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Algebra of Observables and Charge
Superselection Sectors for QED
on the Lattice

by

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Abstract

Quantum Electrodynamics on a finite lattice is investigated within the hamiltonian approach. First, the structure of the algebra of lattice observables is analyzed and it is shown that the charge superselection rule holds. Next, for every eigenvalue of the total charge operator a canonical irreducible representation is constructed and it is proved that all irreducible representations corresponding to a fixed value of the total charge are unique up to unitary equivalence. The physical Hilbert space is by definition the direct sum of these superselection sectors. Finally, lattice quantum dynamics in the Heisenberg picture is formulated and the relation of our approach to gauge fixing procedures is discussed.

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1 Introduction

This paper is a continuation of [4] and [5]. In [4] we have proved that the classical Dirac-Maxwell system can be reformulated in a spin-rotation covariant way in terms of gauge invariant quantities, for earlier attempts to solve this problem see [6], [7], [8] and [9]. In [5] we have shown that it is possible to perform similar constructions on the level of the (formal) functional integral of Quantum Electrodynamics. For an application of these ideas to nonabelian Higgs models see [10] and for a similar approach to QCD see [16]. The result is a functional integral completely reformulated in terms of local gauge invariant quantities, which differs essentially from the effective functional integral obtained via the Faddeev-Popov procedure [11]. In particular, it turns out that standard perturbation

techniques, based upon a splitting of the effective Lagrangian into a free part (Gaussian measure) and an interaction part (proportional to the bare coupling constant e), are rather not applicable to this functional integral. On the contrary, our formulation seems to be rather well adapted to investigations of nonperturbative aspects of QED. We will show in this paper that there exists a natural lattice version of QED in terms of gauge invariant quantities and that this formulation provides a natural framework for discussing basic structures like the observable algebra and charge superselection sectors. We also refer to [13] – [15], where nonabelian Higgs models and scalar QED have been formulated in terms of lattice gauge invariants.

There are relations to a series of papers, see [20], [21] and [22], by Strocchi and Wightman, where these authors have analyzed the structure of gauge theories within the context of algebraic field theory in the sense of Haag and Kastler [18]. In particular, in [20] Quantum Electrodynamics was considered. It was shown that if one insists on locality and Lorentz covariance, one is rather naturally led to a theory with indefinite metric. Within this scheme, the complicated structure of the physical Hilbert space implies that there is a whole hierarchy of notions of gauge invariance of operators built from elements of the field algebra one starts with. The above authors distinguish between gauge independence, weak gauge invariance, gauge invariance and strict gauge invariance (in order of increasing restriction). The main result of [20] is the proof of the charge superselection rule for QED: In a local covariant gauge every quasilocal observable (coming from a gauge independent operator) commutes with the total electric charge. Thus, it is rather the notion of gauge independence, which is relevant. Unfortunately, the gauge independent operators do not form an algebra, whereas the weakly gauge invariant and the gauge invariant operators do. We see that there is no canonical candidate for the observable algebra. Consequently, a decomposition of the physical Hilbert space into a direct sum of subspaces of vectors carrying definite charge was not given. For a discussion of this decomposition within the context of theories, which do not contain massless particles see [19]. Some progress towards an implementation of these ideas for theories with massless particles has been made, see [23] and [24] and further references therein, but a complete solution of the above mentioned problems for theories of this type has - as to our knowledge - not been obtained until now. The basic idea of [23] was to construct charged states from the vacuum by the action of so-called charged morphisms of an algebra of quasi-local observables (arising from non-local elements of this algebra). Within this context a Haag-Ruelle scattering theory was proposed. In [24] a certain criterion characterizing charged states (infrared minimal states) was taken as a starting point (without a concrete field theoretical setting). Within this framework - using techniques of algebraic quantum field theory - the existence of asymptotic electromagnetic fields in all charge sectors was established. In [25] a concrete lattice model, namely a Z_2 -gauge theory with Z_2 -matter fields, was studied. In particular, the ground state and charged states were found. Using methods of Euclidean quantum field theory, the authors were able to show that - for some regions in the space of coupling constants - the thermodynamic limit for charged states could be controlled. This is a very

interesting result, but it is doubtful, whether these methods can be generalized to more realistic models.

In this paper we discuss some of the above-mentioned problems in the simplified setting of QED on a finite lattice. For basic notions concerning lattice gauge theories (including fermions) we refer to [26] and references therein. In the context of lattice approximation complicated operator theoretic problems arising in (continuum) quantum field theory naturally disappear, whereas problems typical for gauge theories remain and can be, therefore, discussed separately. We consider QED in the hamiltonian approach on a finite cubic (3-dimensional) lattice and work in the so-called non-compact formulation, where the gauge potential remains Lie-algebra-valued on the lattice level. We stress, however, that a completely analogous analysis can be carried out within the more familiar, compact formulation leading essentially to the results presented in this paper – with some slight modifications only. Our starting point is the naive Schrödinger representation of the classical commutation and anti-commutation relations for canonically conjugate pairs of lattice quantities. In this representation the generator of local gauge transformations is given by the “Gauss-law-operator”. Thus, a necessary and sufficient condition for gauge invariance of the wave function is that the Gauss law constraint should be fulfilled. However, wave functions fulfilling this constraint are not square integrable and – in principle – different (may be physically even inequivalent) Hilbert space structures, see [27], may be chosen. In the present paper we follow a different strategy: In a first step we define the algebra of observables as the algebra of gauge invariant operators fulfilling the Gauss law and show that the charge superselection rule holds. Basically, the observable algebra is generated by electric and magnetic flux operators, together with gauge invariant operators bilinear in fermion fields. These gauge invariant operators fulfill a number of algebraic identities, which, however, by using the technical tool of a lattice tree become tractable. This tool enables us to classify all irreducible representations of the observable algebra and to obtain the physical Hilbert space as a direct sum of representation spaces labeled by the total charge. This is the main result of the present paper. We underline that these representations are explicitly constructed in purely algebraic terms. A posteriori, an interpretation in terms of gauge invariant wave function turns out to be possible. Concerning sectors with total charge different from zero, there is a remarkable difference comparing with the continuum theory. As in the continuum theory, nonvanishing total charge arises from nonvanishing electric flux “on the boundary”. In the continuum theory there is an infinite number of additional superselection rules corresponding to such fluxes [24]. On the (finite) lattice one can easily see that representations with different fluxes giving the same total charge are equivalent. We also stress that the above mentioned subtleties concerning the very notion of gauge invariance disappear in our approach.

Our paper is organized as follows: In Sections 2 and 3 we discuss the standard second quantization procedure on the lattice and – in some detail – gauge invariance and constraints. In Section 4 we discuss the generators of the algebra of observables to be constructed and derive basic identities relating these generators. In Section 5 we intro-

duce the notion of a tree and show that using this notion, the number of identities reduces essentially. We prove that, given the tree data, we are able to reconstruct the whole set of generators. This enables us to define the algebra of observables as the tensor product of the electromagnetic part (Heisenberg-algebra generated by a finite number of tree elements) and a finite dimensional part generated by invariants built from (on-tree) bilinear combinations of the fermion fields (pair creation and annihilation operators). The latter invariants inherit a number of relations and it is the main technical point of this paper to analyze this subalgebra. In Section 6 we show that the total charge defines a *superselection rule* in the observable algebra. In Section 7 we find all its irreducible representations (up to unitary equivalence) and prove that they are labeled by the eigenvalues of the total charge operator. Some technical details of this proof are shifted to Appendix B. In a next step, see Section 8, we show that the above mentioned representations can be given an ordinary wave function interpretation and, finally, lattice quantum field dynamics in the Heisenberg picture is formulated (Section 9). The paper is completed by two sections, where we point out the relation of our scheme with gauge fixing approaches and outline our philosophy of passing to the continuum limit.

2 Second Quantization on the Lattice

Let us begin with the continuum field configuration, which consists of a U(1)-gauge potential (A_μ) and a four-component spinor field (ψ^a), where $a, b, \dots = 1, 2, 3, 4$ denote bispinor indices and $\mu, \nu, \dots = 0, 1, 2, 3$ spacetime indices. The Lagrangian of the theory is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m\psi^{a*}\beta_{ab}\psi^b - \hbar \operatorname{Im} \left\{ \psi^{a*}\beta_{ab}(\gamma^\mu)^b{}_c D_\mu\psi^c \right\}, \quad (2.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.2)$$

$$D_\mu\psi^a = \partial_\mu\psi^a + igA_\mu\psi^a, \quad (2.3)$$

are the electromagnetic field strength and the covariant derivative. The star denotes complex conjugation, β_{ab} denotes the canonical Hermitean structure in bispinor space and (γ^μ) are the Dirac matrices. For a given Cauchy hyperplane $\Sigma = \{t = \text{const}\}$ in Minkowski space, the above Lagrangian gives rise to an infinite-dimensional Hamiltonian system in variables $(A_k, E^k, \psi^a, \psi^{a*})$ with the Hamiltonian given by

$$\mathcal{H} = \frac{1}{2} (E_k E^k + B_k B^k) + m\psi^{a*}\beta_{ab}\psi^b + \hbar \operatorname{Im} \left\{ \psi^{a*}\beta_{ab}(\gamma^k)^b{}_c D_k\psi^c \right\}, \quad (2.4)$$

where $B = \operatorname{curl} A$.

The goal of this paper is to approximate the above Hamiltonian system by a system with a finite number of degrees of freedom corresponding to elements of a finite

3-dimensional lattice and to prove that there is one and only one (up to equivalence) quantum version of this theory on the lattice.

Let Λ be a finite regular cubic lattice contained in Σ and let a be its lattice spacing. To each lattice site x of Λ we assign the value of the continuous matter field at x scaled in the following way:

$$\psi_x^a := a^{\frac{3}{2}} \psi^a(x) . \quad (2.5)$$

As we shall see in the sequel, the factor $a^{\frac{3}{2}}$ has been chosen in order to simplify the canonical anticommutation relations of the lattice degrees of freedom of ψ . It is implied by the geometric character of ψ as a half-density.

Moreover, to each lattice link $(x, x + \hat{k})$ we assign the value of the line integral:

$$A_{x, x+\hat{k}} := \int_{(x, x+\hat{k})} A_k dl . \quad (2.6)$$

Here, $\hat{k}$ is a vector of length a and direction of the oriented k -th axis ($\hat{k}$ may thus assume the following values: $\hat{1}, -\hat{1}, \hat{2}, -\hat{2}, \hat{3}, -\hat{3}$). Finally, to each link we assign the value of the electric flux through the corresponding plaquette $\sigma(x, x + \hat{k})$ of the dual lattice

$$E_{x, x+\hat{k}} := \int_{\sigma(x, x+\hat{k})} E^k d\sigma_k . \quad (2.7)$$

The plaquette $\sigma(x, x + \hat{k})$ is orthogonal to the link $(x, x + \hat{k})$ and they both have common center.

The second quantization of the continuum theory is based upon the following canonical (anti-) commutation relations for the field operators:

$$[\hat{\psi}^a(x), \hat{\psi}^{b*}(y)]_+ = \delta^{ab} \delta(x - y) , \quad (2.8)$$

$$[\hat{A}_k(x), \hat{E}^l(y)] = i\hbar \delta_k^l \delta(x - y) . \quad (2.9)$$

The remaining (anti-) commutators vanish. In the first equation the factor $i\hbar$ disappears, because, according to the convention used in (2.1), the quantity “canonically conjugate” to ψ^a is equal to $i\hbar\psi^{a*}$.

In the discrete version of the theory the Dirac distribution $\delta(x - y)$ has to be replaced by $a^{-3}\delta_{xy}$, where the symbol δ_{xy} stands for the Kronecker symbol labeled by pairs (x, y) of lattice sites. Indeed, integration is replaced by summation with the weight a^3 corresponding to the volume of each lattice cube. This implies that the “discrete integration” of $a^{-3}\delta_{xy}$ with respect to y over the lattice gives the correct value 1.

Replacing the Dirac delta in (2.9) by its lattice counterpart and multiplying both sides by a^3 we obtain, according to (2.5), the following anticommutation relations for the lattice quantities:

$$[\hat{\psi}_x^a, \hat{\psi}_y^{b*}]_+ = \delta^{ab} \delta_{xy} . \quad (2.10)$$

Analogously, we get from (2.9):

$$[\hat{A}_{x,x+\hat{k}}, \hat{E}_{y,y+\hat{l}}] = i\hbar\delta_{xy}\delta_{\hat{k}\hat{l}}. \quad (2.11)$$

Here, the weight a^3 disappears, because the quantity $A_{x,x+\hat{k}}$ in (2.6) contains the factor a as a line integral over a link of Λ and the quantity $E_{x,x+\hat{k}}$ in (2.7) contains the factor a^2 as a 2-dimensional integral over a plaquette of the dual lattice. The remaining (anti-)commutators have to vanish.

All irreducible representations in the strong (Weyl) sense of the above algebra are equivalent (see [1] and [2]). In particular, the bosonic quantities (A, E) may be described by the Schrödinger representation, in the Hilbert space of wave functions Ψ depending on parameters A . Operators $\hat{A}$ are thus multiplication operators. Momenta canonically conjugate are represented by derivatives

$$\hat{E}_{x,x+\hat{k}} := \frac{\hbar}{i} \frac{\partial}{\partial A_{x,x+\hat{k}}}. \quad (2.12)$$

For each of the anticommuting degrees of freedom ψ^a we may take either the “holomorphic” or the (equivalent) “anti-holomorphic” representation. To match our notation with the notation commonly used in textbooks, we split ψ into a pair of Weyl spinors. We use the anti-holomorphic representation for the upper part and the holomorphic representation for the lower part. We denote

$$\psi^a = \begin{pmatrix} \phi^K \\ \varphi_L^* \end{pmatrix}, \quad (2.13)$$

where $K, L = 1, 2$, and have

$$\psi^{a*} = \begin{pmatrix} \phi^{K*} \\ \varphi_L \end{pmatrix}. \quad (2.14)$$

Assuming that the “classical” quantities (ϕ^{K*}, φ_L^*) are (anticommuting) Grassmann variables, we represent them as multiplication operators $(\hat{\phi}^{K*}, \hat{\varphi}_L^*)$ in the space of all functions (polynomials) of these variables. It follows that the adjoint operators $(\hat{\phi}^K, \hat{\varphi}_L)$ satisfy relations (2.10). They may be represented as derivatives (see [17]):

$$\hat{\phi}^K := \frac{\partial}{\partial \phi^{K*}}, \quad (2.15)$$

$$\hat{\varphi}_L := \frac{\partial}{\partial \varphi_L^*}. \quad (2.16)$$

The tensor product of all these representations is, therefore, defined in the space of wave functions

$$\Psi = \Psi(\{A_{x,x+\hat{k}}\}, \{\phi_x^{K*}\}, \{\varphi_{x;L}^*\}). \quad (2.17)$$

Such wave functions are polynomials in the anticommuting variables (ϕ^*, φ^*) with coefficients being functions of variables A . The Hilbert space structure is defined by the L^2 -norm. Integration over the Grassmann variables is understood in the sense of Berezin, which means that the set of all different monomials in these variables forms an orthonormal basis.

We have obtained a representation of the whole algebra generated by (2.10) and (2.11). This algebra contains a lot of unphysical (gauge-dependent) elements. Moreover, the above electric field operators do not satisfy the Gauss law. We conclude, that the quantum theory constructed this way is unphysical. In what follows we will present an explicit construction of the algebra of observables $\mathcal{O}$ together with a complete classification of its irreducible representations.

For simplicity, we have omitted the discussion of nontrivial boundary data here, this will be done in the next section.

3 Gauge Invariance, Constraints and Boundary Data

A local gauge transformation of the continuum theory is given by

$$\tilde{\psi} = \xi \psi, \quad (3.1)$$

$$\tilde{A} = A + \frac{i}{g} \xi^{-1} d\xi, \quad (3.2)$$

where ξ is a $U(1)$ -valued function on Minkowski space M .

To describe a lattice version of these transformations, we define the group G_Λ as the group of *equivalence classes of continuous gauge transformations* acting in the same way on lattice quantities $(\psi_x, A_{x,x+\hat{k}})$. The commonly used lattice description of gauge transformations is obtained by restricting ξ to lattice sites, $\xi_x := \xi(x)$. This way one obtains the following transformations for lattice quantities:

$$\tilde{\psi}_x = \xi_x \psi_x, \quad (3.3)$$

$$\exp(ig\tilde{A}_{x,x+\hat{k}}) = \xi_x \exp(igA_{x,x+\hat{k}}) (\xi_{x+\hat{k}})^{-1}. \quad (3.4)$$

Of course, such a description *does not* provide a parameterization of G_Λ .

