The quantum consistency of sigma-models describing the dynamics of extended objects in a curved background requires the cancellation of their world-volume anomalies, which are conformal anomalies for the heterotic string and \(SO(1, 5)\) Lorentz-anomalies for the heterotic five-brane, and of their ten dimensional target space anomalies. In determining these anomalies in a \(D = 10\) Lorentz-covariant background gauge we find that for the heterotic string the worldvolume anomalies cancel for 32 heterotic fermions while for the conjectured heterotic five-brane they cancel for only 16 heterotic fermions, this result being in contrast with the string/five-brane duality conjecture. For what concerns the target space anomalies we find that the five-brane eight-form Lorentz-anomaly polynomial differs by a factor of \(1/2\) from what is expected on the basis of duality. Possible implications of these results are discussed.

1 Introduction

It is by now clear that \(p\)-branes are fated to play a central role in the duality relations occurring in string theories and \(M\)-theory. Among these dualities an
interesting one is the heterotic string/heterotic five-brane strong–weak-coupling duality [1]. Unfortunately until now no consistent five-brane theory, based on a classical action triggering also its quantum dynamics, does exist. The principal problems are the following:

1) Whereas there exists a \( \kappa \)-invariant classical action for the gravitational sector of the five-brane, for its heterotic sector no \( \kappa \)-invariant classical action is known.

2) The heterotic five-brane sigma model appears to be power counting non-renormalizable.

3) Is there a ten-dimensional space–time interpretation for the physical modes of the gravitational sector of the heterotic five-brane? These physical modes are four fermionic plus four bosonic modes which do not span a representation of \( SO(8) \), the little ten-dimensional Lorentz group.

4) What is the quantum heterotic five-brane? A classification of the six-dimensional topologies is not available, and, moreover a term like \( \int \sqrt{g} R \varphi \), which furnishes in the case of the NSR-string the quantum expansion parameter, seems not available in the case of Green–Schwarz (GS)-extended objects.

5) How many fermions are there in the heterotic sector?

6) Do the anomalies in the heterotic five-brane cancel?

7) Can the resulting heterotic five-brane be dual to the heterotic string?

The problems 1) – 4) will not be addressed in this talk. For what concerns 1), if one chooses fermions as basic fields for the heterotic sector and constructs a simply minded action – for example introducing a minimal coupling with the external gauge fields – \( \kappa \)-invariance is destroyed.

For what concerns renormalizability, point 2), from a dimensional point of view the theory, living in six dimensions, does not seem renormalizable; but if eventually a \( \kappa \)-invariant formulation will be found it is possible that \( \kappa \)-invariance prevents the appearance of non-renormalizable divergences in the effective action. A similar conjecture has, in fact, been made for the eleven dimensional membrane by Paccanoni et al [2]. Actually, the analysis of this paper is not complete since an exhaustive classification of all the possible divergences is very difficult to realize and has not been made. On the other hand, GS-strings conformal invariance, which is fundamental for its quantum consistency, is entailed by \( \kappa \)-invariance and it may be that for five–branes, and other GS-extended objects, \( \kappa \)-invariance is just as fundamental for their quantum consistency as conformal invariance is for strings.

