)} e^{j \frac{w}{c} d} \cos \frac{w}{c} (l - d) \right\} \frac{\varphi_0 b \cos m \varphi}{1 - \beta_w^2}, \tag{2.8b}
\]

and

\[
a^+ = -\frac{2\sigma_{1\beta_w}}{c D} \left\{ \text{re}^{-j k z s (1 - \beta_w \beta)} e^{j \frac{w}{c} (l - d)}
\right.
\]
\[\vspace{0.5cm}\]
\[-j k z_\infty (\beta_w - \beta) e^{j \frac{w}{c} d}
\right.
\[\vspace{0.5cm}\]
\[\left. + e^{-j k z_\infty (\beta_w - \beta)} e^{j \frac{w}{c} (l - d)} \cos \frac{w}{c} d \right\} \frac{\varphi_0 b \cos m \varphi}{1 - \beta_w^2}, \tag{2.9a}
\]
\[ b^+ = - \frac{2a}{1 - \beta^2} \left\{ \begin{array}{c} \frac{j k z}{c} \left( 1 - \beta^2 \right) e^{-j \frac{\omega}{c} (z - d)} \\ + e^{j k z \cos \left( \beta_w - \beta \right)} \frac{-j \frac{\omega}{c} (z - d)}{\cos \frac{\omega}{c} d} \end{array} \right. \]

\[ - \frac{j k (z + l)}{1 - \beta^2 \beta_w^2} \left[ \frac{j \frac{\omega}{c} d}{\cos \frac{\omega}{c} d} \right] \frac{\varphi_0 b \cos m \phi}{1 - \beta_w^2} \quad (2.9b) \]

where \( D \) is the determinant of the system:

\[ D = \cos \frac{\omega}{c} (z - 2d) + \cos \frac{\omega}{c} \lambda + 2j \sin \frac{\omega}{c} \lambda . \quad (2.10) \]

### 2.3 Potential distribution in the pipe

We take the surface of the vacuum chamber outside the plates to be the surface of zero scalar and vector potentials. At the wall surface, \( \rho = b \), we take the following expression of the potentials \( V \) and \( A \) around the accelerator circumference:

\[ V(b, \phi, z, t) = \Omega(z) \Phi(\phi) V_1(z, t) \cos m \phi \quad (2.11a) \]

\[ A(b, \phi, z, t) = \Omega(z) \Phi(\phi) A_1(z, t) \cos m \phi \quad (2.11b) \]

The functions \( \Omega(z) \) and \( \Phi(\phi) \) are zero everywhere except on the plates, where they are 1. The expressions for \( V_1 \) and \( A_1 \) on the right hand side of Eqs. (2.11a, b) are given by Eqs. (2.7a and b).

The potential \( V \) at any point inside the vacuum chamber satisfies the wave equation

\[ \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0 \quad (2.12) \]

and similarly for \( A \). Equation (2.12) is satisfied by the following expression:
\[ V(\rho, \phi, z, t) = \sum_{p,h} \phi_p V_h \frac{I_p(q_0)}{I_p(q_h)} \cos p \phi e^{-j\left(\frac{h}{R} z - wt\right)} , \quad (2.13a) \]

and

\[ A(\rho, \phi, z, t) = \sum_{p,h} \phi_p A_h \frac{I_p(q_0)}{I_p(q_h)} \cos p \phi e^{-j\left(\frac{h}{R} z - wt\right)} . \quad (2.13b) \]

The coefficients \( \phi_p, V_h \) and \( A_h \) are obtained by a double Fourier expansion of Eqs. (2.11a,b). As all modes \( p \neq m \) and \( h \neq n \) are orthogonal to the fundamental mode of the perturbation, the forces produced by these modes are ineffective over a complete turn in the machine. Therefore we retain the terms with \( h = n \) and \( p = m \), i.e. the field components having the same wave numbers as the perturbation.

We consider \( M \) plates around the circumference of the accelerator, all of them having the same \( d \). One obtains

\[
V_n = \frac{M \beta}{2\pi R} \frac{\sigma_1}{(1 - \beta_w^2)C} \left[ (\beta_v - \beta) - \frac{(\beta_w c_{1} + c_2)}{j\delta D(1 - \beta_w^2)} \right] \quad (2.14a)
\]

\[
A_n = \frac{M \beta}{2\pi R} \frac{\sigma_1}{(1 - \beta_w^2)C} \left[ (\beta_v - \beta) - \frac{\beta_w (c_{1} + c_2 \beta)}{j\delta D(1 - \beta_w^2)} \right] \quad (2.14b)
\]

where

\[
c_1 = j(\delta - \beta_w) \sin 2\phi - (1 - \beta_w) \left[ e^{-j\delta(1-\delta)} \cos 2\phi(1+\delta) - e^{j\delta(1+\delta)} \cos 2\phi(1-\delta) \right]. \quad (2.15a)
\]
\[ c_2 = (\phi_w - \phi) \left[ e^{j\phi(1-\delta)} \cos\phi(1+\delta) - e^{-j\phi(1+\delta)} \cos\phi(1-\delta) \right] + (1-\phi_w) \left[ 2r (\cos 2\theta - \cos 2\phi) - j \sin 2\phi \right]. \] (2.15b)

In the above equations we have introduced the dimensionless parameter
\[ \theta = \frac{\lambda}{2} \quad \phi = \frac{\omega l}{2c} = \phi_w \theta \] (2.16)
\[ \delta = \frac{(2d-1)/\lambda}{(-1 \leq \delta \leq 1)}. \]

In the case where the plates extend from \( -\phi_0 \) to \( +\phi_0 \) (Fig. 1) one has:
\[ \phi_0 = \frac{2\phi^2 b}{\pi} \quad \phi_1 = \frac{2\phi b}{\pi} \left( \phi_0 + \sin \phi_0 \cos \phi_0 \right) \] (2.17)

The expressions for \( c_1 \) and \( c_2 \) show that the \( n^{th} \) Fourier components of the vector and scalar potentials are independent on the position of the plates. This fact justifies the factor \( M \) in Eqs. (2.14a,b), that is the additivity of the responses of the electrodes. This is a consequence of the fact that the beam is at the same time the subject and the object of the perturbation and will therefore load all the plates in the same way.