To obtain a correct description of classes of equivalent transformations let us observe that, given a continuum gauge transformation ξ , the following quantity is well defined:

$$\lambda_x := -\frac{1}{g} \left\{ \arg(\xi_{x_0}) + \int_{x_0}^x \frac{d\xi}{i\xi} \right\}. \quad (3.5)$$

The integral does not depend upon the choice of a path connecting x_0 with x , because ξ is *globally defined* on a topologically trivial domain M . Its value gives us the (uniquely defined) phase increment of ξ between x_0 and x . Hence, the phase λ_x of ξ_x is uniquely

defined up to a global additive constant $2n\pi$ contained in the (arbitrarily chosen) phase $\arg(\xi_{x_0})$. This way we obtain a parameterization of elements of G_Λ (up to a global additive constant $2n\pi$) by collections of *real* numbers λ_x assigned to the lattice sites. Obviously, in this parameterization gauge transformations take the form:

$$\tilde{\psi}_x = \exp(-ig\lambda_x)\psi_x, \quad (3.6)$$

$$\tilde{A}_{x,x+\hat{k}} = A_{x,x+\hat{k}} + \lambda_{x+\hat{k}} - \lambda_x. \quad (3.7)$$

In particular, $\tilde{\phi}_x = \exp(-ig\lambda_x)\phi_x$ and $\tilde{\varphi}_x = \exp(ig\lambda_x)\varphi_x$.

We stress at this point that instead of our lattice description of gauge potentials $A_{x,x+\hat{k}}$ we could have also used the compact description, in which the lattice gauge potentials are group-valued quantities $\exp(igA_{x,x+\hat{k}})$ (parallel transporters). On the level of physical observables, this would lead to replacing the magnetic field B , see forthcoming formula (4.1), by $\exp(igB)$ (the so called *Wilson loop*). This change leads only to an obvious and straightforward modification of our results: the so called θ -representations for the electromagnetic field operators $\exp(igB)$ and E would occur in the classification of irreducible representations of the observable algebra. This means that the uniqueness theorem proved in the present paper would no longer be true. To reestablish uniqueness, we would have to exclude these θ -representations *a priori*. That is why we believe that the description we are using is – in the case of electrodynamics – simpler than the Wilson description. It does not introduce any artificial compactification of the physical degrees of freedom and is, therefore, closer to the topological structure of the continuum theory.

Local gauge transformations act on wave functions in the following way:

$$(U(\{\lambda_x\})\Psi)(\{A_{x,x+\hat{k}}\}, \{\phi_x^{K*}\}, \{\varphi_{x,L}^*\}) = \Psi(\{\tilde{A}_{x,x+\hat{k}}\}, \{\tilde{\phi}_x^{K*}\}, \{\tilde{\varphi}_{x,L}^*\}). \quad (3.8)$$

Observe that (3.8) induces the transformation law for the field operators $\hat{A}$ and $\hat{\psi}$, formally identical with (3.6) and (3.7). For a deeper discussion of possible gauge fixings see Section 10.

To calculate the generator $\mathcal{G}_x$ of infinitesimal local gauge transformation at x we take the derivative of $\Psi(\{\tilde{A}_{x,x+\hat{k}}\}, \{\tilde{\phi}_x^{K*}\}, \{\tilde{\varphi}_{x,L}^*\})$ with respect to the gauge parameter λ_x at $\lambda_x = 0$. This way we obtain

$$\begin{aligned} \mathcal{G}_x \Psi &:= - \sum_{\hat{k}} \frac{\partial \Psi}{\partial A_{x,x+\hat{k}}} + ig \sum_K \phi_x^{K*} \frac{\partial \Psi}{\partial \phi_x^{K*}} - ig \sum_L \varphi_{x,L}^* \frac{\partial \Psi}{\partial \varphi_{x,L}^*} = \\ &= \frac{i}{\hbar} \left\{ - \sum_{\hat{k}} \hat{E}_{x,x+\hat{k}} + g\hbar \sum_K (\hat{\phi}_x^{K*} \hat{\phi}_x^K - \hat{\varphi}_{x,K}^* \hat{\varphi}_{x,K}) \right\} \Psi. \end{aligned} \quad (3.9)$$

We conclude that the generator $\mathcal{G}_x$ of local gauge transformations is given by

$$\mathcal{G}_x := -\frac{i}{\hbar} \left\{ \sum_{\hat{k}} \hat{E}_{x,x+\hat{k}} - \hat{j}_x^0 \right\}. \quad (3.10)$$

Here, the operator $\hat{j}_x^0$ of electric charge at x is automatically obtained in the “normally ordered” form:

$$\hat{j}_x^0 = : e \sum_a \hat{\psi}_x^{a*} \hat{\psi}_x^a : = e \sum_K \left(\hat{\phi}_x^{K*} \hat{\phi}_x^K - \hat{\phi}_{x;K}^* \hat{\phi}_{x;K} \right) , \quad (3.11)$$

where $e := g\hbar$ is the elementary charge. We define the operator $\hat{Q}$ of total electric charge putting

$$\hat{Q} := \sum_{x \in \Lambda} \hat{j}_x^0 . \quad (3.12)$$

Due to (3.11), we can interpret $\hat{\phi}_x^{K*}$ as the creation operator of a positron at x , carrying the charge $+e$, and $\hat{\phi}_{x;K}^*$ as the creation operator of an electron at x , carrying the charge $-e$. The index $K = 1, 2$ describes the two possible helicity states.

The necessary and sufficient condition for gauge invariance of the wave function is provided by the following “Gauss law constraint”:

$$\mathcal{G}_x \Psi = 0 , \quad (3.13)$$

which is equivalent to

$$\Psi(\{\tilde{A}_{x,x+\hat{k}}\}, \{\tilde{\phi}_x^{K*}\}, \{\tilde{\phi}_{x;L}^*\}) = \Psi(\{A_{x,x+\hat{k}}\}, \{\phi_x^{K*}\}, \{\phi_{x;L}^*\}) . \quad (3.14)$$

Unfortunately, wave functions fulfilling (3.13) *are not* square integrable with respect to the standard measure on the configuration space, because they are constant on non-compact gauge orbits. One possible strategy would consist in looking for an appropriate Hilbert space structure in the space of gauge invariant wave functions. This strategy was used e. g. by K. Kuchař (see [27]), who argued that the ambiguity in the choice of such a structure in the space of physical wave functions may lead to physically inequivalent formulations of quantum electrodynamics.

Here we will follow a different strategy: In a first step we investigate the algebra $\tilde{\mathcal{O}}$ of gauge invariant operators and derive a set of identities relating its generators. Next we take these identities, supplemented by the Gauss law, as the defining relations for the algebra of observables $\mathcal{O}$. Then we explicitly construct $\mathcal{O}$ and prove that every its faithful, irreducible representation is equivalent to a certain canonical irreducible representation. Finally, it turns out that these representations can be described in terms of gauge invariant wave functions, see Section 8. In this approach there is *no ambiguity* in the definition of the scalar product in the space of physical, gauge-invariant functions. It is *uniquely* implied by the structure of the algebra of observables. Such a strategy was first implemented in [15] in the case of quantum electrodynamics with scalar (bosonic type) matter.

If we sum up equations (3.13) over all $x \in \Lambda$ we see that, heuristically, the total charge $\hat{Q}$ should vanish, when acting on gauge invariant wave functions Ψ :

$$\hat{Q}\Psi = 0 . \quad (3.15)$$

This is a trivial consequence of the fact that each operator $\hat{E}_{x,x+\hat{k}}$ enters twice with opposite signs into the sum. However, we want to include also nontrivial values of the total charge $\hat{Q}$. They can only arise from nontrivial boundary data, which we are now going to introduce. For this purpose we consider also external links of our finite lattice Λ , connecting lattice sites belonging to the boundary $\partial\Lambda$ with “the rest of the world”. This way we can treat Λ as part of a bigger (maybe infinite) lattice. We denote these external links by (x,∞) and allow the wave functions Ψ to depend from the beginning on the corresponding potentials $A_{x,\infty}$. Now, gauge invariance does not imply vanishing of the total charge, because the electric fields on external links remain when we sum up equations (3.13) over all sites of Λ :

$$\hat{Q}\Psi = \left(\sum_{x \in \Lambda} \hat{j}_x^0 \right) \Psi = \left(\sum_{x \in \partial\Lambda} \hat{E}_{x,\infty} \right) \Psi . \quad (3.16)$$

We stress that the external fluxes $\hat{E}_{x,\infty}$ are not dynamical quantities, they play the role of *prescribed* boundary conditions. It will be shown in the sequel that the charge operator $\hat{Q}$ defines a superselection rule. Thus, we have $\hat{Q} = Q\mathbb{1}$ on every superselection sector. Consequently, the only consistent choice for the external fluxes is $\hat{E}_{x,\infty} = E_{x,\infty} \mathbb{1}$ on every superselection sector, where $E_{x,\infty}$ are *c*-numbers fulfilling

$$Q = \sum_{x \in \partial\Lambda} E_{x,\infty} . \quad (3.17)$$

Now, in principle, we could distinguish representations characterized by the same value Q , but corresponding to different external flux distributions fulfilling (3.17). This would lead to additional superselection rules. In the present paper we have chosen another option, which is motivated by the fact that – as will be shown in Section 8 – different boundary conditions corresponding to the same value Q give equivalent representations. Thus, for any value of Q we have an equivalence class of boundary data and we choose a representative, e.g. the “most symmetric” distribution of the values $\hat{E}_{x,\infty}$ on $\partial\Lambda$. Another option would consist in taking for each value of Q as many copies of *the same* representations as there are different distributions of external fluxes, satisfying (3.17). It is, therefore, straightforward, to rewrite all our results in the framework of these, more general, superselection rules.

4 Algebra of Gauge Invariant Operators

In this section we investigate the algebra $\tilde{\mathcal{O}}$ of gauge invariant operators. In particular, we find a set of algebraic relations between generators of this algebra. This set, supplemented by the Gauss law, will be taken as a set of axioms for the generators of the algebra of observables, which we are going to construct in the next Section. We start with the

auxiliary Hilbert space $\mathcal{H}^0$ of square integrable wave functions (2.17), admitting (possibly) nontrivial boundary data $\hat{E}_{x,\infty}$ on $\partial\Lambda$.

Definition 1 *The algebra $\tilde{\mathcal{O}}$ of gauge invariant operators is defined as the commutant of $\{U(\{\lambda_x\})\}$ in $B(\mathcal{H}^0)$.*

Since the set generated by $\{U(\{\lambda_x\})\}$ is $*$ -invariant, $\tilde{\mathcal{O}}$ is a von Neumann algebra. We see that arbitrary bounded functions of (self-adjoint) electric flux operators $\hat{E}$ are gauge invariant. To each oriented plaquette $(x; \hat{k}, \hat{l})$ we may assign the (self-adjoint) magnetic flux operator:

$$\hat{B}_{x;\hat{k},\hat{l}} := \hat{A}_{x,x+\hat{k}} + \hat{A}_{x+\hat{k},x+\hat{k}+\hat{l}} + \hat{A}_{x+\hat{k}+\hat{l},x+\hat{l}} + \hat{A}_{x+\hat{l},x} . \quad (4.1)$$

Again, arbitrary bounded operator functions of $\hat{B}$ are gauge invariant – due to (3.7). Observe that magnetic flux operators are subject to the following constraint: the total magnetic flux through the boundary of a lattice cube vanishes as a consequence of (4.1),

$$\text{div}_{x;\hat{k},\hat{l},\hat{n}} \hat{B} := \hat{B}_{x;\hat{k},\hat{l}} + \hat{B}_{x;\hat{n},\hat{k}} + \hat{B}_{x;\hat{l},\hat{n}} + \hat{B}_{x+\hat{n},\hat{l},\hat{k}} + \hat{B}_{x+\hat{l},\hat{k},\hat{n}} + \hat{B}_{x+\hat{k},\hat{n},\hat{l}} = 0 . \quad (4.2)$$

To any oriented lattice path γ connecting two lattice points x and y we may assign the following gauge-invariant, bilinear combination of the fermionic operators $\hat{\psi}$:

$$\hat{\mathcal{W}}_\gamma^{ba} := \hat{\psi}_y^{b*} \exp(-ig \int_\gamma \hat{A}) \hat{\psi}_x^a , \quad (4.3)$$

where the integral denotes the oriented sum over all links belonging to γ . In particular, γ may be trivial, in that case the integral over γ is equal to zero. Obviously, any $\hat{\mathcal{W}}$ is bounded in $\mathcal{H}^0$ and hence it belongs to $\tilde{\mathcal{O}}$.

Proposition 1 *If γ is a path connecting x with y , β is a path connecting y with z and if the degrees of freedom $(x, a), (y, b)$ and (z, c) differ from each other, then we have the following identity:*

$$[\hat{\mathcal{W}}_\beta^{cb}, \hat{\mathcal{W}}_\gamma^{ba}] = \hat{\mathcal{W}}_{\beta\gamma}^{ca} , \quad (4.4)$$

where $\beta\gamma$ is the composition of γ and β , a path connecting x with z .

Proof:

$$\begin{aligned} [\hat{\mathcal{W}}_\beta^{cb}, \hat{\mathcal{W}}_\gamma^{ba}] &= (\hat{\psi}_z^{c*} \hat{\psi}_y^b \hat{\psi}_y^{b*} \hat{\psi}_x^a - \hat{\psi}_y^{b*} \hat{\psi}_x^a \hat{\psi}_z^{c*} \hat{\psi}_y^b) \exp(-ig \int_{\beta\gamma} \hat{A}) \\ &= (\hat{\psi}_y^b \hat{\psi}_y^{b*} + \hat{\psi}_y^{b*} \hat{\psi}_y^b) \hat{\mathcal{W}}_{\beta\gamma}^{ca} = \hat{\mathcal{W}}_{\beta\gamma}^{ca} . \end{aligned} \quad \square$$

Observe that – due to (3.6) and (3.7) – gauge invariant operators constructed from $\hat{A}$ and $\hat{\psi}$ may be expressed as bounded combinations of $\{\hat{E}, \hat{B}, \hat{\mathcal{W}}, \hat{\mathcal{W}}^*\}$ only.

There are three types of operators $\hat{\mathcal{W}}$:

$$\hat{\mathcal{L}}_\gamma^{LK} := \hat{\phi}_y^{L*} \exp(-ig \int_\gamma \hat{A}) \hat{\phi}_x^K, \quad (4.5)$$

$$\hat{\mathcal{M}}_{\gamma;L}^K := \hat{\phi}_{y;L} \exp(-ig \int_\gamma \hat{A}) \hat{\phi}_x^K, \quad (4.6)$$

$$\hat{\mathcal{R}}_{\gamma;LK} := \hat{\phi}_{y;L} \exp(-ig \int_\gamma \hat{A}) \hat{\phi}_{x;K}^*, \quad (4.7)$$

(the fourth type coincides with the adjoint operators $\hat{\mathcal{M}}_{\gamma;L}^{K*}$). Observe that $\hat{\mathcal{M}}_{\gamma;L}^K$ annihilates a positron at x and an electron at y , whereas $\hat{\mathcal{M}}_{\gamma;L}^{K*}$ creates such a pair. We will call them “pair annihilation operators” (respectively, “pair-creation operators”). A non-diagonal operator $\hat{\mathcal{L}}_\gamma^{LK}$ (i. e. such that $(x, K) \neq (y, L)$, where x is the beginning and y is the end of γ) annihilates a positron at x and creates another one at y , whereas $\hat{\mathcal{R}}_{\gamma;LK}$ does the same with electrons. They may be called “flip operators”. Finally, we have diagonal operators corresponding to trivial paths. We denote them by $\hat{\mathcal{L}}_x^{KK}$ and $\hat{\mathcal{R}}_{y;LL}$ respectively. They give us the “particle number” operators. More precisely, the number of positrons at x with helicity K is given by the projector $\hat{\mathcal{L}}_x^{KK} = \hat{\phi}_x^{K*} \hat{\phi}_x^K$, whereas the number of electrons at x with helicity L is given by the projector $\mathbb{1} - \hat{\mathcal{R}}_{x;LL} = \hat{\phi}_{x;L}^* \hat{\phi}_{x;L}$.