The points 5) – 7) will be addressed in this talk and we concentrate on the possible quantum consistency of the heterotic super–five–brane sigma model embedded in an \( N = 1, D = 10 \) target superspace, i.e. on the derivation and cancellation of its anomalies. This analysis will give us a concrete information on the field content of the heterotic sector and shed some new unexpected light on string/five–brane duality. The results presented in this talk have been obtained in refs. [3, 4] to which we refer the reader for the details of their derivation and for more detailed references.
Since \( p \)-brane sigma models are defined by GS–type actions, like the GS–heterotic string, a natural attempt in the five–brane anomaly analysis consists in trying to extend, as much as possible, the techniques we use in GS–string theory to the five–brane sigma model. We will start with the world–volume anomalies which are conformal anomalies for the string and \( SO(1, 5) \) local Lorentz anomalies for the five–brane. Actually, for the GS heterotic string the conformal anomaly is cohomologically tied to the \( SO(1, 1) \) local Lorentz anomaly and the \( \kappa \)-anomaly, while for the five–brane the \( SO(1, 5) \) anomaly is cohomologically tied to the \( \kappa \)-anomaly, via the Wess–Zumino consistency conditions. This means that it is sufficient to worry about \( SO(1, 1) \) (\( SO(1, 5) \)) anomalies only: once they cancel all other worldsheet (worldvolume) anomalies will cancel automatically. So first of all we have to have a good understanding of the \( SO(1, 1) \) anomaly cancellation in the string. This leads us to face the "conformal anomaly puzzle" i.e. a naif counting of the chiral fermionic degrees of freedom in the GS heterotic string leaves a non vanishing anomaly there: the left handed \( \vartheta \)-fields are 16 which by \( \kappa \)-symmetry are reduced to 8, while the right handed heterotic fermions are 32, so the \( \vartheta \)'s are by a factor of 4 to short to cancel the \( SO(1, 1) \) anomaly. For what concerns the conformal anomaly in the non–supersymmetric sector, the \( X^m \) plus \((b, c)\)-fields count \( 10 - 26 = -16 \) while the \( \vartheta \)'s count \( \frac{1}{2} \cdot 8 = 4 \) and again their contribution should in some way be multiplied by 4 to lead to a cancellation. The conformal anomaly cancellation mechanism for the GS–string has been discovered in the flat case, in a \( D = 10 \) Lorentz covariant background gauge, by Wiegmann [5]. Our procedure for the \( SO(1, 1) \) anomaly cancellation mechanism in the sigma–model is based on this paper.

The cancellation of the target space anomalies (via the GS–mechanism) is necessary for the quantum consistency of the string/five–brane sigma–models since they are cohomologically tied to genuine sigma–model worldvolume \( \kappa \)-anomalies [3, 4, 6] i.e. which vanish only in the flat limit.\(^1\)

In section 2 we will show that the above mentioned quadruplication is intimately related to the target space \( SO(1, 9) \) local Lorentz anomaly in the GS–string, and obtain the expected complete four–form anomaly polynomial for the heterotic string. Encouraged by this result we will in section 3 extend this method to the five–brane sigma model and compute its complete eight–form anomaly polynomial, under the assumption that the heterotic sector is made out of a certain number of fermions. Our principal results are the following. The number of fermions needed to cancel the worldvolume anomaly is sixteen rather than the expected thirtytwo. On the other hand the coefficient of the \( D = 10 \) target space Lorentz anomaly carries a factor of \( \frac{1}{2} \) with respect to what is expected on the basis of duality. Section 4 is devoted to a brief discussion of our results.

\(^1\)Apart from these one expects additional genuine sigma–model \( \kappa \)-anomalies which are \( SO(1, 9) \) and \( SO(1, 1)/SO(1, 5) \) invariant and can be cancelled by modifying the target superspace constraints [6, 7], which we do not consider here.
2 Heterotic string anomalies

The sigma-model action for the heterotic Green-Schwarz string with gauge group $SO(32)$ in ten target space–time dimensions is given by

$$S_2 = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left( \frac{1}{2} \sqrt{g} g^{ij} V_i^a V_j a + \tilde{B}_2 - \frac{1}{2} \sqrt{g} e^j_i \psi (\partial_j - A_j) \psi \right). \quad (1)$$

Here the string fields are the supercoordinates $Z^M = (X^\mu, \vartheta^\rho)$, the 32 heterotic fermions $\psi$ and the worldsheet zweibeins $e_+^j, g^{ij} = e_i^j e^j_+$. The induced zehn-beins are given by $V_i^A = \partial_i Z^M E_M^A(Z)$ and $\tilde{B}_2$ is the pullback on the string worldsheet of the supergravity two-superform.