### 2.4 The forces acting on the beam

The forces per unit charge acting upon the beam at \( \rho = 0 \) are:
\[ f_{\text{long}} = j k \left[ \frac{V_n - \phi A_n}{I_0(qb)} \right] \phi_0 e^{-j(kz - \omega t)} \] (2.18a)
\[ f_{\text{trans}} = \frac{a/2}{I_1(qb)} \left[ V_n - \phi A_n \right] \phi_1 e^{-j(kz - \omega t)}. \] (2.18b)
Inserting Eqs. (2.14a,b) and taking into account Eqs. (2.3a,b,c,d), one obtains for the forces expressions of the type:

\[ F_{\text{long}} = + j\beta \frac{M_o}{2\pi R} (k_{\text{long}})^2 g^*_{\text{long}} F_{\text{long}} \frac{\lambda}{\kappa c} \]  

(2.19a)

\[ F_{\text{trans}} = - \beta \frac{M_o}{2\pi R} (k_{\text{trans}})^2 g^*_{\text{trans}} F_{\text{trans}} \frac{b}{b^2 c} . \]  

(2.19b)

The coefficient \( \frac{M_o}{2\pi R} \) gives the fraction of the accelerator circumference occupied by the plates. If there are two plates, symmetrical to the beam, a factor 2 must be added.

The coupling factors \( k_{\text{long}} \) and \( k_{\text{trans}} \) are related to the transverse size of the electrodes and depend on the geometry of the pipe. One has:

\[ (k_{\text{long}})^2 = \frac{2}{\pi^2} \]  

(2.20a)

and, in the case where the plate extends from \(-\varphi_o\) to \(+\varphi_o\), i.e. the beam oscillates normally* to the plate:

\[ (k_{\text{trans}})^2 = \frac{(\varphi + \sin \varphi_o \cos \varphi_o) \varphi_o}{(2\pi)^2} \]  

(2.20b)

* If the beam oscillates parallel to the plate, i.e. if the plate ranges from \(\pi/2 - \varphi_o\) to \(\pi/2 + \varphi_o\), one has

\[ (k_{\text{trans}})^2 = \frac{(\varphi_o - \sin \varphi_o \cos \varphi_o) \varphi_o}{(2\pi)^2} \]

We point out that this factor \( k_{\text{trans}} \neq 0 \) indicates the possibility of transverse beam instabilities parallel to the plate.
The quantities $g^{\text{long}}$ and $g^{\text{trans}}$ depend only on $a$, $b$ and $q$:

\[ g^{\text{long}} = \frac{g}{I_0(qb)} \quad (2.21c) \]

\[ g^{\text{trans}} = \frac{a b g}{2 I_1(qb)} \quad (2.21d) \]

where the $g$-factors are given by Eqs. (2.3c,d). An analysis of the $g^{\text{long}}$-factors and of $g^{\text{trans}}$ is given in Appendix II. In the limit \( q b < 1 \) the $g^{\text{long}}$-factors equal unity. For all frequencies \( g^{\text{trans}} \approx g^{\text{p}} \).

The coefficients \( f^{\text{long}} \) and \( f^{\text{trans}} \) are related to the electrical properties of the plates and to their termination. They are expressed by:

\[ f^{\text{long}} = - \frac{j c_2 g^{\text{p}}}{2 D (1 - \beta^2_w)}, \quad (\text{with } \beta = \beta_w) \]

\[ f^{\text{trans}} = \frac{1}{1 - \beta^2_w} \left\{ - \frac{j \beta}{(1 - \beta^2_w) D} \left[ (\beta_w - \beta^2_p) c_1 + (1 - \beta_p \beta_w) c_2 \right] - (\beta_w - \beta_p)^2 \right\}. \]

(\text{with } \beta = \beta^2_p)

Introducing the expressions $c_1$ and $c_2$, one has:

\[ f^{\text{long}} = \frac{\beta_w}{2} \frac{2 j r (\cos 2\phi - \cos 2\phi) - \sin 2\phi}{\cos 2\phi + \cos 2\phi + 2 j r \sin 2\phi} \quad (2.22a) \]
and:

\[
p\text{trans} = \frac{1}{1-\beta_w^2} \left\{ \begin{array}{c}
\beta_w \frac{-\sin2\psi \left[ (\beta_w - \beta_p)^2 + (1-\beta_w) \beta_p^2 \right] + 2jr(1-\beta_w) \beta_p^2 (\cos2\psi - \cos2\theta)}{(1-\beta_w^2)(\cos2\psi + \cos2\theta + 2jr \sin2\theta)^3} \\
+ \frac{2(\beta_w - \beta_p)(1-\beta_w) \beta_p}{(1-\beta_w^2)(\cos2\psi + \cos2\theta + 2jr \sin2\theta)^3} \left[ \cos(\psi(1+\delta)) \sin(\theta(1-\delta)) + \cos(\psi(1-\delta)) \sin(\theta(1+\delta)) \right] - (\beta_p - \beta_w)^2 \end{array} \right\}. \quad (2.22b)
\]

One can observe the symmetry of \(P\text{long}\) and \(P\text{trans}\) with respect to the distance of the termination from the centre of the electrode. In the case of one centrally located termination, i.e. \(\delta = 0\), one has for \(P\text{trans}\) the expression given by Laslett (Ref 1), Eq. (11b). In the limit of the long wavelengths with respect to the length of the electrode \(0 \ll 1, |\psi| \ll 1\), in linear approximation, from Eqs. (2.22a,b) we obtain

\[
P\text{long} = \frac{\theta}{2} \cdot \frac{2jr\psi(1 - \beta_w^2) - \beta_w^2}{1 + 2jr \psi} \quad (2.23a)
\]

\[
P\text{trans} = \frac{2jr\psi(1 - \beta_p^2) - \beta_p^2}{1 + 2jr \psi} \quad (2.23b)
\]

In the long wavelength limit the forces do not depend on the position of the termination. Equation (2.23a) is in agreement with Eq. (V.7a) of Ref. 2.
2.5 Dynamics of the beam

The stability of an intense coasting beam has been treated (Refs. 3 and 4) in terms of parameters $U$ and $V$ which are related to the real and imaginary parts of the forces acting on the beam. Because of the convention used in the above papers, in order to compare the terms $U_\perp$ and $V_\perp$ due to the forces $F_{\text{long}}$ and $F_{\text{trans}}$, given by Eqs. (2.22a,b), to $U$ and $V$ given by Eqs. (3.9a,b) of Ref. 3 and by Eqs. (3.17a,b) of Ref. 4, one must change the sign in front of the $j$ in the expression for $F_{\text{long}}$ and $F_{\text{trans}}$. This is equivalent to introducing the imaginary quantity $i = -j = \sqrt{-1}$.