Proposition 2 *The operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ can be expressed in terms of the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}^*$.*

Proof: For nondiagonal elements we have immediately from Proposition 1:

$$\hat{\mathcal{L}}_\alpha^{PK} = [(\hat{\mathcal{M}}_{\beta;L}^P)^*, \hat{\mathcal{M}}_{\gamma;L}^K], \alpha = \beta^{-1}\gamma, \quad (4.8)$$

$$\hat{\mathcal{R}}_{\alpha;LQ} = [\hat{\mathcal{M}}_{\gamma;L}^K, (\hat{\mathcal{M}}_{\beta;Q}^K)^*], \alpha = \gamma\beta^{-1}, \quad (4.9)$$

where β and γ are arbitrary paths chosen in such a way that $\alpha = \beta^{-1}\gamma$ or $\alpha = \gamma\beta^{-1}$ respectively. Obviously, the left hand sides of (4.8) and (4.9) do not depend upon this arbitrary splitting of α . It remains to consider the case of diagonal elements $\hat{\mathcal{L}}_x^{KK}$ and $\hat{\mathcal{R}}_{x;LL}$, that means elements $\hat{\mathcal{L}}_\alpha^{KK}$ and $\hat{\mathcal{R}}_{\alpha;LL}$ with α being a trivial path at x . For that purpose we define the following projectors:

$$\hat{P}_{x,y;L}^K = \hat{\mathcal{M}}_{\gamma;L}^K (\hat{\mathcal{M}}_{\gamma;L}^K)^* = \hat{\phi}_x^K \hat{\phi}_x^{K*} \hat{\phi}_{y;L} \hat{\phi}_{y;L}^*, \quad (4.10)$$

$$\hat{Q}_{x,y;L}^K = (\hat{\mathcal{M}}_{\gamma;L}^K)^* \hat{\mathcal{M}}_{\gamma;L}^K = \hat{\phi}_x^{K*} \hat{\phi}_x^K \hat{\phi}_{y;L}^* \hat{\phi}_{y;L}, \quad (4.11)$$

where γ is an arbitrary path from x to y . Obviously, these operators depend only upon the endpoints x and y , and not on the chosen path. Moreover, they all commute with each other. It is also useful to introduce the following operators:

$$\hat{K}_{x,y;L}^K := \hat{Q}_{x,y;L}^K - \hat{P}_{x,y;L}^K = [(\hat{\mathcal{M}}_{\gamma;L}^K)^*, \hat{\mathcal{M}}_{\gamma;L}^K]. \quad (4.12)$$

One may easily check the following identity:

$$\hat{K}_{x,y;L}^K = \hat{L}_x^{KK} - \hat{R}_{y;LL} . \quad (4.13)$$

These equations may be easily solved with respect to the operators $\hat{L}$ and $\hat{R}$, provided the total charge $\hat{Q}$ is given. For this purpose observe that the total number of positrons $\hat{N}^p$ and the total number of electrons $\hat{N}^e$ are equal to

$$\hat{N}^p = \sum_{K,x} \hat{L}_x^{KK} , \quad (4.14)$$

$$\hat{N}^e = N\mathbb{1} - \sum_{L,y} \hat{R}_{y;LL} , \quad (4.15)$$

where N is the number of positron (and also electron) degrees of freedom, (twice the number of lattice sites). Summing up equations (4.13) over all indices we obtain

$$\hat{N}^p + \hat{N}^e = \frac{1}{N} \sum_{K,x} \sum_{L,y} \hat{K}_{x,y;L}^K + N\mathbb{1} . \quad (4.16)$$

On the other hand, we have

$$\hat{N}^p - \hat{N}^e = \frac{1}{e} \hat{Q} . \quad (4.17)$$

This enables us to calculate both $\hat{N}^p$ and $\hat{N}^e$ and, finally, to obtain $\hat{L}_x^{KK}$ by summing up equation (4.13) with respect to (L, y) and $\hat{R}_{y;LL}$ by summing it up with respect to (K, x) . The final result is:

$$\hat{L}_x^{KK} = \frac{1}{N} \sum_{L,y} \hat{K}_{x,y;L}^K - \frac{1}{2N^2} \sum_{R,u} \sum_{S,v} \hat{K}_{u,v;S}^R + \frac{1}{2} \mathbb{1} + \frac{1}{2Ne} \hat{Q} , \quad (4.18)$$

$$\hat{R}_{y;LL} = -\frac{1}{N} \sum_{K,x} \hat{K}_{x,y;L}^K + \frac{1}{2N^2} \sum_{R,u} \sum_{S,v} \hat{K}_{u,v;S}^R + \frac{1}{2} \mathbb{1} + \frac{1}{2Ne} \hat{Q} . \quad (4.19)$$

In the last step, we will express also the total charge operator $\hat{Q}$ in terms of the projectors (4.11) and (4.10). For that purpose denote by $\mathcal{N}$ the set of all positron (or electron) degrees of freedom, defined as the product of the set of all lattice sites by the set $\{1, 2\}$ of all possible helicity values. For each pair (V, W) of subsets of $\mathcal{N}$ we are going to construct projectors $\hat{Q}_{(V,W)}$ (or $\hat{P}_{(V,W)}$, respectively), which correspond to the following, physical questions: “Are all positron degrees of freedom in V and all electron degrees of freedom in W *fully occupied* (or *fully unoccupied*, respectively)?”. If none of (V, W) is empty, we define

$$\hat{Q}_{(V,W)} := \prod_{(K,x) \in V} \prod_{(L,y) \in W} \hat{Q}_{x,y;L}^K , \quad (4.20)$$

$$\hat{P}_{(V,W)} := \prod_{(K,x) \in V} \prod_{(L,y) \in W} \hat{P}_{x,y;L}^K . \quad (4.21)$$

(The order of factors is irrelevant, due to the commutativity of operators $\hat{P}$ and $\hat{Q}$.) If both V and W are empty, we put $\hat{Q}_{(\emptyset,\emptyset)} = \hat{P}_{(\emptyset,\emptyset)} = 1$. Finally, if only one of them is empty, we proceed as follows: For V not empty (resp. empty) and W empty (resp. not empty), we choose any pair $((K_0, x_0), (L_0, y_0))$ such that $(K_0, x_0) \notin V$ (resp. is arbitrary) and (L_0, y_0) is arbitrary (resp. $(L_0, y_0) \notin W$). Next, we choose an arbitrary path γ from x_0 to y_0 and put

$$\hat{Q}_{(V,\emptyset)} := \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right) \hat{Q}_{(V,\{(L_0,y_0)\})} \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right)^* , \quad (4.22)$$

$$\hat{P}_{(V,\emptyset)} := \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right)^* \hat{P}_{(V,\{(L_0,y_0)\})} \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right) , \quad (4.23)$$

$$\hat{Q}_{(\emptyset,W)} := \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right) \hat{Q}_{(\{(K_0,x_0)\},W)} \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right)^* , \quad (4.24)$$

$$\hat{P}_{(V,\emptyset)} := \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right)^* \hat{P}_{(\{(K_0,x_0)\},W)} \left(\hat{\mathcal{M}}_{\gamma;L_0}^{K_0} \right) . \quad (4.25)$$

For any integer $Z \in [-N+1, N-1]$ we define the following projector:

$$\hat{P}_Z := \sum_{\#V - \#W = Z} \hat{Q}_{(V,W)} \hat{P}_{(\mathcal{N} \setminus V, \mathcal{N} \setminus W)} , \quad (4.26)$$

where by $\#V$ and $\#W$ we denote the number of elements of the sets V and W , respectively. It is easy to check – using the wave function representation of $\mathcal{H}^0$ – that the above projectors give the spectral decomposition of the total charge, i. e. that the following is true:

$$\hat{Q} = e \sum_Z Z \hat{P}_Z . \quad (4.27)$$

□

It follows from this Proposition that also every local charge operator $\hat{j}_x^0$ can be expressed in terms of $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}^*$:

$$\begin{aligned} \hat{j}_x^0 &= e \left(\sum_K \hat{L}_x^{KK} - \sum_L (1 - \hat{R}_{y;LL}) \right) \\ &= \frac{e}{N} \left(\sum_{K,L,y} \hat{K}_{x,y;L}^K - \sum_{K,L,y} \hat{K}_{y,x;L}^K \right) + \frac{2}{N} \hat{Q} , \end{aligned} \quad (4.28)$$

where $\hat{Q}$ is given by (4.27).

We see, therefore, that the algebra $\tilde{\mathcal{O}}$ is generated by the family $\{\hat{E}, \hat{B}, \hat{\mathcal{M}}, \hat{\mathcal{M}}^*\}$. These generators are, however, not independent. There is a number of operator identities between them, which we list below.

Proposition 3 *We have*

1.

$$\text{div}_{x;\hat{k},\hat{l},\hat{n}} \hat{B} = 0 . \quad (4.29)$$

2. Any pair of operators $\hat{\mathcal{M}}$ commutes:

$$[\hat{\mathcal{M}}_{\alpha;L}^K, \hat{\mathcal{M}}_{\beta;N}^M] = 0 . \quad (4.30)$$

3. Pair-annihilation operators along two different paths γ and β , having common ends x and y , are related by:

$$\hat{\mathcal{M}}_{\gamma;L}^K = \exp(-ig\hat{B}_{(\gamma\beta^{-1})})\hat{\mathcal{M}}_{\beta;L}^K , \quad (4.31)$$

where $\hat{B}_{(\gamma\beta^{-1})}$ denotes the magnetic flux through the closed path $\gamma\beta^{-1}$.

4. The operators $\hat{\mathcal{M}}_{\alpha;L}^K$ are nilpotent. Moreover, for two paths α and β

$$\hat{\mathcal{M}}_{\alpha;L}^K \hat{\mathcal{M}}_{\beta;M}^K = 0 , \quad (4.32)$$

if α and β have the same starting point x , and

$$\hat{\mathcal{M}}_{\alpha;L}^K \hat{\mathcal{M}}_{\beta;L}^M = 0 , \quad (4.33)$$

if α and β have the same end point y .

5. Let α connect x with y , β connect z with u and γ connect t with w . Denote by $\delta_{(x,K)(z,M)}$ the Kronecker symbol, which vanishes if $(x,K) \neq (z,M)$ and takes the value equal to one if both points and indices coincide. Then:

$$\begin{aligned} [\hat{\mathcal{M}}_{\alpha;L}^K, \hat{\mathcal{M}}_{\beta;N}^{M*}, \hat{\mathcal{M}}_{\gamma;R}^P] &= \delta_{(x,K)(z,M)}\delta_{(u,N)(w,R)}\hat{\mathcal{M}}_{\alpha\beta^{-1}\gamma;L}^P \\ &+ \delta_{(y,L)(u,N)}\delta_{(t,P)(z,M)}\hat{\mathcal{M}}_{\gamma\beta^{-1}\alpha;R}^K , \end{aligned} \quad (4.34)$$

where we have denoted $[\cdot, \cdot, \cdot] := [[\cdot, \cdot], \cdot] \equiv [\cdot, [\cdot, \cdot]]$, (both definitions are equivalent, because the first and the last variable commute).

6.

$$[\hat{\mathcal{M}}_{\alpha;L}^K, \hat{B}_{x;\hat{k},\hat{l}}] = 0 , \quad (4.35)$$

$$[\hat{\mathcal{M}}_{\alpha;L}^K, \hat{E}_{x,x+\hat{k}}] = e\hbar\delta_{\alpha,(x,x+\hat{k})}\hat{\mathcal{M}}_{\alpha;L}^K , \quad (4.36)$$

$$[\hat{B}_{x;\hat{k},\hat{l}}, \hat{E}_{y,y+\hat{n}}] = i\hbar\delta_{\partial(x;\hat{k},\hat{l}), (y,y+\hat{n})} . \quad (4.37)$$

Here $\delta_{\alpha,(x,x+\hat{k})} = 0$ if $(x,x+\hat{k}) \notin \alpha$, $\delta_{\alpha,(x,x+\hat{k})} = 1$ if $(x,x+\hat{k}) \in \alpha$ and has the same orientation as α and $\delta_{\alpha,(x,x+\hat{k})} = -1$ otherwise. In the last formula we have denoted by $\partial(x;\hat{k},\hat{l})$ a closed path – the boundary of the oriented plaquette $(x;\hat{k},\hat{l})$.

Proof: Identity (4.29) was already shown above, see (4.2). Identities (4.30) – (4.36) follow easily from the canonical (anti-) commutation relations (2.10) and (2.11) and the commutation relations (4.37) are an immediate consequence of the commutation relations (2.11). $\square$

Remark 1 *The commutators containing unbounded operators $\hat{E}$ and $\hat{B}$ have to be understood in their “bounded version”. In particular, the canonical commutation relations (4.37) have to be understood in the sense of the Weyl commutation relations for the one-parameter groups generated by $\hat{E}$ and $\hat{B}$.*

5 Algebra of Observables and Tree Decomposition

The algebra $\tilde{\mathcal{O}}$ is, of course, unphysical, because it does not respect the Gauss law. Its representation in $\mathcal{H}^0$ is highly reducible. To construct the physical observable algebra $\mathcal{O}$ we additionally impose the Gauss law

$$\operatorname{div}_x \hat{E} = \hat{j}_x^0, \quad (5.1)$$

where $\hat{j}_x^0$ is defined by $\hat{\mathcal{M}}$ according to formulae (4.28) and (4.27). We take this equation as an additional relation between generators.

Thus, let us start with the $*$ -algebra generated by the family of abstract elements $\{\hat{E}, \hat{B}, \hat{\mathcal{M}}, \hat{\mathcal{M}}^*\}$ (with $\hat{E}$ and $\hat{B}$ being self-adjoint), satisfying axioms (4.29) – (4.37) together with (5.1), where the charges $\hat{j}_x^0$ are given by (4.28) and (4.27). We will prove in this section that these axioms (with canonical commutation relations understood in the sense of Weyl) define, indeed, uniquely a von Neumann algebra, which we shall call *Algebra of Observables* of our model and denote by $\mathcal{O}$.

Our main tool will be the notion of a *tree*. By a tree we mean a pair $(x_0, \mathcal{T})$, where $x_0 \in \Lambda$ is a lattice site, which we call the *root* of the tree and $\mathcal{T}$ is a subset of links of the lattice, having the following property: for every site $x \in \Lambda$ there is one and only one path composed of links belonging to $\mathcal{T}$, which connects the root x_0 with x . We denote this path by $(x_0, x)_{\mathcal{T}}$.

The simplest example of a tree is obtained as follows. Choose any root and take as $\mathcal{T}$ the following collection of links

1. all the links $(x, x + \hat{3})$ belonging to the x^3 -axis passing through the root,
2. all the links $(x, x + \hat{2})$, belonging to the two-dimensional plane (x^2, x^3) passing through the root,
3. all the links $(x, x + \hat{1})$.

With every *off-tree* link, $(x, x + \hat{k}) \notin \mathcal{T}$, we associate the unique closed path composed of:

1. the tree-path $(x_0, x)_\mathcal{T}$,
2. the link $(x, x + \hat{k})$ itself,
3. the inverse tree-path $(x + \hat{k}, x_0)_\mathcal{T}$.

For any surface (finite number of plaquettes), such that the above closed path is its boundary, we denote the operator of total magnetic flux through it, i.e. the sum of all operators $\hat{B}_{x;\hat{k},i}$ corresponding to this surface, by $\hat{B}_{x,x+\hat{k}}(\mathcal{T})$. Of course, this quantity does not depend upon the choice of the surface, because the divergence of $\hat{B}$ vanishes. We call these quantities along-tree magnetic fluxes. From (4.37) we have

$$[\hat{B}_{x,x+\hat{k}}(\mathcal{T}), \hat{E}_{y,y+\hat{l}}] = i\hbar\delta_{xy}\delta_{\hat{k}\hat{l}}, \quad (5.2)$$

which means that the operators canonically conjugate to the along-tree magnetic fluxes are equal to the off-tree electric fluxes. From now on we denote them by $\hat{E}_{y,y+\hat{l}}(\mathcal{T})$.

Finally, let us describe all electron and positron degrees of freedom as follows: For operators $\hat{\phi}_x^K$ and $\hat{\varphi}_{y,L}$ we write $\hat{\phi}_i$ and $\hat{\varphi}_j$, where $i = (x, K)$, $j = (y, L)$ label all possible values of indices K and L , and all lattice sites x and y . We denote by $\hat{m}_{ij}$ those generators $\hat{\mathcal{M}}$, which correspond to on-tree paths:

$$\hat{m}_{ij} := \hat{\mathcal{M}}_{\gamma,L}^K, \quad (5.3)$$

where $i = (x, K)$, $j = (y, L)$ and γ denotes the unique “on-tree” path connecting the lattice site x with the lattice site y .

Definition 2 For a given tree $\mathcal{T}$ and a given family $\{\hat{E}, \hat{B}, \hat{\mathcal{M}}, \hat{\mathcal{M}}^*\}$ of generators fulfilling our axioms, we call the family $\{\hat{E}_{y,y+\hat{l}}(\mathcal{T}), \hat{B}_{x,x+\hat{k}}(\mathcal{T}), \hat{m}_{ij}, \hat{m}_{ij}^*\}$ the tree data of $\{\hat{E}, \hat{B}, \hat{\mathcal{M}}, \hat{\mathcal{M}}^*\}$.