This action is invariant under $d = 2$ diffeomorphisms, local $SO(1,1)$ Lorentz transformations, conformal and $\kappa$-transformations. Diffeomorphisms anomalies can always be eliminated at the expense of conformal/$SO(1,1)$ anomalies, so we will not dwell upon them. Since the coefficient of the conformal and $\kappa$–anomalies is tied for cohomological reasons to the $SO(1,1)$ anomaly we will now concentrate on the last one. Since this is an ABBJ–anomaly only fermions will contribute, in our case the $\vartheta^a$'s and the $\psi$. The contribution of the latter is standard, so we will now consider in detail the formers. It is most convenient to use the background field method together with a normal coordinate expansion; calling the quantum $\vartheta^a$'s $y^a$ where $a = 1, \cdots, 16$ the relevant part of the expanded action becomes

$$I(V, \Omega, y) = \frac{1}{2} \int d^2\sigma \sqrt{g} g^{ij} V_i^a y \Gamma_a \frac{1 - \Gamma}{2} D_j \frac{1 - \Gamma}{2} y \quad (2)$$

where $D_j \equiv \partial_j - \frac{1}{4} \Gamma_{cd} \partial_j \varpi^{cd}$, $\Omega_{j}^{\cdot cd}$ is the $SO(1,9)$ target space Lorentz connection, the $\Gamma^a$ are ten dimensional Dirac matrices and we defined the matrix $\Gamma^a_\beta = \frac{1}{2} \sqrt{g} V_i^a V_j^b (\Gamma_{ab})^\alpha_\beta$. An $SO(1,9)$-covariant background gauge fixing can now be achieved by imposing $\frac{1 + \Gamma}{2} y = 0$, which reduces the physical $y$’s from 16 to 8, but the problematic feature of (2) is that the kinetic term for the $y$’s is not canonical in that it is multiplied by the external (classical) fields $V_i^a$ and one can not define a propagator. Eq. (2) can be transformed to an action with a canonical kinetic term, taking advantage from its manifest classical $SO(1,9)$ invariance, by applying a convenient $SO(1,9)$ Lorentz rotation with group element $\Lambda_a^b$. But, since the integration measure $\int \{ Dy \}$ under local $SO(1,9)$ transformations is not invariant [6], this rotation gives in general rise to a Wess-Zumino term. The $SO(1,9)$ Lorentz anomaly, contrary to the $SO(1,1)$ anomaly, can be computed with standard techniques and the corresponding polynomial turns out to be [6]

$$X_L^{(2)} = \frac{1}{8\pi} tr R^2 = \frac{1}{8\pi} d \omega_3(\Omega), \quad (3)$$

4
where \( R_{\alpha \beta} \) is the \( D = 10 \) Lorentz curvature two-form and \( \omega_3(\cdot) \) is the standard Chern-Simons three-form. Therefore, for a generic rotation, \( \Lambda \), the measure \( \int \left\{ D\gamma \right\} \), and hence the effective action, change by a Wess-Zumino term given by

\[
\Gamma_{WZ} = \frac{1}{8\pi} \int_{D_3} \left( \omega_3(\Omega) - \omega_3(\Omega^1) \right),
\]

where the boundary of \( D_3 \) is the worldsheet. The crucial point is that for the particular \( \Lambda_{\alpha \beta} \) which renders the kinetic term of the \( \gamma \)'s canonical [3] one has \( \omega_3(\Omega^1) = \omega_3(\omega^{(2)}) + Y_2 + dY_2 \), where \( \omega^{(2)} \) is the two-dimensional Lorentz connection, \( Y_2 \) is a local form and can therefore be disregarded and \( Y_2 \) is an \( SO(1,1) \) and \( SO(1,9) \)-invariant form. The Wess-Zumino term (4) contributes therefore to the \( SO(1,1) \) anomaly with a polynomial which is given by

\[
X_{WZ}^{(2)} = -\frac{1}{8\pi} tr R^2 = -\frac{1}{192\pi} \cdot 24 tr R^2,
\]

where \( R \) is the two-dimensional Lorentz curvature two-form (all traces are in the fundamental representations of the orthogonal groups). The functional integral over the (transformed) \( \gamma \)'s is now canonical and corresponds to eight Weyl-Majorana fermions with effective action given by 8 \( \ln \det^{1/2}(\sqrt{g} \partial_\mu) \); this entails a contribution to the anomaly given by [8]

\[
X_{naif}^{(2)} = -\frac{1}{192\pi} \cdot 8 tr R^2.
\]

The total contribution of the quantum \( \gamma \)'s to \( SO(1,1) \) and \( SO(1,9) \) anomalies is thus obtained by summing up (3),(5) and (6):

\[
X_\gamma^{(2)} = \frac{1}{2\pi} \left( \frac{8 + 24}{96} tr R^2 + \frac{1}{4} tr R^2 \right).
\]

We see that the Wess-Zumino term leads to a quadruplication of the "naif" \( SO(1,1) \) anomaly.