With the same procedure of Refs. 3 and 4 we have from Eqs. (2.20a,b):

\[
(U_\perp - iV_\perp)_{\text{long}} = i Ne^2 \frac{F_{\text{long}}}{\lambda} = 8 Ne^2 \left(\frac{Me}{2\pi R}\right) \left(K_{\text{long}}\right)^2 \frac{F_{\text{long}}}{\lambda c} g_{\text{long}},
\]  

(2.24a)

and

\[
(U_\perp + iV_\perp)_{\text{trans}} = \frac{Ne^2}{Q \omega_o m_o \gamma} \frac{F_{\text{trans}}}{4\pi R} = -\frac{2Ne^2}{Q \omega_o m_o \gamma} \left(\frac{Me}{2\pi R}\right) \left(K_{\text{trans}}\right)^2 \frac{F_{\text{trans}}}{b^2 c} g_{\text{trans}},
\]  

(2.24b)

where $Q$ is the number of betatron oscillations per revolution, $\omega_o$ is the revolution angular frequency and $m_o \gamma$ is the particle relativistic mass.
3. **TRANSMISSION LINE RESONANT MODES**

As pointed out in Ref. 2, the response of the plate (i.e. the P-factors) has the feature of a current parallel circuit RLC, namely:

\[
\frac{-\frac{R}{\omega C} + \frac{L}{C}}{R + j(\omega L - \frac{1}{\omega C})} \quad (3.1)
\]

The resonant condition for the P-factors, as for the device represented by the above response, occurs when the reactance in the denominator is zero. We write:

\[ r = r' + j x' \quad (3.2) \]

Thus the resonant conditions for the P-factor are given by the equation:

\[ \cos 2\psi + \cos 2\psi - 2x' \sin 2\psi = 0 \quad (3.3) \]

Equation (3.3) includes two different types of resonances:

i) **Loaded resonances** (Section 3.1), which are characterized by the condition:

\[ \sin 2\psi \neq 0 \quad (3.4) \]

The figure of quality of these resonances is predominantly determined by the dissipation on the termination, unless \( r' \) is small enough to be comparable to the adimensional parameter \( \alpha \), which represents the dissipation on the plate itself and is defined in Section 4.
ii) **Unloaded resonances** (Section 3.2) which we characterized by the condition:

$$\sin 2\phi = 0$$ \hspace{1cm} (3.5)

For these resonances the electrode appears as floating. Hence the figure of quality is determined by the dissipation on the plate.

In the two limiting cases \( r' = 0 \) (purely reactive load) and \( r' = \infty \) (floating electrode) the resonant conditions are expressed by equations (3.3) and (3.5), respectively; however, for both cases the figure of quality at resonance is determined by the resistivity of the plate.

### 3.1 Loaded transmission line resonant modes

Let us first consider the case \( r' = 0 \). In this case the P-factors are real, that is the response is purely reactive. The condition expressed by Eq. (3.3) can be written in the form:

$$x' = \left[ \frac{1}{\tan\delta(1 + \delta) + \tan\delta(1 - \delta)} \right] = 0$$ \hspace{1cm} (3.6a)

Physically, Eq. (3.6) expresses the condition that the reactances of the termination, of the right hand side and of the left hand side of the plate cancel each other. In Figs.2 we have plotted the bracketed term \( X \) in Eq. (3.6a) as a function of \( 2\delta/\pi \) and for various values of \( \delta \). The intersections with the curve representing \( x' \) give the resonant frequencies: the condition (3.6a) is fulfilled and the P-factors go to infinity, just like a lossless resonant circuit. The presence of a resistance \( r' \neq 0 \) will damp these resonances according to the figure of quality of the circuit. In the particular case of purely resistive load \( (x' = 0) \) the solution of equation (3.6a) is:
(3.6b)

\[ \phi = \frac{\pi}{2} \left( \frac{2k - 1}{1 + \delta} \right) \]

(k positive integer).

Inserting Eq. (3.6a) into Eq. (3.33a), one obtains for \( p_{\text{long}} \) at resonance

\[ p_{\text{long}} = j \frac{\beta_w}{4\tau'} + \beta_w (1 + j \frac{x'}{\tau'}) \frac{\cos 2\phi - \cos 2\theta}{2 \sin 2\phi} \]  

(3.7)

In the limit:

\[ |1 - \beta_w^{-1}|\phi \ll 1 \]  

(3.8)

one has:

\[ p_{\text{long}} = j \frac{\beta_w}{4\tau'} + (1 + j \frac{x'}{\tau'}) \phi (1 - \beta_w) \left[ 1 + (1 - \beta_w) \cotg 2\phi \right] \]  

(3.9)

For sufficiently large values of \( \gamma \) the values of the maxima are independent of \( \phi \), namely:

\[ p_{\text{long}} = j \frac{1}{4\tau'} \]  

(3.10)

For the transverse case one obtains:
\[ p_{\text{trans}} = -\frac{(\beta_p - \beta_w)^2}{1 - \beta_w^2} \]

\[ + j \frac{\beta_w^2}{r' \sqrt{1 + \beta_w^2}} \left\{ \frac{(1 + \beta_p)^2}{2} + \frac{(\beta_w - \beta_p)(1 - \beta_w \beta_p)}{(1 - \beta_w)^2} \right\} \left[ 1 - \cos(\theta - \phi) \cos(\theta - \phi) - 4x' \sin(\theta - \phi) \sin(\theta - \phi) \right] \]

\[ \cos(\theta - \phi) - \frac{\sin 2\delta}{\sin 2\phi} \sin(\theta - \phi) \sin(\theta - \phi) \]} + \left[ \frac{\beta_w(1 - \beta_w \beta_p)}{1 - \beta_w^2} \right]^2 (1 + j \frac{x'}{r'}) \]

\[ \cos 2\phi = \cos 2\theta \frac{\phi}{\sin 2\phi} \]