Observe that the tree data inherit the following properties from axioms (4.30) – (4.37):

Proposition 4

1.

$$[\hat{m}_{ij}, \hat{m}_{kl}] = 0, \quad (5.4)$$

2.

$$\hat{m}_{ij}\hat{m}_{il} = 0 = \hat{m}_{ij}\hat{m}_{kj}, \quad (5.5)$$

3.

$$[\hat{m}_{ij}, \hat{m}_{kl}^*, \hat{m}_{rs}] = \delta_{ki}\delta_{ls}\hat{m}_{rj} + \delta_{kr}\delta_{lj}\hat{m}_{is}. \quad (5.6)$$

4.

$$[m_{ij}, \hat{B}_{x;x+\hat{k}}(\mathcal{T})] = 0, \quad (5.7)$$

$$[m_{ij}, \hat{E}_{x,x+\hat{k}}(\mathcal{T})] = 0, \quad (5.8)$$

$$[\hat{B}_{x;x+\hat{k}}(\mathcal{T}), \hat{E}_{y,y+\hat{l}}(\mathcal{T})] = i\hbar\delta_{xy}\delta_{\hat{k}\hat{l}}. \quad (5.9)$$

Actually, (5.4) and (5.5) may be replaced by the following (apparently stronger) condition:

$$\hat{m}_{ij}\hat{m}_{kl} = -\hat{m}_{il}\hat{m}_{kj} = \hat{m}_{kl}\hat{m}_{ij}. \quad (5.10)$$

Of course, (5.10) contains (5.4) and (5.5). It is, however, easy to check that (5.4) and (5.5), together with (5.6), imply (5.10). Moreover, let us observe that the composition law (5.6) contains, as special cases, the following identities, which will be used in what follows:

$$\hat{m}_{ij}\hat{m}_{ij}^*\hat{m}_{ij} = \hat{m}_{ij}, \quad (5.11)$$

$$\hat{m}_{ij}\hat{m}_{kj}^*\hat{m}_{ij} = 0 \text{ if } k \neq i, \quad (5.12)$$

$$\hat{m}_{ij}\hat{m}_{il}^*\hat{m}_{ij} = 0 \text{ if } l \neq j, \quad (5.13)$$

$$\hat{m}_{ij} = [\hat{m}_{il}, \hat{m}_{kl}^*, \hat{m}_{kj}] \text{ if } k \neq i \text{ or } l \neq j. \quad (5.14)$$

It will be seen in the next Section that the $*$ -algebra $\mathcal{O}_m$ generated by operators $\hat{m}$ and $\hat{m}^*$, fulfilling axioms (5.4) – (5.6), is finite dimensional.

Remark 2 *In the present paper we conclude this fact from the classification of all irreducible representations of $\mathcal{O}_m$. Alternatively, one can deduce it from an additional structure carried by this algebra, which will be discussed in a separate paper. More precisely, $\mathcal{O}_m$ can be constructed in terms of the enveloping algebra of a Lie algebra generated by $\{\hat{m}, \hat{m}^*, \hat{l}, \hat{r}\}$, where $\hat{l}$ and $\hat{r}$ are constructed from $\hat{m}$ and $\hat{m}^*$ in analogy to formulae (4.8), (4.9), (4.18) and (4.19).*

Theorem 1 *Let there be given a tree $\mathcal{T}$. Every family $\{\hat{E}, \hat{B}, \hat{\mathcal{M}}, \hat{\mathcal{M}}^*\}$ of generators fulfilling axioms (4.29) – (4.37) together with (5.1) and (4.28) is in one-to-one correspondence with its tree data, fulfilling identities (5.4) – (5.9).*

Proof: It is enough to show that every family fulfilling our axioms can be uniquely reconstructed from the tree data, fulfilling (5.4) – (5.9). First, it is obvious that the magnetic flux through every plaquette of Λ may be calculated provided we know all fluxes $\hat{B}_{x,x+\hat{k}}(\mathcal{T})$. For this purpose we first calculate the values $\hat{B}_{x;\hat{k},\hat{l}}$ corresponding to

the “along-tree-plaquettes”, as the difference of the subsequent fluxes $\hat{B}_{x,x+\hat{k}}(\mathcal{T})$, starting from the root and moving towards the end-points of the tree. Finally, we use identity (4.2) to calculate the remaining (“off-tree”) fluxes. This way this identity is automatically fulfilled by the reconstructed full magnetic field configuration.

Next, using formula (4.31), all operators $\hat{\mathcal{M}}$ may be expressed in terms of operators $\hat{m}$ and $\hat{B}$. This way identity (4.31) is automatically fulfilled by the reconstructed quantities. Also the remaining identities for the operators $\hat{\mathcal{M}}$ are trivially implied by the corresponding identities of the operators $\hat{m}$. Now, we get from equation (4.28) all charges $\hat{j}_x^0$. But knowing all off-tree quantities $\hat{\mathcal{E}}_{x,x+\hat{k}}(\mathcal{T})$ and all charges $\hat{j}_x^0$ we are able to solve the equation $\text{div} \hat{E} = \hat{j}^0$ and to calculate all the remaining (i. e. on-tree) electric operators $\hat{E}$. For this purpose we have to solve the equation at each site x , starting from the tree end-points and moving towards the tree root. Indeed, an end-point x of the tree is characterized by the condition that *only one* among the lattice links starting from x belongs to the tree. Hence, knowing $\hat{E}$ on the remaining links and knowing $\hat{j}_x^0$ we may calculate $\hat{E}$ on this particular link. Obviously, this procedure may be continued when we move towards the root. At points belonging to the boundary $\partial\Lambda$ we must use also the values of external fluxes $\hat{E}_{x,\infty} = E_{x,\infty} \mathbf{1}$, which are supposed to be fixed *a priori*. The compatibility condition at the root is automatically fulfilled for consistent boundary data in the sense of formula (3.17). This way the Gauss law constraint will be satisfied by the reconstructed quantities.

It is also obvious that (4.35) is fulfilled by the reconstructed data. Thus, it remains to show that identities (4.36) and (4.37) hold. The proof is given in Appendix A. $\square$

Due to this Theorem, the $*$ -algebra generated by $\{\hat{E}, \hat{B}, \hat{\mathcal{M}}, \hat{\mathcal{M}}^*\}$ fulfilling axioms (4.29) – (4.37) together with (5.1) and (4.28), is isomorphic to the $*$ -algebra generated by the tree data, fulfilling identities (5.4) – (5.9). Due to (5.2), the first two components, $\hat{\mathcal{E}}(\mathcal{T})$ and $\hat{\mathcal{B}}(\mathcal{T})$, fulfill the commutation relations of a finitely generated Heisenberg algebra. We take the corresponding Weyl-algebra generated by them and denote its strong closure by $\mathcal{O}_{\text{e-m}}(\mathcal{T})$. This algebra may be represented as $B(L^2(\hat{\mathcal{B}}(\mathcal{T})))$, the algebra of bounded operators acting on the Hilbert space of L^2 -integrable functions depending on classical variables $\hat{\mathcal{B}}(\mathcal{T})$, with $\hat{\mathcal{B}}_{x,x+\hat{k}}(\mathcal{T})$ defined as multiplication and $\hat{\mathcal{E}}_{x,x+\hat{k}}(\mathcal{T})$ as differentiation operators (Schrödinger representation). It follows from (5.7) and (5.8) that $\mathcal{O}_{\text{m}}$ and $\mathcal{O}_{\text{e-m}}(\mathcal{T})$ commute. Thus, the C^* -algebra generated by $(\hat{\mathcal{E}}(\mathcal{T}), \hat{\mathcal{B}}(\mathcal{T}), \hat{m}, \hat{m}^*)$ is the tensor product of $\mathcal{O}_{\text{e-m}}(\mathcal{T})$ and $\mathcal{O}_{\text{m}}$.

Consequently, we have the following

Definition 3 *The observable algebra $\mathcal{O}$ is defined as*

$$\mathcal{O} := \mathcal{O}_{\text{e-m}}(\mathcal{T}) \otimes \mathcal{O}_{\text{m}}. \quad (5.15)$$

Remark 3 *Due to Theorem 1 we may identify elements of $\mathcal{O}$ reconstructed from tree data corresponding to different trees. It is easy to check that this identification is an*

isomorphism of algebras. Hence, the algebras $\mathcal{O}$ generated from data corresponding to different trees coincide. Consequently, the definition of $\mathcal{O}$ does not depend upon the tree.

Remark 4 *The generators $\hat{m}$ are non-local. But due to (5.6) they can be expressed in terms of local quantities, namely annihilation operators of pairs located at the same lattice site or pairs separated by one link. This aspect will be discussed in more detail in Section 9.*

There is an interesting relation to a theory of C^* -algebras generated by unbounded elements recently developed by Woronowicz, see [30]. The following fact is a simple consequence of a general theorem contained in his paper. Let us take the C^* -algebra $\mathcal{A}$ of all compact operators acting on the Hilbert space $L^2(\mathcal{B}(\mathcal{T}))$. Due to [30] – this algebra is generated (in a quite complicated sense) by $(\hat{\mathcal{E}}(\mathcal{T}), \hat{\mathcal{B}}(\mathcal{T}))$ and these generators are affiliated with $\mathcal{A}$ in the C^* -sense, see [31]. This way the canonical commutation relations (5.9) can be considered as relations among affiliated elements and we can extend the notion of observables to generators treated as affiliated elements in the sense just mentioned. A natural question arises, namely, whether or not one can omit our construction based on the above tree decomposition and directly define the algebra generated by our axioms, considered as relations between unbounded generators. This is an interesting question, which we are going to consider in the future.

6 Charge Superselection Rule

In this section we prove that the total charge operator defined by (4.27) is a central element of the observable algebra. For this purpose we first list some properties of $\mathcal{O}$.

Lemma 1 *Operators $\hat{m}$ satisfy the following identity:*

$$[\hat{m}_{kl}^*, \hat{m}_{ij}] = 0 \quad \text{if } k \neq i \text{ and } l \neq j. \quad (6.1)$$

Proof. Due to (5.14) we have

$$\hat{m}_{kl}^* \hat{m}_{ij} = \hat{m}_{kl}^* (\hat{m}_{il} \hat{m}_{kl}^* \hat{m}_{kj} + \hat{m}_{kj} \hat{m}_{kl}^* \hat{m}_{il} - \hat{m}_{il} \hat{m}_{kj} \hat{m}_{kl}^* - \hat{m}_{kl}^* \hat{m}_{il} \hat{m}_{kj}) = -\hat{m}_{kl}^* \hat{m}_{il} \hat{m}_{kj} \hat{m}_{kl}^*, \quad (6.2)$$

because the three other terms vanish by virtue of the nilpotency properties of $\hat{m}$'s, together with identities (5.12) and (5.13). Similarly,

$$\hat{m}_{ij} \hat{m}_{kl}^* = (\hat{m}_{il} \hat{m}_{kl}^* \hat{m}_{kj} + \hat{m}_{kj} \hat{m}_{kl}^* \hat{m}_{il} - \hat{m}_{il} \hat{m}_{kj} \hat{m}_{kl}^* - \hat{m}_{kl}^* \hat{m}_{il} \hat{m}_{kj}) \hat{m}_{kl}^* = -\hat{m}_{kl}^* \hat{m}_{il} \hat{m}_{kj} \hat{m}_{kl}^*, \quad (6.3)$$

which ends the proof. $\square$

In the notation used now, the projectors corresponding to (4.10) and (4.11) take the form:

$$\hat{P}_{kl} := \hat{m}_{kl} \hat{m}_{kl}^* , \quad (6.4)$$

$$\hat{Q}_{kl} := \hat{m}_{kl}^* \hat{m}_{kl} . \quad (6.5)$$

Physically, $\hat{Q}_{ij}$ corresponds to the question “Are the degrees of freedom (i, j) fully occupied?” and $\hat{P}_{ij}$ corresponds to the question “Are the degrees of freedom (i, j) fully unoccupied, i. e. is there vacuum in (i, j) ?”

It follows from (5.11) and (5.5) that they obey the following, obvious identities:

$$\hat{Q}_{kl} \hat{m}_{kj} = \hat{m}_{kl} \hat{P}_{kj} = 0 , \quad (6.6)$$

$$\hat{Q}_{kj} \hat{m}_{ij} = \hat{m}_{kj} \hat{P}_{ij} = 0 , \quad (6.7)$$

$$\hat{P}_{kj} \hat{Q}_{kl} = \hat{Q}_{kl} \hat{P}_{kj} = 0 , \quad (6.8)$$

$$\hat{P}_{ij} \hat{Q}_{kj} = \hat{Q}_{kj} \hat{P}_{ij} = 0 , \quad (6.9)$$

$$\hat{m}_{kl} \hat{Q}_{kl} = \hat{P}_{kl} \hat{m}_{kl} = \hat{m}_{kl} , \quad (6.10)$$

$$\hat{m}_{kl}^* \hat{P}_{kl} = \hat{Q}_{kl} \hat{m}_{kl}^* = \hat{m}_{kl}^* . \quad (6.11)$$

Moreover, we have

Lemma 2 *The following identities hold:*

$$[\hat{P}_{ij}, \hat{Q}_{kl}] = 0 , \quad (6.12)$$

$$[\hat{P}_{ij}, \hat{P}_{kl}] = 0 , \quad (6.13)$$

$$[\hat{Q}_{ij}, \hat{Q}_{kl}] = 0 , \quad (6.14)$$

$$\hat{P}_{ij} \hat{P}_{kl} = \hat{P}_{il} \hat{P}_{kj} , \quad (6.15)$$

$$\hat{Q}_{ij} \hat{Q}_{kl} = \hat{Q}_{il} \hat{Q}_{kj} . \quad (6.16)$$

Proof. Identity (6.1) implies that $\hat{P}_{ij}$ and $\hat{Q}_{ij}$ commute with $\hat{P}_{kl}$ and $\hat{Q}_{kl}$ if $k \neq i$ and $l \neq j$. They also commute if $k = i$ and $l = j$. Identities (6.8) and (6.9) imply that $\hat{P}_{ij}$ commutes with $\hat{Q}_{kl}$ also in the case of one common index. It is, therefore, sufficient to prove that $\hat{P}_{ij}$ commutes with $\hat{P}_{kl}$ and that $\hat{Q}_{ij}$ commutes with $\hat{Q}_{kl}$ in the case of one common index. Below we give the proof for one of the four possible cases:

$$\begin{aligned} \hat{P}_{kj} \hat{P}_{kl} &= \hat{m}_{kj} \hat{m}_{kj}^* \hat{m}_{kl} \hat{m}_{kl}^* = (\hat{m}_{kl} - \hat{m}_{kl} \hat{m}_{kj}^* \hat{m}_{kj}) \hat{m}_{kl}^* \\ &= \hat{m}_{kl} \hat{m}_{kl}^* - \hat{m}_{kl} (\hat{m}_{kl}^* - \hat{m}_{kl}^* \hat{m}_{kj} \hat{m}_{kj}^*) = \hat{P}_{kl} \hat{P}_{kj} , \end{aligned} \quad (6.17)$$

where we have used identity (5.14) twice.

To prove identity (6.15) observe, that it follows from (6.13) if $k = i$ or $l = j$. If the contrary is true, we use Lemma 1 together with (5.5):

$$\begin{aligned}\hat{P}_{ij}\hat{P}_{kl} &= \hat{m}_{ij}\hat{m}_{ij}^*\hat{m}_{kl}\hat{m}_{kl}^* = \hat{m}_{il}\hat{m}_{kj}(\hat{m}_{il}\hat{m}_{kj})^* \\ &= \hat{m}_{il}\hat{m}_{il}^*\hat{m}_{kj}\hat{m}_{kj}^* = \hat{P}_{il}\hat{P}_{kj} .\end{aligned}\tag{6.18}$$

The proof of (6.16) is completely analogous. $\square$

Theorem 2 *The total charge operator defines a superselection rule in $\mathcal{O}$. We have*

$$\mathcal{O} = \bigoplus_Q \{ \mathcal{O}_{\text{e-m}}(\mathcal{T}) \otimes \mathcal{O}_{\text{m}}(Q) \} ,\tag{6.19}$$

where $\mathcal{O}_{\text{m}}(Q)$ are the central decomposition components of $\mathcal{O}_{\text{m}}$.

Proof: It is sufficient to show that all generators $\hat{m}$ commute with $\hat{Q}$ or, equivalently, with every projector $\hat{P}_Z$ given by formula (4.26). For this purpose let us observe that for $k \in V$ and $l \notin W$ and, similarly, for $k \notin V$ and $l \in W$, formulae (6.6) and (6.7) imply:

$$\hat{m}_{kl}\hat{Q}_{(V,W)}\hat{P}_{(\mathcal{N}\setminus V, \mathcal{N}\setminus W)} = \hat{Q}_{(V,W)}\hat{P}_{(\mathcal{N}\setminus V, \mathcal{N}\setminus W)}\hat{m}_{kl} = 0 .\tag{6.20}$$

The only nontrivial contribution to the commutator of $\hat{m}_{kl}$ with $\hat{P}_Z$ may come from those terms of the right-hand side of (4.26) which correspond to $k \in V$ and $l \in W$ or those for which $k \notin V$ and $l \notin W$. Let us begin with the first possibility. Due to (6.16) we have

$$\hat{Q}_{(V,W)} = \hat{Q}_{kl}\hat{Q}_{(V\setminus\{k\}, W\setminus\{l\})} .\tag{6.21}$$

Hence, we have

$$\begin{aligned}[\hat{m}_{kl}, \hat{Q}_{(V,W)}\hat{P}_{(\mathcal{N}\setminus V, \mathcal{N}\setminus W)}] &= \hat{m}_{kl}\hat{Q}_{kl}\hat{Q}_{(V\setminus\{k\}, W\setminus\{l\})}\hat{P}_{(\mathcal{N}\setminus V, \mathcal{N}\setminus W)} \\ &= \hat{P}_{kl}\hat{m}_{kl}\hat{Q}_{(V\setminus\{k\}, W\setminus\{l\})}\hat{P}_{(\mathcal{N}\setminus V, \mathcal{N}\setminus W)} .\end{aligned}\tag{6.22}$$

Let us denote $V' := V \setminus \{k\}$ and $W' := W \setminus \{l\}$. Due to (6.15) we have

$$\hat{P}_{(V',W')} = \hat{P}_{kl}\hat{P}_{(\mathcal{N}\setminus V, \mathcal{N}\setminus W)} .\tag{6.23}$$

Hence, we have

$$[\hat{m}_{kl}, \hat{Q}_{(V,W)}\hat{P}_{(\mathcal{N}\setminus V, \mathcal{N}\setminus W)}] = \hat{Q}_{(V',W')}\hat{P}_{(\mathcal{N}\setminus V', \mathcal{N}\setminus W')}\hat{m}_{kl} .\tag{6.24}$$

Now we consider the second possibility. If $k \notin V'$ and $l \notin W'$, then equations (6.6) and (6.7) imply:

$$[\hat{m}_{kl}, \hat{Q}_{(V',W')}\hat{P}_{(\mathcal{N}\setminus V', \mathcal{N}\setminus W')}] = -\hat{Q}_{(V',W')}\hat{P}_{(\mathcal{N}\setminus V', \mathcal{N}\setminus W')}\hat{m}_{kl} .\tag{6.25}$$

We see that these contributions cancel each other in the commutator of $\hat{m}_{kl}$ with $\hat{P}_Z$. Consequently, $\hat{Q}$ is a central element of $\mathcal{O}$. Thus, the algebra $\mathcal{O}_{\text{m}}$ splits – due to the central decomposition theorem [32] – into a direct sum of components $\mathcal{O}_{\text{m}}(Q)$, corresponding to fixed eigenvalues Q of the total charge. $\square$

It will be seen in the next section that the total charge $\hat{Q}$ generates the whole center of $\mathcal{O}$.