The contribution of \( N_\psi \) right-handed heterotic Majorana-Weyl fermions, which contribute only to \( SO(1,1) \) and Yang-Mills anomalies, can be read directly from the index theorem [8], \( X_\psi^{(2)} = \frac{1}{2\pi} \left( \frac{N_\psi}{96} tr R^2 - \frac{1}{4} tr F^2 \right) \). Summing up this and (7) we obtain the total worldsheet and target space anomaly polynomial for the heterotic string as

\[
X^{(2)} = \frac{1}{2\pi} \left( \frac{N_\psi - (8 + 24)}{96} tr R^2 + \frac{1}{4} (tr R^2 - tr F^2) \right).
\]

The worldsheet anomaly cancels for 32 heterotic fermions, the gauge group can therefore be taken to be \( SO(32) \) and the remaining target space anomaly can be cancelled by modifying the \( B_2 \) Bianchi identity to

\[
dH_3 = -2\pi \alpha' \cdot \frac{1}{8\pi} (tr R^2 - tr F^2) \equiv -2\pi \alpha' \cdot I_4,
\]

in agreement with the GS mechanism.
3 Heterotic five–brane anomalies

The action for the super-fivebrane sigma-model [9] embedded in an $N = 1$, $D = 10$ target space supergravity background is given by

$$S_6 = -\frac{1}{(2\pi)^3 \beta i} \int d^6 \sigma \left( \frac{1}{2} e^{-\frac{3}{2} \sqrt{g} g^{ij} V_i V_j - \tilde{B}_6 - 2\sqrt{g}} \right),$$

where $\tilde{B}_6$ is the pullback on the six–dimensional worldvolume of the dual supergravity six–superform $B_6$. $S_6$ is invariant under $\kappa$–transformations, $d = 6$ diffeomorphisms and $SO(1,5)$ local Lorentz transformations if one replaces the metric $g_{ij}$ with sechsineis. As in the case of the string it is sufficient to worry about $SO(1,5)$ and $SO(1,9)$ anomalies only. As we will see, the action in eq. (10) will give rise to a non–vanishing $SO(1,5)$ anomaly, therefore one must add a heterotic sector to cancel this anomaly. Despite the difficulties mentioned in the introduction we will assume that this sector is made out of a certain number $N_\psi$ of $d = 6$ complex Weyl fermions, minimally coupled to Yang–Mills fields of a gauge group $G$. A part from this, the derivation of the anomalies follows mainly the strategy we adopted in section 2 for the string, so we will only report the results referring to [4] for the details of their derivation.

The total $SO(1,5)$ and $SO(1,9)$ anomaly due to the $\psi$'s is again a sum of three terms, like (3), (5) and (6), $X^{(6)}_\psi = X^{(6)}_L + X^{(6)}_{WZ} + X^{(6)}_{naif}$, and the formula analogous to (7) is

$$X^{(6)}_\psi = \frac{1}{192(2\pi)^3} \left( -1 - 15 \right) \left( \frac{1}{30} tr R^4 + \frac{1}{24} (tr R^2)^2 \right)$$

$$+ tr R^2 tr R^2 - \frac{3}{8} (tr R^2)^2 + \frac{1}{2} tr F^4 .$$

In this case the Wess–Zumino term (counting for 15 complex Weyl fermions) amounts to multiply the naif $SO(1,5)$ anomaly (corresponding to 1 fermion, i.e. the 8 physical real $\psi$'s) by a factor of 16. The index theorem gives for the heterotic fermions, with chirality opposite to that of the $\psi$'s,

$$X^{(6)}_\psi = \frac{1}{192(2\pi)^3} \left( N_\psi \left( \frac{1}{30} tr R^4 + \frac{1}{24} (tr R^2)^2 \right) - 2 tr F^2 tr R^2 + 8 tr F^4 \right).$$