(3.11c)

In the limit expressed by Eq. (3.8), one has:

\[ p_{\text{trans}} = -\frac{(\beta_p - \beta_w)^2}{1 - \beta_w^2} + \frac{j \beta_w^2}{(1 + \beta_w)^2} \left\{ \frac{(1 + \beta_p)^2}{2r' \phi} - \frac{(\beta_w - \beta_p)(1 - \beta_w \beta_p)}{(1 - \beta_w) \beta_w} \right\} \left[ 1 + \frac{x'}{r'} \right] 

\[ + \frac{5}{r'} \frac{\sin 2\delta}{\sin 2\phi} \left( 1 + \beta_w \beta_p \right) \left[ 2 \frac{\beta_w(1 - \beta_w \beta_p)^2}{(1 + \beta_w)^2 (1 - \beta_w)} \right] (1 + j \frac{x'}{r'}) \left[ 1 + (1 - \beta_w) \theta \cot 2\phi \right] \] (3.11b)

For \( \gamma^{-2} \gg Q/n \) one has:

\[ p_{\text{trans}} = j \frac{1}{2r' \phi} \]

(3.12)
3.2 Unloaded transmission line resonant modes

With reference to Eqs. (3.3) and (3.5), we investigate the existence of combined values of $\delta$ and $\phi$ which may cause unloaded resonances of the electrodes. To illustrate the situation, let us first consider the typical case which gives unloaded resonances, namely a floating plate. In this case Eqs. (2.22a,b) give

$$p_{\text{long}} = \beta_{w} \frac{\cos 2\phi - \cos 2\theta}{2 \sin 2\phi}$$

$$p_{\text{trans}} = \beta_{w} \frac{(1-\beta_{w}^2 p)^2}{(1-\beta_{w}^2)^2 \cos 2\phi - \cos 2\theta - (\beta_{w}^2 p)^2} \frac{\cos 2\phi - \cos 2\theta}{\theta \sin 2\phi - \frac{\beta_{w}^2 p}{1-\beta_{w}^2}}$$

The resonances occur when and only when Eq. (3.5) is satisfied.

In the general case similar resonances occur when the denominator $D$, common to both $p_{\text{long}}$ and $p_{\text{trans}}$ is zero, i.e.

$$D = (\cos 2\phi + \cos 2\theta - 2x' \sin 2\phi) + 2jr' \sin 2\phi = 0 \quad \text{(3.13)}$$

The condition that the imaginary part is zero is satisfied by the resonant modes of the floating plate, given by Eq. (3.5); the additional condition for the real part, i.e. Eq. (3.3), impose the relation between the frequency of the perturbation and the position of the termination. This means that the loaded electrode will resonate unloaded only for some of the modes of the floating plate. The solutions of Eq. (3.5) are:

$$\phi = \frac{\pi}{2} \nu \quad (\nu \text{ positive integer}) \quad \text{(3.14a)}$$

which combined with Eq. (3.3) give:
\[ |\delta| = \frac{2p-v-1}{v} \quad (p < v \text{ and positive integer}) \quad (3.14b) \]

The conditions (3.14a) and (3.14b) can also be found by simple physical arguments. An unloaded transmission line has a resonant mode when its length is \( v \) times half the wavelength \( \lambda \), i.e.:

\[ \lambda = \frac{\lambda}{2} ; \quad v = 1, 2, 3 \ldots \]

In this situation the potential on the electrodes has the form of a standing wave with maxima at the ends. The termination impedance does not absorb power if at its position there is a node in the standing wave, that is:

\[ d = (2p - 1) \frac{\lambda}{4} \quad p = 1, 2, 3 \ldots \]

In this case the transmission line structure formed by the electrode with its termination appears as unloaded*. Introducing the parameter \( \delta \) (Eq. 2.16) and taking into account that \( |\delta| \leq 1 \), one sees that the two above conditions coincide with Eqs. (3.14a) and (3.14b).

The smallest frequency which satisfies Eqs. (3.14a,b) is an increasing function of \( |\delta| \). Hence it would seem advantageous to place the termination close to one of the ends of the plate, so that damping is introduced because of the finite size of the termination.

* This strictly applies only for wavelengths much larger than the size of the connection of the terminations on the plate; otherwise the unloaded transmission line situation cannot be reached, since current will flow into the termination. This effect gives an additional damping of the resonances.
4. EFFECT OF FINITE CONDUCTIVITY

We have seen that a perfectly conductive electrode plate might have a catastrophic effect on the stability of the beam. For some frequencies the forces acting on the beam would be infinite and there would be no possibilities to compensate them. However, these transmission line resonant modes are damped according to the figure of quality of the device involved: for this reason we must take into account the finite resistivity of the electrode plate.

As the conductivity of the material does not affect the charge conservation law, Eq. (2.4a) remains unchanged. Equation (2.4b) for a dissipative transmission line becomes:

\[
\frac{\partial V_1}{\partial z} + \left( L \frac{\partial}{\partial t} + Z \right) \frac{A_1}{Z_0} = 0
\]  

(4.1)

where \( Z \) is the longitudinal impedance per unit length introduced by the resistivity of the plate. Taking into account the skin effect, one has:

\[
Z = \frac{2\pi\sigma}{\sigma_{bc}}
\]

(4.2)

where

\[
\zeta = (1 + j)\sigma_c = (1 + j)\sqrt{\frac{\omega_{bc}}{\omega_{ns}}}
\]

(4.3)

is the surface impedance and \( \sigma \) the conductivity of the plate.
From Eqs. (4.1), (4.2) and (4.3) one obtains

\[ \frac{\partial V}{\partial z} + \frac{1}{c} \frac{\partial A}{\partial t} = - \frac{\zeta}{\tau} A \]  

(4.4)

where \( \tau \) is the equivalent electrical distance of the plate from ground, as calculated from equation

\[ C = \frac{2\varepsilon_0 b}{4\pi \tau} \]