7 Uniqueness of Irreducible Representations and Charge Superselection Sectors

Concerning representations of the algebra $\mathcal{O}_{e-m}(\mathcal{T})$ we restrict ourselves to strongly continuous representations of the Weyl relations. Due to [1] all strongly continuous representations of the Weyl relations are unitarily equivalent to at most a countable sum of copies of the Schrödinger representation. Thus, all representations of the observable algebra $\mathcal{O}$ may be constructed from the Schrödinger representation and from representations of $\mathcal{O}_m$. In this Section we will construct all irreducible representations of the latter algebra. We are going to list only *faithful* representations, i.e. such that none of the generators $\hat{m}$ is represented trivially by the zero operator. If a representation is not faithful, e. g. if $\hat{m}_{i_0 j_0}$ vanishes identically for some pair (i_0, j_0) of indices, then due to formula (5.14) we have also $\hat{m}_{i_0 l} \equiv 0$ and $\hat{m}_{k j_0} \equiv 0$, for any k and l . This means that the degrees of freedom i_0 and j_0 may be neglected and the representation is in fact a faithful representation of a smaller number of degrees of freedom. Thus, our list contains *all* irreducible representations of the algebra.

In a first step, we are going to define a family of *canonical* representations of $\mathcal{O}_m$. We will show in the sequel that every irreducible representation is isomorphic to one of them. For this purpose we denote $\mathcal{N} = \{1, 2, \dots, N\}$, where N is – as in Section 4 – the number of all positron degrees of freedom $\hat{\phi}_i$ (and also the number of all electron degrees of freedom $\hat{\varphi}_j$). For any integer $Z \in [-N + 1, N - 1]$ we define a finite-dimensional representation of $\mathcal{O}_m$ in the following way.

We denote by $S_{\mathcal{N}}$ the set of all subsets of $\mathcal{N}$ and take the free vector space

$$\mathcal{H}_m(\mathcal{T}) := \bigoplus_{(I,J) \in S_{\mathcal{N}} \times S_{\mathcal{N}}} \mathcal{H}_{(I,J)}, \quad \mathcal{H}_{(I,J)} \cong \mathbb{C},$$

over $S_{\mathcal{N}} \times S_{\mathcal{N}}$ (see [29]). Next we restrict ourselves to the subspace $\mathcal{H}_Z$ of such vectors that the number $\#I$ of elements of I differs from the number $\#J$ of elements of J exactly by Z :

$$\mathcal{H}_Z := \bigoplus_{\#I - \#J = Z} \mathcal{H}_{(I,J)}. \quad (7.1)$$

We endow $\mathcal{H}_Z$ with a Hilbert space structure by choosing in each subspace $\mathcal{H}_{(I,J)} \cong \mathbb{C}$ the unit number $\Omega_{(I,J)} := 1_{(I,J)} \in \mathcal{H}_{(I,J)}$ and treating $\{\Omega_{(I,J)}\}$ as an orthonormal basis. We have, of course,

$$\mathcal{H}_m(\mathcal{T}) = \bigoplus_Z \mathcal{H}_Z. \quad (7.2)$$

We define the irreducible representation of the operators $\hat{m}$ and $\hat{m}^*$ on $\mathcal{H}_Z$ as follows:

$$\hat{m}_{ij} \Omega_{(I,J)} = \begin{cases} \text{sgn}_{((i,j), (I \setminus \{i\}, J \setminus \{j\}))} \Omega_{(I \setminus \{i\}, J \setminus \{j\})} & \text{if } i \in I \text{ and } j \in J \\ 0 & \text{otherwise,} \end{cases} \quad (7.3)$$

$$\hat{m}_{kl}^* \Omega_{(I,J)} = \begin{cases} \text{sgn}_{((k,l),(I,J))} \Omega_{(I \cup \{k\}, J \cup \{l\})} & \text{if } k \notin I \text{ and } l \notin J \\ 0 & \text{otherwise} \end{cases} \quad (7.4)$$

By $\text{sgn}_{((k,l),(I,J))}$ (where $k \notin I$ and $l \notin J$) we denote the parity (± 1) of the permutation, which is necessary to reestablish the canonical order of the sequence $(k, l, i_1 \dots i_k, j_1 \dots j_l)$. It is easy to check that the operators $\hat{m}$ represented this way fulfill the defining relations (5.6) and (5.10) of the algebra $\mathcal{O}_m$.

The above constructed representations will be called Z -representations.

Remark 5 The Hilbert space $\mathcal{H}_Z$ constructed above may also be treated as a subspace of the fermionic Fock space defined by N (abstract) positron degrees of freedom ϕ_i and N electron degrees of freedom φ_j :

$$\mathcal{H}_Z \subset \mathcal{F}(\mathbb{C}^N) \otimes \mathcal{F}(\mathbb{C}^N), \quad (7.5)$$

where by $\mathcal{F}(\mathbb{C}^N)$ we denote the fermionic Fock space with N generators, satisfying the canonical anticommutation relations. In this space, vectors $\Omega_{(I,J)}$ may be represented as canonically ordered monomials $\phi_{i_1}^* \dots \phi_{i_k}^* \varphi_{j_1}^* \dots \varphi_{j_l}^*$, (where $i_1 < \dots < i_k$ and $j_1 < \dots < j_l$ are all elements of I and J respectively) of Grassmannian (anticommuting) variables ϕ^* and φ^* . They are obtained from the Fock vacuum by the action of creation operators:

$$\phi_{i_1}^* \dots \phi_{i_k}^* \varphi_{j_1}^* \dots \varphi_{j_l}^* := \hat{\phi}_{i_1}^* \dots \hat{\phi}_{i_k}^* |\omega_p\rangle \otimes \hat{\varphi}_{j_1}^* \dots \hat{\varphi}_{j_l}^* |\omega_e\rangle, \quad (7.6)$$

and $|\omega_p\rangle$ (respectively $|\omega_e\rangle$) denotes the (fermionic) Fock vacuum for positrons (resp. electrons). The subspace $\mathcal{H}_Z$ corresponds to the value $Q = eZ$ of the total charge, as will be seen from Lemma 4. In the polynomial representation of $\mathcal{H}_Z$, operators $\hat{m}^*$ may be identified with multiplication operators by the Grassmannian 2-nd order quantity $m_{ij}^* := \phi_i^* \varphi_j^*$. Correspondingly, the coefficient arising in (7.4) is chosen in such a way that it reestablishes the canonical order in the product $\phi_k^* \varphi_l^* \phi_{i_1}^* \dots \phi_{i_k}^* \varphi_{j_1}^* \dots \varphi_{j_l}^*$. In terms of the tensor product of creation operators we may also write:

$$\hat{m}_{ij}^* = (\hat{\phi}_i^* \otimes \mathbb{1})(\mathbb{1} \otimes \hat{\varphi}_j^*). \quad (7.7)$$

The following three Lemmas characterize Z -representations.

Lemma 3 A vector $\Omega \in \mathcal{H}_Z$ belongs to $\mathcal{H}_{(I,J)}$ if and only if

$$\hat{Q}_{ij} \Omega = \Omega \text{ for } i \in I \text{ and } j \in J, \quad (7.8)$$

$$\hat{P}_{kl} \Omega = \Omega \text{ for } k \notin I \text{ and } l \notin J, \quad (7.9)$$

$$\hat{P}_{ij} \Omega = 0 \text{ for } i \in I \text{ or } j \in J, \quad (7.10)$$

$$\hat{Q}_{kl} \Omega = 0 \text{ for } k \notin I \text{ or } l \notin J. \quad (7.11)$$

Proof. The proof follows immediately by acting with m_{ij} (or respectively m_{ij}^*) onto the left-hand sides of equations (7.3) and (7.4). $\square$

Lemma 4 *The projector P_Z given by formula (4.26) reduces to unity on the representation space $\mathcal{H}_Z$ and vanishes on all the representation spaces $\mathcal{H}_{\tilde{Z}}$, for $\tilde{Z} \neq Z$.*

Proof: The value of the operator (4.26) on $\mathcal{H}_Z$ may be easily checked by inspection. $\square$

This Lemma shows that the canonical representations constructed above are labeled by the values of the total charge only: the space $\mathcal{H}_Z$ is an eigenspace of $\hat{Q}$ given by (4.27), with eigenvalue $Q = eZ$.

Lemma 5 *We have*

$$\hat{m}_{ij}\mathcal{H}_{(I,J)} = \begin{cases} \mathcal{H}_{(I\setminus\{i\}, J\setminus\{j\})} & \text{if } i \in I \text{ and } j \in J \\ 0 & \text{otherwise,} \end{cases} \quad (7.12)$$

and the operator m_{ij} , treated as a mapping from $\mathcal{H}_{(I,J)}$ to $\mathcal{H}_{(I\setminus\{i\}, J\setminus\{j\})}$ is a unitary (intertwining) isomorphism. Equivalently, we have

$$\hat{m}_{kl}^*\mathcal{H}_{(I,J)} = \begin{cases} \mathcal{H}_{(I\cup\{k\}, J\cup\{l\})} & \text{if } k \notin I \text{ and } l \notin J \\ 0 & \text{otherwise,} \end{cases} \quad (7.13)$$

and the operator m_{ij}^* , treated as a mapping from $\mathcal{H}_{(I,J)}$ to $\mathcal{H}_{(I\cup\{k\}, J\cup\{l\})}$ is a unitary (intertwining) isomorphism.

Proof. The proof follows immediately from (7.3) and (7.4). $\square$

Now we start to analyze arbitrary faithful, irreducible representations of $\mathcal{O}_m$.

Definition 4 *Let there be given a faithful, irreducible representation of $\mathcal{O}_m$ with representation space H . For every pair $(I, J) \in S_N \times S_N$ we define the subspace $H_{(I,J)} \subset H$, as the space of all vectors $\Omega \in H$ fulfilling conditions (7.8) – (7.11).*

Remark 6 *Among four equations (7.8) – (7.11) only two are independent:*

a) (7.8) and (7.10) in the case $I = N = J$. Equations (7.9) and (7.11) are trivially redundant.

b) (7.9) and (7.11) in the case $I = \emptyset = J$. Equations (7.8) and (7.10) are trivially redundant.

c) (7.8) and (7.9) in the case $\emptyset \neq I \neq N$ and $\emptyset \neq J \neq N$. They imply equations (7.10) and (7.11). To show, for instance, that (7.8) implies (7.10), suppose that $i \in I$. Then, due to (7.8) we can represent $\Omega = \hat{Q}_{il}\Omega$, for some $l \in J$. Acting with $\hat{P}_{ij}$ onto this equation and using (6.8) gives (7.10). A similar argument may be used if $j \in J$.

d) (7.8) and (7.11) in the case $I = \mathcal{N}$ and $\emptyset \neq J \neq \mathcal{N}$. Equation (7.9) is redundant and (7.8) implies (7.10). To prove it we represent $\Omega = \hat{Q}_{il}\Omega$, for some $l \in J$. Acting with $\hat{P}_{ij}$ onto (7.8) and using (6.8) we obtain (7.10). The same is true for the case $J = \mathcal{N}$ and $\emptyset \neq I \neq \mathcal{N}$.

e) (7.9) and (7.10) in the case $I = \emptyset$ and $\emptyset \neq J \neq \mathcal{N}$. Equation (7.8) is redundant and – using the same argument as above – (7.9) implies (7.11). Again, the same is true for the case $J = \emptyset$ and $\emptyset \neq I \neq \mathcal{N}$.

Theorem 3 *For every faithful, irreducible representation of $\mathcal{O}_m$ there is a nontrivial subspace $H_0 \subset H$ satisfying conditions (7.8) – (7.11) for some $(I_0, J_0) \in S_{\mathcal{N}} \times S_{\mathcal{N}}$.*

In this Section we give the proof for the simplest case $Z = 0$ (the complete proof is contained in Appendix B). It is based on the analysis of the “long projectors”, i. e. the products of projectors $\hat{P}_{ij}$ and $\hat{Q}_{kl}$ having different indices. Let K_- , K_+ , L_- and L_+ be four subsets of $\mathcal{N}$, such that $\#K_- = \#L_-$, $\#K_+ = \#L_+$, $K_- \cap K_+ = \emptyset$ and $L_- \cap L_+ = \emptyset$. We put:

$$P_{(K_-, L_-)(K_+, L_+)} = \prod_{i \in K_-} \prod_{j \in L_-} \hat{P}_{ij} \prod_{k \in K_+} \prod_{l \in L_+} \hat{Q}_{kl} . \quad (7.14)$$

Due to properties (6.15) and (6.16) we are able to reorganize the product on the right-hand side in such a way that each index i , j , k and l appears only once. It may happen that some of these projectors vanish identically on the representation space H . Consider a non-vanishing projector of *maximal* length $\#K = \#L$, where $K := (K_- \cup K_+)$ and $L := (L_- \cup L_+)$ and define $|Z| := N - \#K = N - \#L$. This means that every product of projectors P and Q having *different indices*, which is longer than $N - |Z|$, vanishes identically on H . In particular, $Z = 0$ means that there is at least one non-vanishing long projector of length N (as we shall see later on, none of the long P -operators of length N vanishes in this case).

Proof. Let (7.14) be a non-vanishing projector of length N . We take

$$H_0 := P_{(K_-, L_-)(K_+, L_+)} H , \quad (7.15)$$

and put $I_0 := K_+$, $J_0 := L_+$. Properties (7.8) – (7.11) follow immediately from (6.8), (6.9) and (6.12) – (6.16). $\square$

Of course, we have $|Z| < N$ in the general case, because projectors of “length 1” do not vanish. Indeed, vanishing of $\hat{P}_{ij}$ would imply vanishing of $\hat{m}_{ij}^*$ and vanishing of $\hat{Q}_{ij}$ would imply vanishing of $\hat{m}_{ij}$, according to identities:

$$\begin{aligned} \|\hat{m}_{ij}^* \Omega\|^2 &= (\Omega | \hat{P}_{ij} \Omega) \\ \|\hat{m}_{ij} \Omega\|^2 &= (\Omega | \hat{Q}_{ij} \Omega) . \end{aligned}$$

This is excluded by the faithfulness of the representation.

Every nonvanishing long projector of maximal length defines a non-trivial subspace

$$H_{(K_-, L_-)(K_+, L_+)} := P_{(K_-, L_-)(K_+, L_+)} H . \quad (7.16)$$

In the general case this space cannot be taken directly as H_0 of Theorem 3. From the physical point of view, we know only that the degrees of freedom corresponding to K_+ and L_+ are occupied and those corresponding to K_- and L_- are unoccupied. Hence, we have to check what happens with the remaining degrees of freedom. We prove in Appendix B that the situation in which there are in $\mathcal{N} \setminus K$ and $\mathcal{N} \setminus L$ degrees of freedom having the same status (i. e. both occupied or both unoccupied) is excluded by the assumption that all the projectors longer than $N - |Z|$ vanish identically on H . Hence, there are only two possible situations: either all positron degrees of freedom in $\mathcal{N} \setminus K$ are occupied and all electron degrees of freedom in $\mathcal{N} \setminus L$ are unoccupied, or *vice versa*. In the first case we take $(I_0, J_0) := (\mathcal{N} \setminus K_-, L_+)$ and put $Z := |Z|$, whereas in the second case we take $(I_0, J_0) := (K_+, \mathcal{N} \setminus L_-)$ and put $Z := -|Z|$. Finally, we prove that the space (7.16) fulfills conditions (7.8) – (7.11).