The total heterotic five–brane anomaly, which is gotten summing up (11) and (12), becomes:

$$X^{(6)} = \frac{1}{192(2\pi)^3} \left( (N_\psi - 1 - 15) \left( \frac{1}{30} tr R^4 + \frac{1}{24} (tr R^2)^2 \right) \right.$$

$$+ (2tr R^2 - tr R^2) \left( \frac{1}{2} tr R^2 - tr F^2 \right)$$

$$+ \frac{1}{2} \left( tr R^4 + \frac{1}{4} (tr R^2)^2 \right) - tr F^2 tr R^2 + 8tr F^4 \right).$$

6
4 Discussion

One aspect of the string/five-brane duality conjecture emerges from the factorization of the $N = 1, D = 10$ supergravity anomaly polynomial, $I_{12} = \frac{1}{2\pi} I_4 \cdot I_8$, where for the gauge group $SO(32)$ $I_4$ is given in eq. (9) and

$$I_8 = \frac{1}{192(2\pi)^5} \left( trR^4 + \frac{1}{4}(trR^2)^2 - trR^2 trF^2 + 8 tr F^4 \right),$$

where the Yang–Mills curvature $F$ belongs to the fundamental representation of $SO(32)$. According to the conjecture, once in (13) the worldvolume anomaly cancels, the remaining target space anomaly polynomial should coincide with (14). To cancel the worldvolume anomaly one needs $N_\psi = 16$, i.e. sixteen heterotic fermions, and therefore the gauge group cannot be $SO(32)$ (not even $E_6 \otimes E_6$) and one cannot identify $F$ with $F$. Moreover, there are mixed terms in (13), $2tr R^2 \cdot \left(\frac{1}{2}tr R^2 - tr F^2\right)$, which can be cancelled in no way, and the weights of the leading target space Lorentz anomaly, $tr R^4$, in $X^{(6)}$ and $I_8$ differ by a factor of 1/2.

To quantify these discrepancies let us assume that the $\vartheta$'s count for two, instead of one, complex Weyl fermions. In this case the total anomaly polynomial would be given by $\widetilde{X}^{(6)} = 2 \cdot X_\vartheta^{(6)} + X_\varphi^{(6)}$ which can be written as

$$\widetilde{X}^{(6)} = I_8 + \frac{1}{48(2\pi)^5} \left( 2tr R^2 - tr R^2 \right) \cdot I_4 + \frac{N_\psi - 32}{192(2\pi)^5} \left( \frac{1}{30} tr R^4 + \frac{1}{24} (tr R^2)^2 \right).$$

In this case one would need 32 heterotic fermions in the fundamental representation of $SO(32)$, the term proportional to $I_4$ would correspond to a trivial anomaly thanks to (9), and $\widetilde{X}^{(6)}$ would reduce to $I_8$ in complete agreement with duality – which could be eliminated by modifying the Bianchi identity of $B_6$ to $dH_7 = (2\pi)^3 \beta' I_8$. Since, according to duality, $H_7$ has to be the Hodge–dual of $H_3$, this Bianchi identity, together with (9), would imply a relation between the charges of strings and five–branes involving the ten–dimensional Newton’s constant $\kappa$, i.e. $2\kappa^2 = (2\pi)^3 \beta' I_8$, which corresponds to a Dirac–like quantization condition [10] with $n = 1$.

So our principal conclusion is that the five–brane $\vartheta$–anomaly is only half of what is expected on the basis of string/five–brane duality, adding a new problem to the ones already mentioned in the introduction. We can nevertheless mention that if we set in $X^{(6)}$ and $I_8$ the gauge fields to zero, $F = F = 0$, then the worldvolume anomaly cancels for sixteen heterotic fermions and by subtracting a suitable trivial anomaly, as above, $X^{(6)}$ would reduce to $\frac{1}{2} \cdot I_8$. This would imply the quantization condition $2\kappa^2 = \frac{1}{2} (2\pi)^3 \beta' I_8$, i.e. $n = \frac{1}{2}$ which signals the presence of half–charged five–branes. Half–charged five–branes arose, actually, in ref. [11] where they appear, however, always in pairs such that their total charge is always integer. Half integral magnetic charges have arisen also on fixed
points of $Z_2$-orbifold compactifications of $N = 1, D = 11$ Supergravity in ref. [12].

References