The solution of the system of differential equations (2.4a) and (4.4) is of the form:

\[ V = e^{\left[-j \frac{\omega c}{ae} (z-z_0) + j \frac{\omega c}{be} (z-z_0)\right]} e^{j\omega t} + \frac{\sigma c}{2} \frac{2\varepsilon_0 \beta \psi_0 b}{\left[1 - \beta^2 \varepsilon_0^2\right]} (4.5a) \]

\[ A = e^{\left[-j \frac{\omega c}{ae} (z-z_0) - j \frac{\omega c}{be} (z-z_0)\right]} e^{j\omega t} + \frac{\sigma c}{2} \frac{2\varepsilon_0 \beta \psi_0 b}{\left[1 - \beta^2 \varepsilon_0^2\right]} (4.5b) \]

where \( \epsilon = \sqrt{1 + \alpha} \)

---

* Eq. (4.4) can also be obtained relating at the plate surface the electric field \( E_z \), represented by the l.h.s. term of Eq. (4.4), to the magnetic field by the boundary condition

\[ E_z = - \zeta \frac{\partial A}{\partial \rho} \bigg|_{\rho = b} \]

and assuming

\[ A = A_1 \frac{\rho}{\tau} \]
\[ \alpha = \frac{L_0}{j\omega} = (1 - j)\sqrt{\frac{2\pi}{\omega s}} \frac{1}{2\pi \omega \beta_0} \] (4.5c)

Different values of the coefficient \(a, b\) still apply to the two sides of the feed-point; they are obtained by again using Eqs. (2.5) and (2.6).

Proceeding in the same way as in Section 2 one obtains for the forces the same expressions as given by Eqs. (2.19a,b); however, the P-factors for the finite conductivity case are expressed by

\[ p_{\text{long}} = -j\beta_w \frac{\beta_w \alpha_1 + \epsilon(1-\beta_w^2)c_2}{2(1-\beta_w^2)\epsilon^2 D} \] (4.6a)

\[ p_{\text{trans}} = \frac{1}{1-\beta_w^2} \left\{ \frac{-j\beta_w}{(1-\beta_w^2)\epsilon^2 D} \left[ (\beta_w \epsilon^2 - \beta_p) c_1 + \epsilon(1-\beta_w \beta_p) c_2 \right] \right. 

\left. \quad - (\beta_w - \beta_p)(\beta_w \epsilon^2 - \beta_p) \right\} \] (4.6b)

The new expressions of \(c_1, c_2\) and \(D\) can be obtained by replacing in Eqs. (2.15a,b)

\[ \omega \quad \text{by} \quad \omega_s \]

\[ \beta_w \quad \text{"} \quad \beta_w \epsilon \]

\[ \beta \quad \text{"} \quad \beta \epsilon \]

\[ \alpha \quad \text{"} \quad \alpha \epsilon \]

\[ Z_T \quad \text{"} \quad Z_T / \epsilon^2 \]

and \( r \quad \text{"} \quad r / \epsilon \)

In fact the above substitution transform Eqs. (2.7a,b) into Eqs. (4.5a,b).
The response of the electrodes as a function of the frequency is essentially determined by the figure of quality of the device involved. The unloaded resonant frequencies are characterized (see Appendix III) by a very high figure of quality

$$Q = \frac{1}{\text{Re}(\alpha)} = \frac{\omega T}{\Omega c} \quad (4.7)$$

so that the effect of the surface resistivity is limited to a very narrow band. The response in this band will be studied in the following section. Outside of this band the response is practically insensitive of the surface sensitivity, such that the results of Sections 2 and 3 remain valid. In Appendix IV we will compute the reduction of the $Q$ value given by a spread in the resonant frequency of the $M$ resonators, just like a Landau damping, (Ref. 5).

4.1 Damping of unloaded transmission line resonant modes

We calculate the expressions of $P$-factors at the limit of the bandwidth, where $\text{Re}(F)$ is maximum, namely at the frequency where the condition (3.14a) holds (see Appendix III). At these frequencies in the limit of $\alpha \ll 1$, linearizing in $\alpha$ we get

$$D = (-1)^n \frac{\pi}{\epsilon} \pi \nu \alpha \quad (4.8a)$$

$$\frac{c_1}{D} = \frac{\epsilon}{2r} \frac{(1 - \epsilon \beta^2_w)(1 + \epsilon \beta)}{2r} \quad (4.8b)$$

$$\frac{c_2}{D} = \frac{\pi}{\alpha \epsilon} (1 - \beta_w \epsilon^2) \frac{(1 - \epsilon \beta^2_w)(\epsilon - \beta^2_w)}{\beta^2_w} - \frac{c_1}{D} \quad (4.8c)$$

In the limit expressed by Eq. (3.8), one obtains from the above equations:
$$p_{\text{long}} = \frac{i}{4} \left[ \frac{1}{r} - j \phi \alpha^{-1} \gamma^{-2} (\alpha + \gamma^{-2}) \right]$$  \hspace{1cm} (4.9a)

$$p_{\text{trans}} = \frac{i}{2 \theta} \left[ \frac{1}{r} - j \phi \alpha^{-1} (\gamma^{-2} + \frac{q}{n})(\gamma^{-2} + \frac{2q}{n} + \alpha) \frac{\gamma^{-2} + \frac{q}{n} - \alpha}{\gamma^{-2} + \frac{2q}{n} - \alpha} \right]$$  \hspace{1cm} (4.9b)

The above expressions clearly show the terms which go to infinity when the plate resistivity tends to zero.

4.2 Purely reactive load

An interesting particular case occurs when \( r' = 0 \), i.e. for example when the plate is directly connected to ground. By simple physical arguments we can realize that the essential features of the response are essentially independent of the position of the termination. For simplicity we take \( \delta = \pm 1 \). The resonant condition is:

$$x' = \cot \theta \phi$$  \hspace{1cm} (4.10)

The resonant frequency can then be deduced from Fig. 2e. Proceeding as in Section 4.1, one obtains for the \( P \)-factors at the limit of the bandwidth (\( \alpha \ll 1 \) and Eq. (3.8) satisfied):

$$p_{\text{long}} = \frac{1}{4 \phi \alpha (1 + x'^2)}$$  \hspace{1cm} (4.11a)

$$p_{\text{trans}} = \frac{1}{2 \theta \alpha (1 + x'^2)}$$  \hspace{1cm} (4.11b)

Although the damping mechanism is provided by the plate resistivity as for the unloaded resonances, here there is no \( \gamma \)-dependence and a strong coupling to the beam. This is because the cancellation effect described in Section 5 does not apply.
5. SUMMARY AND DISCUSSION