Lemma 6 *Let there be given a subspace $H_0 \in H$ fulfilling conditions (7.8) – (7.11) for some $I = I_0$ and $J = J_0$. For $(i, j) \in (I_0, J_0)$ we define $H_1 := \hat{m}_{ij} H_0$ and for $(k, l) \notin (I_0, J_0)$ we define $H_2 := \hat{m}_{kl}^* H_0$. These spaces have the following properties:*

1. H_1 fulfills conditions (7.8) – (7.11), for $I = I_0 \setminus \{i\}$ and $J = J_0 \setminus \{j\}$.
2. H_2 fulfills conditions (7.8) – (7.11), for $I = I_0 \cup \{k\}$ and $J = J_0 \cup \{l\}$.
3. H_0 , H_1 and H_2 are mutually orthogonal.
4. We have

$$\hat{m}_{ij}^* H_1 = H_0 = \hat{m}_{kl} H_2. \quad (7.17)$$

The operators $\hat{m}$ and $\hat{m}^$ are (mutually inverse) isometries between these spaces.*

Proof.

1. According to Remark 6, we have to consider different cases. Let us consider the generic case $\emptyset \neq I_0 \setminus \{i\} \neq \mathcal{N}$ and $\emptyset \neq J_0 \setminus \{j\} \neq \mathcal{N}$, leaving the remaining cases to the reader. Take any vector $\omega = \hat{m}_{ij} \Omega \in H_1$, where $\Omega \in H_0$. If $k \in I_0 \setminus \{i\}$ and $l \in J_0 \setminus \{j\}$, then

$$\hat{Q}_{kl} \omega = \hat{Q}_{kl} \hat{m}_{ij} \Omega = \hat{m}_{ij} \hat{Q}_{kl} \Omega = \hat{m}_{ij} \Omega = \omega ,$$

which proves (7.8). To prove (7.9) we notice that, for $k \notin I_0 \setminus \{i\}$ and $l \notin J_0 \setminus \{j\}$, we have three possibilities: 1) $k \neq i$ and $l \neq j$, 2) $(k, l) = (i, j)$, 3) only one pair of indices coincides. In the first case the proof of (7.9) is similar to the proof of (7.8) above. In the second case (7.9) is implied by identity (6.10). Suppose, therefore,

that one pair of indices coincides, say $k = i$, whereas $l \neq j$. Then, due to (5.14), we have

$$\hat{P}_{il}\omega = \hat{m}_{il}\hat{m}_{il}^*\hat{m}_{ij}\Omega = (\hat{m}_{ij} - \hat{m}_{ij}\hat{Q}_{il})\Omega = \omega .$$

A similar argument works also if $k \neq i$ and $l = j$.

2. The proof for H_2 is analogous to the proof under point 1.

3. We have

$$(\hat{m}_{ij}\Omega|\Omega) = (\Omega|\hat{m}_{ij}^*\Omega) .$$

But, $\hat{m}_{ij}^*\Omega = \hat{m}_{ij}^*\hat{P}_{ij}\Omega = 0$ for $\Omega \in H_0$, which proves orthogonality of H_1 and H_0 . Replacing (i, j) with (k, l) we have $\hat{m}_{kl}\Omega = \hat{m}_{ij}\hat{Q}_{kl}\Omega = 0$ for $\Omega \in H_0$, which proves orthogonality of H_0 and H_2 .

4. For any $\omega = \hat{m}_{ij}\Omega \in H_1$, where $\Omega \in H_0$, we have $\hat{m}_{ij}^*\omega = \hat{Q}_{ij}\Omega = \Omega$. Moreover

$$(\hat{m}_{ij}\Omega|\hat{m}_{ij}\Omega) = (\Omega|\hat{Q}_{ij}\Omega) = (\Omega|\Omega) .$$

A similar argument may be used to prove that $\hat{m}_{kl}^*$ is an isometry from H_0 to H_2 .

□

Lemma 7 *For every faithful, irreducible representation of $\mathcal{O}_m$ there is an integer Z ($|Z| < N$) such that the representation space H has the following structure:*

1.

$$H = \bigoplus_{\#I - \#J = Z} H_{(I,J)} , \quad (7.18)$$

with $H_{(I,J)}$ given by Definition 4.

2. We have

$$\hat{m}_{ij}H_{(I,J)} = \begin{cases} H_{(I \setminus \{i\}, J \setminus \{j\})} & \text{if } i \in I \text{ and } j \in J \\ 0 & \text{otherwise} , \end{cases} \quad (7.19)$$

and the operator m_{ij} , treated as a mapping from $H_{(I,J)}$ to $H_{(I \setminus \{i\}, J \setminus \{j\})}$ is a unitary (intertwining) isomorphism.

Remark 7 *Obviously, point 2 of this Lemma may be equivalently formulated in terms of the intertwiner $m_{ij}^* : H_{(I \setminus \{i\}, J \setminus \{j\})} \rightarrow H_{(I,J)}$ (cf. (7.13)):*

$$\hat{m}_{kl}^*H_{(I,J)} = \begin{cases} H_{(I \cup \{k\}, J \cup \{l\})} & \text{if } k \notin I \text{ and } l \notin J \\ 0 & \text{otherwise} . \end{cases} \quad (7.20)$$

These mappings are mutually inverse, because $\hat{m}_{ij}^*\hat{m}_{ij} = \hat{Q}_{ij}$ reduces to the identity on $H_{(I,J)}$ and $\hat{m}_{ij}\hat{m}_{ij}^* = \hat{P}_{ij}$ reduces to the identity on $H_{(I \setminus \{i\}, J \setminus \{j\})}$.

Proof. Given a subspace H_0 fulfilling (7.8) – (7.11), for some $I = I_0$ and $J = J_0$, we define $\tilde{H}_{(I_0, J_0)} := H_0$. Next we use formulae (7.19) and (7.20) as a recursive definition of spaces $\tilde{H}_{(I, J)}$, for the remaining pairs (I, J) , such that $\#I - \#J = Z$. It is easy to check that adding elements to I and J by the action of $\hat{m}_{kl}^*$ or removing elements by the action of $\hat{m}_{ij}$ does not depend upon the order of these operations. This independence is implied by properties (5.6) and (5.10) of $\hat{m}$'s. It follows from Lemma 6 that the subspaces $\tilde{H}_{(I, J)}$ defined this way fulfill equations (7.8) – (7.11) and are mutually isomorphic via the corresponding intertwiners $\hat{m}$ and $\hat{m}^*$. They are orthogonal to each other because of an argument, similar to that used in the proof of point 3 of Lemma 6. It follows that the direct sum $\tilde{H}$ of these spaces carries a nontrivial representation of our algebra. Irreducibility of the representation implies that $\tilde{H}$ coincides with the whole representation space H . Consequently, we have $\tilde{H}_{(I, J)} = H_{(I, J)}$ i. e. formula (7.18) holds. $\square$

Theorem 4 *Every faithful, irreducible representation of $\mathcal{O}_m$ is unitarily equivalent to one of the Z -representations defined by (7.3) and (7.4).*

Proof. It remains to prove that the spaces $H_{(I, J)}$ are one-dimensional. Choosing a unit vector $\omega_{(I_0, J_0)}$ in one of them, say $H_{(I_0, J_0)}$, and constructing for each (I, J) one vector $\omega_{(I, J)} \in H_{(I, J)}$ by acting with $\hat{m}$'s and $\hat{m}^*$'s on $\omega_{(I_0, J_0)}$, according to formulae (7.3) and (7.4), we obtain an invariant subspace of the representation spanned by all these ω 's. This subspace is obviously isomorphic to the canonical representation space $\mathcal{H}_Z$. The isomorphism is given by identification of ω 's with Ω 's of the canonical representation. On the other hand, since the representation we started with was irreducible, this subspace must be equal to the entire Hilbert space H . $\square$

As a result of this discussion, the physical Hilbert space $\mathcal{H}$ is the unique (up to unitary equivalence) representation space of the observable algebra $\mathcal{O}$. Due to (7.2) we have

$$\mathcal{H} = \bigoplus_Z \{ \mathcal{H}_{e-m}(\mathcal{T}) \otimes \mathcal{H}_Z \} , \quad (7.21)$$

where $\mathcal{H}_{e-m}(\mathcal{T})$ is the space of bosonic wave functions depending on the magnetic variables $\mathcal{B}_{x, x+\hat{k}}(\mathcal{T})$, equipped with the standard L^2 -Hilbert space structure. This is the decomposition of the physical Hilbert space into *charge superselection sectors*, corresponding to the direct sum decomposition (6.19) of $\mathcal{O}$. This means that $\hat{Q}$ acts on each sector $\mathcal{H}_{e-m}(\mathcal{T}) \otimes \mathcal{H}_Z$ as $eZ\mathbb{1}$ and that it generates the whole center of $\mathcal{O}$.

8 Wave Function Description of Physical States

As already mentioned above, $\mathcal{H}_{e-m}(\mathcal{T})$ is the space of bosonic wave functions depending on the magnetic variables $\mathcal{B}_{x, x+\hat{k}}(\mathcal{T})$. Treating the magnetic flux variables as the corresponding curl's of the variables $A_{x, x+\hat{k}}$ we may interpret elements of $\mathcal{H}_{e-m}(\mathcal{T})$ as

gauge-invariant wave functions in the sense of Section 3. Here we will also give a wave function interpretation to elements of $\mathcal{H}_m(\mathcal{T})$. For this purpose we denote

$$A_i := \int_{(x_0, x)_\mathcal{T}} A , \quad (8.1)$$

where $x \in \Lambda$ is the site corresponding to i -th degree of freedom. In the simplest case $Q = 0$ we identify the element $\Omega_{(I, J)}$ as the gauge-invariant, canonically ordered product (see definition of $\text{sgn}_{((k, l), (I, J))}$ which follows formula (7.4)):

$$\Omega_{(I, J)} := \left(\prod_{i \in I} \exp(-igA_i) \phi_i^* \right) \left(\prod_{j \in J} \exp(igA_j) \varphi_j^* \right) . \quad (8.2)$$

Tensorising these two types of wave functions we obtain the whole space of gauge-invariant wave functions (3.14) with a uniquely defined Hilbert space structure, derived from the structures of $\mathcal{H}_{e-m}(\mathcal{T})$ and $\mathcal{H}_m(\mathcal{T})$. It is easy to see that the representation of $\mathcal{O}$, which we have constructed coincides with the representation of gauge-invariant combinations of the operators $(\hat{A}, \hat{E}, \hat{\psi}, \hat{\psi}^*)$ acting on such wave functions, in the sense of Sections 2 and 3.

In the case of $Q \neq 0$ the wave function (8.2) would not be gauge invariant with respect to gauge transformations at the tree root x_0 , because at this point the incoming strings $\exp(-igA_i)$ are not balanced by the same number of the outgoing strings $\exp(igA_j)$. To restore this balance we have to multiply the right-hand side by an additional phase factor which, moreover, matches the boundary conditions $E_{x, \infty}$. For this purpose we denote for $x \in \partial\Lambda$:

$$A_{(x_0, x, \infty)_\mathcal{T}} := \left(\int_{(x_0, x)_\mathcal{T}} A \right) + A_{x, \infty} . \quad (8.3)$$

Finally, we take the following definition of the gauge-invariant wave function:

$$\Omega_{(I, J)} := \left(\prod_{i \in I} \exp(-igA_i) \phi_i^* \right) \left(\prod_{j \in J} \exp(igA_j) \varphi_j^* \right) \left(\prod_{x \in \partial\Lambda} \exp\left(\frac{i}{\hbar} E_{x, \infty} A_{(x_0, x, \infty)_\mathcal{T}}\right) \right) , \quad (8.4)$$

which reduces to (8.2) for trivial boundary conditions $E_{x, \infty} \equiv 0$ (and – consequently – vanishing total charge $Q = 0$). Again, observables act on such wave functions as gauge-invariant combinations of the operators $(\hat{A}, \hat{E}, \hat{\psi}, \hat{\psi}^*)$ from Section 2.

We conclude that the Hilbert space $\mathcal{H}$, which was obtained as the unique representation space of $\mathcal{O}$, has a natural realization as the space of gauge-invariant wave functions $\Psi = \Psi(A, \phi^*, \varphi^*)$. Here, among variables A we take into account also the “external” potentials $A_{x, \infty}$ for $x \in \partial\Lambda$ and limit ourselves to wave functions Ψ fulfilling boundary conditions for the electric field:

$$(\hat{E}_{x, \infty} \Psi :=) \frac{\hbar}{i} \frac{\partial}{\partial A_{x, \infty}} \Psi = E_{x, \infty} \Psi . \quad (8.5)$$

Obviously, such gauge invariant wave functions exist if and only if the total “external” electric flux (3.17) through $\partial\Lambda$ equals $Q = eZ$, where $Z \in [-N + 1, N - 1]$. In this definition no tree-decomposition appears. However, the tree notion is useful to define the scalar product in the space of such wave functions:

Definition 5 *Given a tree $\mathcal{T}$ we define the scalar product $(\cdot|\cdot)_{\mathcal{T}}$ in the following way:*

1. *We choose the “tree gauge”, i.e. we fix arbitrarily all the values of on-tree variables $A_{x,x+\hat{k}}$ for $(x, x+\hat{k}) \in \mathcal{T}$ and all the values of external variables $A_{x,\infty}$ for $x \in \partial\Lambda$.*
2. *We treat Ψ as a function of the remaining variables: off-tree potentials $A_{x,x+\hat{k}}$ for $(x, x+\hat{k}) \notin \mathcal{T}$ and Grassmannian variables (ϕ^*, φ^*) .*
3. *We take the L^2 -scalar product with respect to these variables (integration with respect to Grassmannian variables is meant in the sense of Berezin).*

Proposition 5 *The above scalar product does not depend upon the choice of both the tree and the tree gauge. It coincides – after identification (8.4) – with the previously introduced, canonical scalar product in $\mathcal{H}$.*

The proof is obvious: for bosonic variables, integration with respect to off-tree potentials $A_{x,x+\hat{k}}$ gives the same result as integration with respect to off-tree fluxes $\mathcal{B}_{x,x+\hat{k}}(\mathcal{T})$; for fermionic variables, it follows from the Berezin integration that different vectors $\Omega_{(I,J)}$ defined by (8.4) form an orthonormal set.

We conclude this Section by the observation that faithful, irreducible representations of $\mathcal{O}$, corresponding to the same value of Q are equivalent, even if they differ by the boundary data $E_{x,\infty}$. The intertwining operator between two such representations is obtained by replacing the last factor in (8.4), corresponding to the first choice of data, by the factor corresponding to the second choice of data.

9 Local Observables and Quantum Field Dynamics

Operators $\hat{\mathcal{M}}$ are non-local. Having in mind a possible continuum limit of the theory we would like to be able to describe it in terms of local quantities. For this purpose we may restrict ourselves to the creation operators of pairs, which are located at the same point

$$\hat{u}_{x;L}^K := \hat{\varphi}_{x;L} \hat{\phi}_x^K \quad (9.1)$$

or are separated by at most one lattice link:

$$\hat{w}_{x,x+\hat{k};L}^K := \hat{\varphi}_{x+\hat{k};L} \exp(-ig\hat{A}_{x,x+\hat{k}}) \hat{\phi}_x^K \quad (9.2)$$

Using (4.34) we are able to express all pair creation and annihilation operators along any *long* path γ as a multiple commutator of the above *short* pair creation operators assigned to links and points belonging to γ . This way, the observable algebra $\mathcal{O}$ may be viewed as the algebra generated by the following set of local generators: $\{\hat{E}, \hat{B}, \hat{u}, \hat{w}\}$.

Of course, the above generators are not independent. They fulfill conditions (4.30) – (4.37), whenever both sides of the equation, with $\hat{\mathcal{M}}$ replaced by $\hat{u}$'s and $\hat{w}$'s, make sense. There are obviously many possible choices of sets of independent generators, e. g., we may choose any three $\hat{u}$'s at each lattice site and one $\hat{w}$ on each *along-tree* link. Due to the composition formula (5.14), all the other operators $\hat{m}$ may be reconstructed from these. This way we obtain $2N - 1$ independent generators. Indeed, adding *one* operator $\hat{w}$ on each link to the *three* operators $\hat{u}$ at its end we obtain 4 quantities *per* lattice site, except for the tree root, i. e. the total number of fermionic degrees of freedom minus *one*. The missing *one* degree of freedom is described by the total charge $\hat{Q}$. We see, that the $2N$ operators $\hat{\psi}$ have been replaced in this formulation by the same number of independent generators of the operator algebra $\mathcal{O}_m$. On the other hand, the number of the bosonic operators $(\hat{A}, \hat{E})$ has been reduced by twice the number of independent gauges: first when replacing operators $\hat{A}$ by operators $\hat{B}$ and second when imposing the Gauss law.