Equations (3.10), (3.12), (4.9a and b) show that the ratio between the P-factors is roughly proportional to n

\[ \frac{P_{\text{long}}}{P_{\text{trans}}} = \frac{n}{2} = n \frac{\xi}{4R} \quad (5.1) \]

The ratio between the amount of Landau damping $\Delta S_{\text{long}}$, to stabilize the longitudinal motion, and $\Delta S_{\text{trans}}$, to stabilize the transverse motion, is:

\[ \frac{\Delta S_{\text{long}}}{\Delta S_{\text{trans}}} = n \frac{|k_0|\delta^2}{\Delta Q \omega_0 + (n - Q)2|k_0|\delta} \quad (5.2) \]

where $k_0 = \omega_0 (d\omega_0/d\psi)/2\pi$, $\Delta Q$ is the $Q$-spread and $\omega_0 \delta/\pi$ is the energy spread of the particles. Since the stabilizing procedure presents some similarities in the two cases, we shall focus our attention on the transverse case.

In the long wavelength limit the response of the plate is given by equation (2.24b), which can be rearranged in the following form:

\[ P_{\text{trans}} = \frac{1 \times 24\pi \delta}{1 + 24\delta} - \beta_p^2 \quad (5.3) \]

\[ (2\phi \ll 1) \]

As pointed out in Ref. 2 the above expression is the response of a distributed circuit of the following type:
which corresponds to the following lumped circuit:

The term $-\gamma^2$ takes into account the inductance of the plate and the term 1 takes into account the capacitance of the plate in parallel with the load.

For an almost floating plate

$$|2\gamma| \gg 1$$

we get

$$\rho_{\text{trans}} = 1 - \frac{\gamma^2}{\rho} + j/2\gamma \quad (2 \gamma \ll 1)$$

which for $\gamma \gg 1$ indicates a strong e.m. cancellation of the forces, just like as there were no plate. This result is illustrated by the following configuration of e.m. fields,

where the arrows indicate the electric fields (i.e. the capacitive term 1), and the x's the magnetic field (i.e. the inductive term $-\gamma^2$).

In the limit

$$|2\gamma| \ll 1$$

(5.5a)
for a loaded plate one obtains:

\[ p_{\text{trans}} = -\beta_p^2 + 2j\phi \]  \hspace{1cm} (5.5b)

\( (2\phi \ll 1) \)

where there is no longer any cancellation between electric and magnetic forces. This is explained by the following picture:

```
\[ \begin{array}{c}
\text{plate} \\
\text{wall}
\end{array} \]
```

The electric field lines are shunted by the termination impedance, and then is left only the inductive term, \(-\beta_p^2\), and a small term depending on the termination.

If \( r \) is real, in both cases the antidamping (imaginary) term is very small; the worst situation occurs when

\[ 2r\phi = 1 \]  \hspace{1cm} (5.6a)

In this case we have a weak e.m. cancellation and the highest imaginary term:

\[ P = 1/2 - \beta_p^2 + j/2 \]  \hspace{1cm} (5.6b)

Therefore, in the long wave limit, it would seem advantageous to have floating plates, so that we can recover a strong e.m. cancellation effect. However, in case of clearing field electrodes, this is impossible because they have to be fed by a high voltage generator.
At higher frequencies the response of the plate is characterized (Figs. 3) by two different patterns of resonances, the loaded and the unloaded resonant modes. The first ones occur when the reactance of the load exactly cancels the reactances of the two sides of the transmission line. The maxima of the P-factors (Eq. 3.12) are inversely proportional to $nr!$. Also in this case it seems advantageous to have very high impedance load. In the complex P-plane the P-factor describes approximately a spiral, with the frequency as running parameter (Fig. 3d).

The unloaded resonances are, as pointed out beforehand, some of the floating plate resonances; the position of the termination determines which ones. The maxima of the P-factor are expressed by Eq. (4.9b), and this equation shows a strong inverse $\gamma$-dependence and a weak direct $(ns)$-dependence. Since the figure of quality of these resonances is generally very high, they do not sensibly affect the behaviour of the plate outside of their band. However, in spite of this very high figure of quality, for usual parameters, the beam is weakly coupled to this resonance. An explanation, which has very much appeal although not entirely rigorous, is given recalling that the $\gamma$-dependence is due to the difference

$$\cos 2 \beta_w \theta - \cos 2 \theta$$

The source of the perturbation is a distribution of current

$$\exp [-j(kz-\omega t)]$$

having phase velocity $\beta_w$. This is represented by the solid line in the picture:

![Diagram of plate and source](image-url)
and its effect is taken into account by the term $-(\beta_w - \beta_p)^2/(1-\beta_w^2)$.
To restore the boundary condition at the end B of the plate a secondary current appears in the opposite direction, with space-time dependence $\exp[j\omega(z/c+t)]$ and phase velocity $-1$; this wave is represented by the dashed line. This backward current is ineffective on the beam. As its wavelength differs from that of the source, the boundary conditions are satisfied at B but not at A. Therefore a third wave is reflected back at A; its amplitude is just the difference between the primary wave and the secondary wave at A, i.e. proportional to

$$\cos \beta_w \omega = \cos 2\theta.$$ 

The maxima for this type of resonance are directly proportional to $\sqrt{n}$. However, above the frequencies for which the wavelength $\lambda_0/(2k + 1)$, is of the order of magnitude of the size of the termination, the resonance starts to dissipate on the termination; then the dependence should change from $\sqrt{n}$ to $1/n$.

In the complex P-plane the unloaded resonances give rise to circles almost tangent to the curve of the loaded resonances. An example is given in Fig. 3c,d. The usefulness of having the plate response on the complex P-plane resides in the fact that it can then be directly transferred from this plane to the U-V plane. As discussed in Ref. 6, stability is achieved if on this plane the response lies completely inside a stable region, the size of which is determined by the dispersion relations of the beam dynamics.
APPENDIX I

Derivation of $\beta$ and of the $g$-factor for transverse motion

In cylindrical coordinates the source of the perturbation is in the displacement $\hat{s}$, expressed by:

$$\hat{s} = \xi (\cos \varphi, -\sin \varphi, 0) e^{j(\omega t-kz)} H(a-\rho).$$

The associated perturbed charge current densities in the beam are

$$\sigma = \sigma_0 \xi \delta(a-\rho) \cos \varphi,$$
$$J_\rho = j(\omega - kv) \sigma_0 \xi \cos \varphi H(a-\rho),$$
$$J_\varphi = -j(\omega - kv) \sigma_0 \xi \sin \varphi H(a-\rho),$$
$$J_z = c \beta_\rho \sigma_0 \xi \cos \varphi \delta(a-\rho).$$

In the case of perfect conductivity ($E_z = E_\varphi = 0$ at $\rho = b$), the z-components of the e.m. fields have the form:

$$E_z = \{A_{11}(q_0)H(a-\rho)+B[K_1(q_0)I_1(q_0)-I_1(q_0)K_1(q_0)]H(\rho-a)\} \cos \varphi,$$

$$H_z = \{C_{11}(q_0)H(a-\rho)+D[K_1'(q_0)I_1(q_0)-I_1'(q_0)K_1(q_0)]H(\rho-a)\} \sin \varphi.$$

All the other field components can be derived from $E_z$, $H_z$, $\sigma$ and $J$. The four constants $A$, $B$, $C$ and $D$ have to be determined matching the e.m. field at $\rho = a$ by means of the following four equations:
\[ E_z (a^+) = E_z (a^-) \]
\[ H_z (a^+) = H_z (a^-) \]
\[ H_\phi (a^+) = H_\phi (a^-) + 4\pi j_z c^{-1} \delta^{-1}(0) \]
\[ H_\rho (a^+) = H_\rho (a^-) \]

The charges and associated currents induced on the wall are related to the e.m. field components \( E_\rho \), \( H_\phi \) and \( H_z \) at \( \rho = b \):

\[ q_1 = -\frac{E_\rho}{4\pi} = -\frac{p_1}{ab\pi} \frac{I_1(qa)}{I_1(qb)} \left[ 1 + 2\beta_w - \frac{\beta_w - \beta_p}{1 - \beta_w^2} \frac{I_2(qb)}{I_0(qb) + I_2(qb)} \right] \cos \phi \]

\[ I_1 = -\frac{H_\phi}{4\pi} = -\frac{p_1c}{ab\pi} \frac{I_1(qa)}{I_1(qb)} \left[ \beta_p + 2\frac{\beta_w - \beta_p}{1 - \beta_w^2} \frac{I_2(qb)}{I_0(qb) + I_2(qb)} \right] \cos \phi \]

from which \( \beta_{\text{trans}} \) and \( g_{\text{trans}} \) can be drawn.
APPENDIX II

Study of the $g$-factors

The $g$-factors, Eqs. (2.21c) and (2.21d), take into account the radial distribution of the fields, since they are related to the propagation normal to the beam. Recalling that

$$ q = k \sqrt{1 - \beta_w^2} = k \gamma_w^{-1} \quad (AII.1) $$

$$ \beta_w = \beta_p \quad \text{for longitudinal case} \quad (3) $$

$$ \beta_w = \beta_p \left( 1 - \frac{n}{n} \right) \quad \text{for transverse case} \quad (4) $$

we see that the transverse propagation, and consequently the $g$-factors, are strongly $\gamma$-dependent. In the limit

$$ q_b \ll 1 \quad \text{i.e.} \quad n \ll \frac{R}{b} \quad (AII.2) $$

one has

$$ g_{\text{long}} = 1 \quad \text{and} \quad g_{\text{trans}} = 1 \quad (AII.3a,b) $$

In the limit

$$ q_b \gg 1 \quad \text{i.e.} \quad n \gg \frac{R}{b} \quad (AII.4) $$

the asymptotic behaviour is:

$$ g_{\text{long}} = 2 \sqrt{\frac{b}{a}} \gamma \frac{R}{an} e^{-k(b-a)/\gamma} \quad (AII.5a) $$
and

\[ g_{\text{trans}} = \left( \frac{b}{a} \right)^{1.5} e^{-k(b-a)/\gamma} \]  \hspace{1cm} (AII.5b)

One can see that at high frequencies the transverse distribution of fields reduces the forces acting on the beam, in agreement with Ref. 7. However, for highly relativistic beams, this effect is appreciable at very high frequencies. Fig. 6 gives \(g_{\text{long}}\) as function of \(qb\), for different values of \(a/b\).

Let us write

\[ g_{\text{trans}} = g_1^t(qb, \frac{a}{b}) - g_2^t(qb, \frac{a}{b}) g_3^t(\frac{n}{Q}, \gamma) \]  \hspace{1cm} (AII.6)

where

\[ g_1^t(qb, \frac{a}{b}) = \frac{b}{a} \frac{I_1(qa)}{I_1(qb)} \]  \hspace{1cm} (AII.7b)

\[ g_2^t(qb, \frac{a}{b}) = \frac{b}{a} \frac{I_1(qa)}{I_1(qb)} \frac{I_2(qb)}{I_0(qb) + I_2(qb)} \]  \hspace{1cm} (AII.7b)

\[ g_3^t(\frac{n}{Q}, \gamma) = \frac{28}{\gamma} \left( \frac{\beta_p - \beta_w}{1 - \beta_w^2} \right) \]  \hspace{1cm} (AII.7c)

Figure 4 shows the expression \(\gamma^{-2}\) as a function of \(n/Q\) for various \(\gamma\). The functions of \(g_1^t\) and \(g_2^t\) are plotted in Figs. 5a and 5b as a function of \(qb\), for various values of \(a/b\). The function \(g_3^t\) is plotted in Fig. 5c as a function of \(n/Q\), for various values of \(\gamma\).
APPENDIX III

Resonant Frequency Shift. Figure of Quality.