To describe the quantum evolution of our system we have to define its Hamiltonian. The quantum version of the Hamiltonian (2.4) will contain three terms,

$$H = H_{e-m} + H_m + H_{\text{kin}}, \quad (9.3)$$

built from the field operators. The form of H_{e-m} is obvious: we have to take the sum of squares of all electric and magnetic fluxes on the lattice. Also the mass term H_m may be immediately expressed in terms of the operators $\hat{u}$. In the Dirac representation it is equal to the sum of the local “particle number” operators $\hat{N}_x$ corresponding to all positron and electron degrees of freedom at the point x . Due to formula (4.12), these operators may be expressed in terms of commutators of our local quantities $\hat{u}_x$ and $\hat{u}_x^*$:

$$\hat{N}_x = \sum_K \left(\hat{\mathcal{L}}_x^{KK} + (\mathbb{1} - \hat{\mathcal{R}}_{y,KK}) \right) = \sum_K \hat{\mathcal{K}}_{x,x;L}^K + 2 \cdot \mathbb{1} = [(\text{tr } \hat{u}_x)^*, \text{tr } \hat{u}_x] + 2 \cdot \mathbb{1}. \quad (9.4)$$

Finally, the kinetic term H_{kin} is obtained in the following way. On each lattice link $(x, x + \hat{k})$ we replace ψ^{a*} by the mean value of ψ_x^{a*} and $\psi_{x+\hat{k}}^{a*}$ and replace $D_k \psi^c$ by the difference of $\psi_{x+\hat{k}}^c$ and ψ_x^c . However, both the sum and the difference of these operators have to be calculated in a gauge-invariant way. For this purpose we first transport e.g. $\psi_{x+\hat{k}}^a$ from $x + \hat{k}$ to x using the “parallel transport operator” $\exp(ig\hat{A}_{x,x+\hat{k}})$. After this transport, linear combinations of the operators ψ^a make sense. This way we express H_{kin} in terms of operators $\hat{w}_{x,x+\hat{k};L}^K$. The reader may easily check the following final result (we skip the constant term, proportional to the unit operator; this is usually done by assuming the “normal order” of creation and annihilation operators):

$$H_{e-m} = \frac{1}{2} \sum_{(x,x+\hat{k})} \left(\hat{E}_{x,x+\hat{k}} \right)^2 + \frac{1}{2} \sum_{x;\hat{k},\hat{l}} \left(\hat{B}_{x;\hat{k},\hat{l}} \right)^2, \quad (9.5)$$

$$H_m = m \sum_x [(\text{tr } \hat{u}_x)^*, \text{tr } \hat{u}_x] , \quad (9.6)$$

$$H_{\text{kin}} = -\frac{\hbar}{a} \sum_{x,\hat{k}} \text{Im } \hat{\mathcal{W}}_{x,x+\hat{k}} , \quad (9.7)$$

where

$$\text{tr } \hat{u}_x := \hat{u}_{x;1}^1 + \hat{u}_{x;2}^2 , \quad (9.8)$$

$$\hat{\mathcal{W}}_{x,x+\hat{k}} := \text{sgn } \hat{k} \cdot \sigma^{KL}_K \cdot \hat{w}_{x,x+\hat{k};L}^K . \quad (9.9)$$

The evolution equations for field operators in the Heisenberg picture read

$$\frac{d}{dt} \hat{\mathcal{F}} = \frac{1}{i\hbar} [\hat{\mathcal{F}}, H] . \quad (9.10)$$

In our case we obtain

$$\frac{d}{dt} \hat{B}_{x,\hat{k},\hat{l}} = -\hat{E}_{x,x+\hat{k}} - \hat{E}_{x+\hat{k},x+\hat{k}+\hat{l}} - \hat{E}_{x+\hat{k}+\hat{l},x+\hat{l}} - \hat{E}_{x+\hat{l},x} = -\text{curl } \hat{E} , \quad (9.11)$$

which together with

$$\frac{d}{dt} \hat{E}_{x,x+\hat{k}} = \sum_{\hat{l}} \hat{B}_{x,\hat{k},\hat{l}} - \frac{2}{a} \cdot e \cdot \text{Re} (\hat{\mathcal{W}}_{x,x+\hat{k}} + \hat{\mathcal{W}}_{x+\hat{k},x}) = \text{curl } \hat{B} - \hat{j}_{x,x+\hat{k}} \quad (9.12)$$

give us the lattice quantum version of Maxwell equations. The other two equations,

$$\frac{d}{dt} \hat{u}_{x;L}^K = \frac{2}{i\hbar} \hat{u}_{x;L}^K - \frac{1}{2a} \sum_{\hat{k}} \left\{ [\hat{u}_{x;L}^K, (\hat{\mathcal{W}}_{x,x+\hat{k}})^*] + [\hat{u}_{x;L}^K, (\hat{\mathcal{W}}_{x+\hat{k},x})^*] \right\} \quad (9.13)$$

and

$$\begin{aligned} \frac{d}{dt} \hat{w}_{x,x+\hat{k};L}^K &= \frac{2}{i\hbar} \hat{w}_{x,x+\hat{k};L}^K - \frac{1}{2a} \sum_{\hat{l}} \left\{ [\hat{w}_{x,x+\hat{k};L}^K, (\hat{\mathcal{W}}_{x,x+\hat{l}})^*] \right. \\ &\quad \left. + [\hat{w}_{x,x+\hat{k};L}^K, (\hat{\mathcal{W}}_{x+\hat{l},x})^*] \right\} \end{aligned} \quad (9.14)$$

replace the second quantized Dirac equation.

The above Hamiltonian is bounded from below, because $H_{\text{e-m}}$ is positive definite and the remaining terms are bounded operators. Hence, we may define the *dynamical vacuum* as the minimal energy state in the vacuum sector $Q = 0$. It has nothing to do with the *perturbative vacuum*, which for each decomposition (7.21) may be defined as the tensor product of the “free-electromagnetic-vacuum” and the “free-matter-vacuum” (of course, the result highly depends upon the decomposition). For the sake of convenience one can subtract the vacuum energy from the Hamiltonian.

Similarly, we may define the notion of a *dressed particle* as the minimal energy state in the $Q = e$ sector, with appropriately chosen boundary data $E_{x,\infty}$. Also this state has nothing to do with the *bare particle* in the perturbative approach.

There is, probably, no way to obtain an exact, analytic expression for these states, even on the lattice level, and a numerical analysis will remain as the only tool to investigate the spectrum of the operator H . There is, however, an interesting idea (see [28]), namely to consider H_{kin} as the perturbation to the operator $H_0 := H_{\text{e-m}} + H_{\text{m}}$. We stress that this type of perturbative approach has nothing to do with “switching the interaction off”, because quantum states corresponding to $g = 0$ and to $g \neq 0$ belong to completely different Hilbert spaces. There is *no* canonical identification of these spaces.

To be able to interpret the quantum mechanical system constructed in this paper as an approximation of the complete quantum electrodynamics we have to relate the parameters m and e used in the definition of the Hamiltonian, with the physically observed quantities of mass and charge of the electron. We may call this procedure *renormalization*. To calculate the mass of the physical particles in our theory we cannot use the one-particle states as in the perturbative approach. Indeed, there is no “creation operator” in our theory, because of the superselection rules. Instead, the operator $\hat{w}_{x,x+\hat{k}}$ creates a particle carrying the charge e at x and an antiparticle carrying the charge $-e$ at $x + \hat{k}$. Therefore, one has to proceed as follows. Fix a pair of sites $(x, y) \in \Lambda$. In the subspace of all quantum states, such that $j_x^0 = +e$, $j_y^0 = -e$, and $j^0 = 0$ elsewhere, we find the state corresponding to the lowest energy $\mathcal{E}(x, y)$. The maximal value of the function $\mathcal{E}(x, y)$ with respect to x and y should correspond to the maximal distance between x and y . It will be identified with twice the physical mass m_0 of the particles we want to describe. On the other hand, the behaviour of $\mathcal{E}(x, y)$ for large distances:

$$\mathcal{E}(x, y) \approx 2m_0 - \frac{e_0^2}{4\pi \|x - y\|} , \quad (9.15)$$

should allow us to identify the physical charge e_0 . Consequently, we have to renormalize the field strength E by the ratio $\frac{e_0}{e}$.

Finally, one has to check, whether or not the physical properties of the above system depend considerably on the volume and the spacing a of the lattice used.

10 Relation with Approaches based on Gauge Fixing. Periodic Boundary Conditions

In the above approach, no physically meaningful “potential operators” $\hat{A}_{x,x+\hat{k}}$ have been defined (the theory described in Section (2), formulated in terms of gauge-dependent wave functions, has no direct physical meaning). There is, however, a possibility to code uniquely the physical information about the magnetic field $\hat{B}_{x;\hat{k},i}$ in the potentials. For

this purpose a gauge condition has to be imposed, which enables us to express *uniquely* potentials in terms of magnetic fluxes.

The simplest example of such a gauge fixing condition consists in taking the *tree gauge*, where we choose arbitrarily all values of the “along-tree-potentials”, i. e. operators $\hat{A}_{x,x+\hat{k}}$ for $(x, x+\hat{k}) \in \mathcal{T}$. The remaining off-tree-potentials (for $(x, x+\hat{k}) \notin \mathcal{T}$) are in one-to-one correspondence with the off-tree fluxes

$$\hat{B}_{x,x+\hat{k}}(\mathcal{T}) = \hat{A}_{x,x+\hat{k}} + \hat{A}_{(x_0,x)\mathcal{T}} + \hat{A}_{(x+\hat{k},x_0)\mathcal{T}} , \quad (10.1)$$

because the last two terms of the above sum are given by the gauge fixing condition. The results of our paper may thus be immediately translated into the language of potentials $\hat{A}$ fulfilling this gauge. To deal with self-adjoint operators only, it is sufficient to choose self-adjoint operators (e. g. *c*-numbers) as the along-tree potentials. Moreover, to be able to define the fermionic operators consistently, it is necessary, as we shall see in the sequel, to consider operators commuting with all the magnetic fluxes $\hat{B}$ (again, *c*-number conditions are good from this point of view).

A tree gauge may be easily replaced by any *elliptic* gauge condition, i. e. a condition which enables us to solve uniquely, at a given instant of time, equations (4.1) with respect to potentials. A typical example is provided by the *Coulomb gauge*:

$$\operatorname{div}_x \hat{A} := \sum_{\hat{k}} \hat{A}_{x,x+\hat{k}} = \hat{F}_x , \quad (10.2)$$

where as $\hat{F}_x$ we may choose arbitrary operators, satisfying the global condition

$$\sum_x \hat{F}_x = 0 . \quad (10.3)$$

Again, it is worthwhile to limit ourselves to self-adjoint operators, commuting with all the magnetic fluxes, in order to have all the operators $\hat{A}$ also self-adjoint and to be able to reconstruct also the fermionic operators. For a lattice with non-vanishing boundary $\partial\Lambda \neq \emptyset$ this is not a complete gauge: we still have the freedom to fix boundary values $\hat{A}_{x,x+\hat{k}}$ for $(x, x+\hat{k}) \in \partial\Lambda$. An additional (“residual”) gauge condition may be imposed e. g. in terms of the 2-dimensional divergence on $\partial\Lambda$ for any $x \in \partial\Lambda$:

$$\operatorname{div}_x^{(2)} \hat{A} := \sum_{(x,x+\hat{k}) \in \partial\Lambda} \hat{A}_{x,x+\hat{k}} = \hat{C}_x . \quad (10.4)$$

The consistency of this condition is again provided by the global condition

$$\sum_{x \in \partial\Lambda} \hat{C}_x = 0 . \quad (10.5)$$

Proposition 6 *The above gauge fixing, together with equation (4.1), uniquely defines all the operators $\hat{A}_{x,x+\hat{k}}$ as linear combinations of the magnetic flux operators $\hat{B}_{x;\hat{k},\hat{l}}$ and the gauge fixing operators $\hat{F}_x$ and $\hat{C}_x$.*

Proof. Choose any solution of equations (4.1) (e. g. using the tree-gauge) and denote this solution by $\hat{a}_{x,x+\hat{k}}$. Look for $\hat{A}$ which differs from $\hat{a}$ by a lattice gradient:

$$\hat{A}_{x,x+\hat{k}} = \hat{a}_{x,x+\hat{k}} + \hat{\lambda}_{x+\hat{k}} - \hat{\lambda}_x . \quad (10.6)$$

Operators $\hat{\lambda}$ assigned to lattice sites have to satisfy the lattice Laplace equation:

$$\Delta_x \hat{\lambda} := \sum_{\hat{k}} (\hat{\lambda}_{x+\hat{k}} - \hat{\lambda}_x) = \hat{F}_x - \text{div}_x \hat{a} , \quad (10.7)$$

where the right-hand side is given. This equation has a *unique* solution for any choice of Dirichlet data $\hat{\lambda}_x$, for $x \in \partial\Lambda$. These data may be obtained from the 2-dimensional Laplace equation on $\partial\Lambda$, which is implied by (10.4):

$$\Delta_x^{(2)} \hat{\lambda} := \sum_{(x,x+\hat{k}) \in \partial\Lambda} (\hat{\lambda}_{x+\hat{k}} - \hat{\lambda}_x) = \hat{C}_x - \text{div}_x^{(2)} \hat{a} . \quad (10.8)$$

This equation has a *unique* solution up to an additive constant. We conclude that operators $\hat{\lambda}_x$ may be calculated up to an additive constant and, hence, operators $\hat{A}$ are given uniquely. $\square$

The situation becomes simpler if we are dealing with periodic boundary conditions, i. e. a lattice on the torus T^3 instead of R^3 . Here, $\partial\Lambda = \emptyset$ and no boundary conditions are necessary. In this case we have at our disposal an important tool, namely (discrete) Fourier transformation. The linear part of the field equations may be nicely analyzed in terms of Fourier coefficients. In particular, the Coulomb gauge consists in fixing the longitudinal part of the potential operators. Unfortunately, the nonlinear part of the dynamics (described e. g. by nonlinear terms in Dirac equations (9.13) and (9.14)) cannot be easily handled in terms of the Fourier analysis.

Given a (complete) gauge fixing condition, we may also give a meaning to fermionic operators $\hat{\psi}_x^a$. For simplicity, we start again with a tree gauge. Taking the direct sum

$$H = \bigoplus_Q H_Q$$

of all the charge sectors, we may interpret vectors in H as polynomials of the form $\phi_{i_1}^* \cdots \phi_{i_k}^* \varphi_{j_1}^* \cdots \varphi_{j_l}^*$ with B -dependent coefficients. In this gauge, fermionic operators $\hat{\phi}_i^*$ and $\hat{\varphi}_j^*$ are uniquely defined as multiplication operators, whereas $\hat{\phi}_i$ and $\hat{\varphi}_j$ are differentiation operators. To pass from one gauge fixing to another, we use the above defined “gauge operators” $\hat{\lambda}_x$ and define “primed” operators (operators in the new gauge) according to the following formulae:

$$\hat{\psi}_x' = \exp(-ig\hat{\lambda}_x) \hat{\psi}_x , \quad (10.9)$$

$$\hat{A}_{x,x+\hat{k}}' = \hat{A}_{x,x+\hat{k}} + \hat{\lambda}_{x+\hat{k}} - \hat{\lambda}_x . \quad (10.10)$$

One immediately checks that the operators $\{\hat{E}, \hat{B}, \hat{\mathcal{M}}, \hat{\mathcal{M}}^*\}$ are equal to corresponding combinations of the above operators $\hat{A}$ and ψ_x^a and, therefore, do not depend upon the particular gauge fixing used.

We stress that the above “gauge operators” $\hat{\lambda}_x$ play a completely different role than the c -number valued quantities λ_x occurring in transformations (3.6) and (3.7), which act in the unphysical Hilbert space of gauge-dependent wave functions. Here, these operators are well defined, linear combinations of magnetic field operators, uniquely determined as intertwining operators between two different gauge fixing conditions. They are not c -numbers, even if we used a c -number-valued gauge fixing condition. What is, however, fundamental for the consistency of the above formulae, is that operators $\hat{\lambda}_x$ commute with all the magnetic fluxes.

Given a gauge fixing condition, we may also, in principle, define operators $\hat{A}_0$ according to formula

$$\hat{E}_{x,x+\hat{k}} - \frac{d}{dt} \hat{A}_{x,x+\hat{k}} = \hat{A}_0(x) - \hat{A}_0(x + \hat{k}) . \quad (10.11)$$

The homogeneous Maxwell equation ensures that the left hand side is, indeed, a curl-free lattice function. Hence, it is equal to a lattice gradient (in case of non-trivial T^3 topology, corresponding to periodic boundary conditions, the above equation contains an additional, cohomological condition for the electric field). This way $\hat{A}_0$ is defined up to an additive, time dependent constant $\hat{f}(t)$, which may be eliminated if we impose a residual gauge condition for the value $\hat{A}_0(x_0)$.

Having defined all the components of the potential as self-adjoint operators, we may use also non-elliptic gauge conditions. Typical conditions of this type are: axial gauge $\hat{A}_0 \equiv 0$ and Lorentz gauge $\frac{d}{dt} \hat{A}_0 - \text{div}_x \hat{A} \equiv 0$. These are incomplete conditions and a residual, elliptic gauge condition fixing space components $\hat{A}_{x,x+\hat{k}}$ for a given $t = t_0$ is necessary. This residual gauge gives an initial condition for the gauge function $\hat{\lambda}_x$ at t_0 and enables us to reconstruct uniquely the four potentials $\hat{A}$ in terms of field operators $\hat{B}$ and $\hat{E}$. Whenever we are able to prove that the resulting gauge operators $\hat{\lambda}_x$ commute, at each instant of time, with all the magnetic fluxes, we may use formula (10.9) to define also fermionic operators in a new, non-elliptic gauge.