The resonances are obtained by the minimization of $D$ as a function of $\omega$. For perfectly conducting plates the resonant frequency $\omega_\nu$ is given by the equation (3.14a), namely

$$\omega_\nu = (\pi \nu c) / \ell$$  \hspace{1cm} (AIII.1)

For finite conductivity the denominator has a zero for $2\varepsilon \phi = \pi \nu$; then for real frequencies the minimum of the function occurs for a frequency $\omega_\nu$, given by

$$\text{Re} \left[ 2\varepsilon \phi \right] = \pi \nu, \text{ i.e. } \text{Re} \left[ \omega_\nu \sqrt{1 - \alpha^2} \right] = (\pi \nu c) / \ell$$  \hspace{1cm} (AIII.2)

In the limit $\alpha \ll 1$, one has:

$$\omega_\nu = \pi \nu c \left[ 1 - \text{Re} (\alpha)/2 \right] / \ell$$  \hspace{1cm} (AIII.3)

The frequency shift introduced by the resistivity is then

$$\frac{\omega_\nu - \omega_\nu^0}{\omega_\nu^0} = - \text{Re}(\alpha)/2$$  \hspace{1cm} (AIII.4)

On the other hand the figure of quality $Q$ of this resonant circuit gives the bandwidth

$$B = \frac{1}{Q} = \frac{\text{Re} (Z)}{\omega L}$$  \hspace{1cm} (AIII.5)
Introducing the expressions of $Z$ and $L$ one obtains:

$$B = \text{Re} \left( a \right) \quad \text{(AIII.6)}$$

Equations (AIII.4 and 6) show that the maximum of the reactance occurs at the frequency $\omega_{vo}$, which does not depend on the conductivity of the plate. This is visualized in Fig. 7, which gives the impedance near the resonance.
APPENDIX IV

Resonators with different resonant frequencies

We have seen that the total response of \( M \) identical devices of length \( L \) is obtained just by multiplying the response of the single device by the factor \( M L / (2\pi R) \). We now calculate the total response, when these resonators have different resonant frequencies \( \phi' \), randomly distributed around a central frequency \( \phi_0 \). If the number of resonators is large, as specified below, we are allowed to take a continuous distribution function, normalized to \( M L / (2\pi R) \), rather than a discrete one.

The response of the plate is proportional to

\[
\frac{1}{\alpha + 2(\phi - \phi')/\phi_0}
\]

(AIV.1)

The total response is then

\[
R_{\text{tot}} = \int_{-\infty}^{+\infty} \frac{F(x' - 1)}{\alpha + 2(x - x')} \, dx'
\]

(AIV.2)

where \( x = \phi/\phi_0 \) and \( F(x' - 1) \) is the distribution function of the resonators.
1) Monochromatic distribution: \( F(x' - 1) = \delta(x' - 1) \left( \frac{M_0}{2\pi R} \right) \) (AIV.3a)

The total response is:

\[
R_{tot} = \left( \frac{M_0}{2\pi R} \right) \frac{1}{\alpha + 2(x - 1)} \tag{AIV.3b}
\]

which is nothing less than the result of Section 2 where all devices have the same resonant frequency.

ii) Lorentzian distribution: \( F(x' - 1) = \frac{\sigma}{\pi \left( (x' - 1)^2 + \sigma^2 \right)^2} \left( \frac{M_0}{2\pi R} \right) \) (AIV.4a)

where \( \sigma \) is the spread.

The total response is:

\[
R_{tot} = \frac{1}{\alpha - 2\sigma + 2(x - 1)} \left( \frac{M_0}{2\pi R} \right) \tag{AIV.4b}
\]

This means that the response is equivalent to that of \( M \) identical resonators, which have a smaller figure of quality

\[
\frac{1}{Q} = \text{Re}(\alpha) + 2\sigma \tag{AIV.5}
\]

iii) Gaussian distribution: \( F(x' - 1) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x' - 1)^2}{2\sigma^2} \right] \left( \frac{M_0}{2\pi R} \right) \) (AIV.6a)

In this case we cannot integrate Eq. (AIV.2). We give the value of \( R_{tot} \) at the frequency for which \( R_{tot} \) is maximum:

\[
R_{tot} (x = 1 - \text{Re}(\alpha)/2) = j \left( \frac{M_0}{2\pi R} \right) \frac{1}{2\sigma} \sqrt{\frac{\pi}{2}} \exp(y^2) \left[ 1 - \text{erf}(y) \right] \tag{AIV.6b}
\]
where \( y = \text{Re}(\alpha)/(2\sqrt{3} \sigma) \) and \( \text{erf}(y) \) is the error function. Equation (AIV.6b) means that the response is roughly equivalent to that of \( M \) devices, each of them having the figure of quality

\[
Q = \frac{\sqrt{y}}{\text{Re}(\alpha)} g(y) = \frac{\sqrt{y}}{\text{Re}(\alpha)} \exp(2y) \left[ 1 - \text{erf}(y) \right]
\]  
(AIV.7)

As for the Lorentzian distribution, here in the limiting cases \( y \gg 1 \) and \( y \ll 1 \) the figure of merit is essentially determined by the figure of merit of the device and by the spread of the resonant frequencies, i.e.:

\[
Q = \frac{1}{\text{Re}(\alpha)} \quad \text{and} \quad Q = \frac{1}{2\sigma} \sqrt{\frac{\pi}{2}}
\]

respectively. The function \( g(y) \) is tabulated in Ref. 8 for \( 0.1 < y < 10 \), under the denomination \( \text{Im}[Z(iy)] \) . For example, in the case \( y = 1 \) one obtains \( Q = 0.76/\text{Re}(\alpha) \).

The discrete distribution function can be replaced by a continuous one when

\[
M \gg 1/y.
\]

If this is not the case, the maximum of the response is determined by accidental overlappings between the responses of the single plates.
REFERENCES


FIGURE CAPTIONS

**Fig. 1** Geometry of the beam, of the pipe and of the conducting plates.

**Fig. 2 a,b,c,d,e** Reactance of the plate from the feed point

**Fig. 3 a,b** Transverse and longitudinal P-factors as functions of $2\pi/r$.
Unloaded resonances appear at $(2\pi/r) \approx 4$.

**Fig. 3 c,d** Transverse and longitudinal P-factor in the complex P-plane,
with harmonic number $n$ as running parameter. Unloaded resonances appear as perfect circles.

**Fig. 4** The quantity $\gamma_w^{-2}$ is plotted as a function of $n/Q$ for various values of $\gamma$.

**Fig. 5 a,b** The quantities $g_1^t$ and $g_2^t$ are plotted as functions of $qb$ for various values of $a/b$.

**Fig. 5 c** The quantity $g_3^t$ is plotted as a function of $n/Q$ for various values of $\gamma$.

**Fig. 6** The quantity $g_{\text{long}}$ is plotted as a function of $qb$ for various values of $a/b$.

**Fig. 7** Frequency shift and figure of quality of the unloaded resonant modes.