The formulation of the theory in terms of the electromagnetic potential operators and corresponding fermionic operators, in a suitably chosen gauge, may be useful when solving specific problems. We stress, however, that these operators play only an auxiliary role: they enable us to encode the information about physical fields $\hat{B}$, $\hat{E}$, $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}^*$ in terms of their non-local combinations $\hat{A}_\mu$ and $\hat{\psi}$. The decision, *which* particular non-local combinations we want to use is taken once we have chosen a (complete!) gauge fixing condition.

11 Towards Continuum Theory

Here, we give only some heuristic ideas concerning the possible reconstruction of the full continuum theory. The actual construction may be of several orders of magnitude more difficult than the problems discussed in this paper.

Heuristically, the algebra of observables of the continuum theory should be constructed as an inductive limit of our algebras $\mathcal{O}_\Lambda$, describing a finite number of degrees of freedom, related to the finite lattice Λ (see e. g. [33] and [34]). For this purpose an order relation “ $\prec$ ” in the set of finite lattices has to be chosen. We say that the lattice Λ_2 is “later” than Λ_1 (or $\Lambda_1 \prec \Lambda_2$) if it describes *more* field degrees of freedom than Λ_1 does. This means that all sites of Λ_1 are also sites of Λ_2 , all links of Λ_1 may be composed of links of Λ_2 and, finally, all dual lattice plaquettes of Λ_1 (which we used in (2.7) to define electric fluxes) may be composed of corresponding dual lattice plaquettes of Λ_2 . A “later” lattice may, therefore, be “bigger” (a step towards the thermodynamical limit) and “finer” (a step towards the ultraviolet limit).

Given a pair $\Lambda_1 \prec \Lambda_2$, there is a natural embedding

$$P_{\Lambda_2, \Lambda_1} : \mathcal{O}_{\Lambda_1} \rightarrow \mathcal{O}_{\Lambda_2} . \quad (11.1)$$

Indeed, operators $\hat{\mathcal{M}}$ on Λ_1 may be identified with the corresponding operators on Λ_2 , whereas each of the fluxes $\hat{E}$ and $\hat{B}$ on Λ_1 may be identified with the sum of corresponding fluxes through those “smaller” plaquettes of Λ_2 , which constitute the “bigger” plaquette of Λ_1 under consideration. It is obvious that this embedding preserves the properties (4.29) – (4.37) of the observables. The Gauss law (5.1), however, is no longer true in its original version but in a new version, with the charge $\hat{j}_x^0$ replaced by the sum of all charges $\hat{j}_{x_i}^0$ corresponding to the sites x_i of Λ_2 which are contained in the same cell of the dual lattice as x . The compatibility relation:

$$P_{\Lambda_3, \Lambda_2} P_{\Lambda_2, \Lambda_1} = P_{\Lambda_3, \Lambda_1} \quad (11.2)$$

may be easily verified if $\Lambda_1 \prec \Lambda_2 \prec \Lambda_3$.

The inductive limit of our observable algebras describes, in principle, degrees of freedom of the continuum theory. To avoid singular objects, we may smear the fields $\hat{E}$, $\hat{B}$ and $\hat{u}$ with sufficiently regular test functions and obtain this way “observable-valued-distributions”. The field $\hat{w}$ cannot be smeared *directly*, because of its non-additive character. A natural way to encode the information about the field $\hat{w}$ in an “observable-valued-distribution” consists in defining the field $\hat{v}$ by the formula:

$$\hat{v}_{k;L}^K(x) := \lim_{a \rightarrow 0} \frac{1}{a} \left(\hat{w}_{x, x+\hat{k};L}^K - \hat{u}_{x;L}^K \right) . \quad (11.3)$$

Expansion of the right-hand side of (9.2) with respect to the lattice spacing a shows that, heuristically, we have

$$\hat{v}_{k;L}^K(x) \simeq (D_k \hat{\varphi}_{x;L}) \hat{\phi}_x^K \simeq \left((\partial_k - ig \hat{A}_k(x)) \hat{\varphi}_{x;L} \right) \hat{\phi}_x^K . \quad (11.4)$$

Our results suggest that, probably, there is no way to give any rigorous meaning to this formula, because the fields $\hat{\varphi}$, $\hat{\phi}$ and $\hat{A}$ on the right-hand side *are not* observables. The field $\hat{v}$ defined by (11.3) has, therefore, to be understood as one of the fundamental fields of the theory (see also [4] and [5], where we have proved that the Feynman path integral of QED may be expressed in terms of gauge-invariants, one of them being the above field $\hat{v}$).

Once the observable algebra of the theory is given, its quantum realizations may be constructed *via* the GNS construction. For this purpose a vacuum state is necessary. The idea proposed in [33] is based on a *perturbative* vacuum, constructed in a fully kinematic way (in the case of bosonic degrees of freedom the construction of the Hilbert space proposed in [33] consists, in fact, in choosing the Gaussian wave function as a vacuum state). In our opinion such a choice might possibly lead to an unphysical sector of the theory, because it is not plausible that the perturbative vacuum belongs to the physical sector. A possible way to avoid this difficulty could be based on an idea presented in [34], where one approximates the vacuum state of the continuum theory by the *true* vacuum states of its lattice approximations. For this purpose observe that the space of states $\mathcal{S}_\Lambda$ (not necessarily *pure states*, but all the *mixed states*) may be treated as being dual to the observable algebra $\mathcal{O}_\Lambda$. This implies that we have a family of dual mappings

$$P_{\Lambda_2, \Lambda_1}^* : \mathcal{S}_{\Lambda_1} \leftarrow \mathcal{S}_{\Lambda_2} , \quad (11.5)$$

defined for $\Lambda_1 \prec \Lambda_2$.

Physically, $\mathcal{O}_{\Lambda_1}$ may be thought of as a *subsystem* of a bigger physical system $\mathcal{O}_{\Lambda_2}$, containing more degrees of freedom. The above mapping assigns, to every state of a bigger system, a mixed state of the subsystem. This state is obtained by “forgetting” about those degrees of freedom which are not contained in the subsystem (such a “forgetting” operator was used e. g. by Einstein, Podolski and Rosen when they considered a system composed of two particles and its subsystem consisting of the first particle only; every wave function of two particles uniquely defines a mixed state of one particle).

Let $\omega_\Lambda \in \mathcal{S}_\Lambda$ denote the approximate vacuum state, corresponding to the minimum of the Hamiltonian (9.3). Let us project the vacuum from “finer lattices” backwards to “coarser lattices” and define

$$\omega_{\Lambda_1, \Lambda_2} := P_{\Lambda_2, \Lambda_1}^* \omega_{\Lambda_2} \in \mathcal{S}_{\Lambda_1} . \quad (11.6)$$

Suppose that there exists a limit

$$\Omega_{\Lambda_1} = \lim_{\Lambda_2} \omega_{\Lambda_1, \Lambda_2} . \quad (11.7)$$

If this is true for every Λ_1 , then the compatibility condition

$$\Omega_{\Lambda_1} = P_{\Lambda_2, \Lambda_1}^* \Omega_{\Lambda_2} \quad (11.8)$$

is automatically fulfilled and the state $\Omega := \{\Omega_\Lambda\}$ belongs to the projective limit of spaces $\mathcal{S}_\Lambda$ and, therefore, defines a state on the inductive limit of the algebras $\mathcal{O}_\Lambda$. As a limit of approximate vacuum states, it is a natural candidate for the non-perturbative vacuum of the continuum theory and the starting point for the GNS construction of the Hilbert space of its quantum states.

The existence of the limit (11.7) may be extremely difficult to prove in non-linear theories. A realistic attitude consists, therefore, in a detailed analysis of lattice approximations of the theory, presented in this paper. If the continuum limit of this approximations does exist, the numerical results obtained on the level of a sufficiently “late” lattice Λ should approximate the true physical quantities which – in principle – may be calculated from the continuum theory. On the other hand, if these approximate results did not have anything to do with the real, physical quantities and, in particular, if they did not “stabilize” with respect to Λ , there would be no reason to expect the existence of the continuum limit of the theory.

Appendices

A Proof of the Canonical Commutation Relations for Reconstructed Generators

Let there be given an arbitrary path α , say from u to v , together with the reconstructed operator $\hat{\mathcal{M}}_{\alpha;L}^K$. Consider a link $(y, y + \hat{k})$ together with the reconstructed operator $\hat{E}_{y,y+\hat{k}}$. If $(y, y + \hat{k}) \notin \mathcal{T}$, then (4.36) follows immediately from (5.8) and the fact that $\hat{\mathcal{M}}_{\alpha;L}^K$ is related with the corresponding operator $\hat{m}_{ij}$ via formula (4.31). We are left with the case $(y, y + \hat{k}) \in \mathcal{T}$. Denote by γ the unique tree path from the root x_0 to y and by $\tilde{\gamma}$ an arbitrary path from $y + \hat{k}$ to a tree end point, say x . The above described reconstruction of the on-tree electric field $\hat{E}_{y,y+\hat{k}}$ yields

$$\hat{E}_{y,y+\hat{k}} = \hat{E}_{x,\infty} + \sum_{x \in \tilde{\gamma}} \sum_l \hat{\mathcal{E}}_{x,x+\hat{l}}(\mathcal{T}) - \sum_{x \in \tilde{\gamma}} \hat{j}_x^0, \quad (\text{A.1})$$

where l runs over off-tree directions. Using (4.28) and (4.34) we get

$$\left[\hat{\mathcal{M}}_{\alpha;P}^Q, \sum_{x \in \tilde{\gamma}} \hat{j}_x^0 \right] = e \sum_{x \in \tilde{\gamma}} (\delta_{ux} - \delta_{vx}) \hat{\mathcal{M}}_{\alpha;P}^Q. \quad (\text{A.2})$$

We denote the unique tree paths from the root x_0 to u and v by α_1 and α_2 respectively. Then the closed oriented path $\tilde{\alpha}$, uniquely defined by α , is given by $\tilde{\alpha} = \alpha \alpha_1 \alpha_2^{-1}$. Due to (4.31) we have

$$\hat{\mathcal{M}}_{\alpha;P}^Q = \exp(-ig\hat{B}_{\tilde{\alpha}}) \hat{m}_{ij}, \quad (\text{A.3})$$

with $i = (u, P)$ and $j = (v, Q)$. Thus, using (5.8) we get

$$\left[\hat{\mathcal{M}}_{\alpha;P}^Q, \sum_{x \in \tilde{\gamma}} \sum_l \hat{\mathcal{E}}_{x,x+i}(\mathcal{T}) \right] = \left[\exp(-ig\hat{B}_{\tilde{\alpha}}), \sum_{x \in \tilde{\gamma}} \sum_l \hat{\mathcal{E}}_{x,x+i}(\mathcal{T}) \right] \hat{m}_{ij}. \quad (\text{A.4})$$

Obviously, the commutator (A.2) vanishes except for the following three cases:

1. $u \in \tilde{\gamma}$ and $v \notin \tilde{\gamma}$,
2. $u \notin \tilde{\gamma}$ and $v \in \tilde{\gamma}$,
3. $u \in \tilde{\gamma}$ and $v \in \tilde{\gamma}$.

It is an easy exercise to show that in all three cases the commutator (A.4) coincides exactly with the right-hand side of (A.2), provided we assume that α and γ have no common links. Using the fact that $\hat{E}_{x,\infty} = E_{x,\infty} \mathbb{1}$, we get that $\hat{\mathcal{M}}_{\alpha;P}^Q$ and $\hat{E}_{y,y+k}$ commute for $(y, y+k) \in \mathcal{T}$ and α and γ having no common links. For the case α and γ having common links we get an additional term in the commutator (A.4), which gives exactly the right-hand side of (4.36). Finally, using again (A.1), we get immediately (4.37). $\square$

B Proof of Theorem 2

Consider a long projector (7.14) of maximal length $\#K = \#L = N - |Z|$. Identities (6.10) and (6.11) imply that for $k \in K_+$ and $l \in L_+$ the operator $\hat{m}_{kl}$ maps $H_{(K_-, L_-)(K_+, L_+)}$ into $H_{(K_- \cup \{k\}, L_- \cup \{l\})(K_+ \setminus \{k\}, L_+ \setminus \{l\})}$ and $\hat{m}_{kl}^*$ maps the latter space back into the first one. These mappings are mutually inverse because $\hat{m}_{kl}^* \hat{m}_{kl} = Q_{kl}$ reduces to the identity on the first space and $\hat{m}_{kl} \hat{m}_{kl}^* = P_{kl}$ reduces to the identity on the second one. Hence, these mappings give unitary isomorphisms between the corresponding subspaces. If one of them is nontrivial, then all are nontrivial. We have, therefore, constructed a net of mutually isomorphic subspaces, with the operators $\hat{m}_{kl}$ and $\hat{m}_{kl}^*$ playing the role of unitary intertwining operators between them. However, this construction makes sense only for $k \in K$ and $l \in L$.

Observe that for $\Omega \in H_{(K_-, L_-)(K_+, L_+)}$ and the “external indices” $r \notin K$, $s \notin L$ we have

$$\hat{Q}_{rs} \Omega = 0 = \hat{P}_{rs} \Omega. \quad (\text{B.1})$$

Indeed, the assumption that the contrary is true, i. e. that we have

$$0 \neq \hat{Q}_{rs} \Omega = \hat{Q}_{rs} P_{(K_-, L_-)(K_+, L_+)} \Omega = P_{(K_-, L_-)(K_+ \cup \{r\}, L_+ \cup \{s\})} \Omega \quad (\text{B.2})$$

or

$$0 \neq \hat{P}_{rs} \Omega = \hat{P}_{rs} P_{(K_-, L_-)(K_+, L_+)} \Omega = P_{(K_- \cup \{r\}, L_- \cup \{s\})(K_+, L_+)} \Omega \quad (\text{B.3})$$

would contradict our assumption about vanishing of all the projectors, which are longer than the maximal rank $N - |Z|$.

We conclude that $\hat{m}_{rs}$ and $\hat{m}_{rs}^*$, with “external indices” $r \notin K$, $s \notin L$, annihilate all the above constructed subspaces $H_{(K_-, L_-)(K_+, L_+)}$, because $\|\hat{m}_{rs}\Omega\|^2 = (\Omega|\hat{Q}_{rs}\Omega)$ and $\|\hat{m}_{rs}^*\Omega\|^2 = (\Omega|\hat{P}_{rs}\Omega)$.

Suppose that also all the remaining operators $\hat{m}_{rl}^*$ and $\hat{m}_{ks}^*$, with $r \notin K$, $l \in L$, $k \in K$, $s \notin L$, annihilate all the subspaces $H_{(K_-, L_-)(K_+, L_+)}$. Since $\hat{m}_{rl} = [\hat{m}_{rs}, \hat{m}_{ks}^*, \hat{m}_{kl}]$ we see that also $\hat{m}_{rl}$ annihilates all these subspaces (in the products of the three operators $\hat{m}_{kl}$ flips between our subspaces and the remaining two operators annihilate all of them). The same is also true for $\hat{m}_{ks} = [\hat{m}_{kl}, \hat{m}_{rl}^*, \hat{m}_{rs}]$. Hence, the sum of these subspaces is an invariant subspace for the representation. It follows from the irreducibility of the representation that this sum is equal to the entire Hilbert space, but the representation would not be faithful.

We conclude, that at least one among the operators $\hat{m}_{rl}^*$ and $\hat{m}_{ks}^*$ does not vanish identically on the sum of our subspaces. We restrict ourselves to the discussion of the case that $\hat{m}_{k_0 s_0}^*$, with $k_0 \in K$, $s_0 \notin L$, does not vanish identically on the direct sum of the spaces $H_{(K_-, L_-)(K_+, L_+)}$. The second case can be dealt with in a completely analogous way. Of course, $\hat{m}_{ks}^*$ vanishes on all those subspaces, for which $k \in K_+$, because we have

$$\hat{m}_{ks}^*\Omega = \hat{m}_{ks}^*\hat{Q}_{kl}\Omega = \hat{m}_{ks}^*\hat{m}_{kl}^*\hat{m}_{kl}\Omega = 0 \quad (\text{B.4})$$

for some $l \in L_+$. The identity $\hat{m}_{ks}^*\hat{m}_{kl}^* = 0$ follows from (5.5). We suppose, therefore, that there is a vector Ω belonging to the direct sum of the subspaces (7.16) having $k_0 \in K_-$, and such that

$$\hat{m}_{k_0 s_0}^*\Omega \neq 0 \quad (\text{B.5})$$

for some $s_0 \notin L$.

Lemma 8 *For $k_0 \in K_-$ and $s_0 \notin L$, the operator $\hat{m}_{k_0 s_0}^*$ does not vanish identically on any of the subspaces $H_{(K_-, L_-)(K_+, L_+)}$.*

Proof. Suppose, on the contrary, that $\hat{m}_{k_0 s_0}^*$ vanishes on one of these subspaces. We know that there exists a polynomial combination of operators $\hat{m}_{ij}^*$ and $\hat{m}_{ij}$, with $i \in K$, $j \in L$ and $i \neq k_0$, giving a unitary isomorphism between any two such spaces. Since these operators commute with $\hat{m}_{k_0 s_0}^*$, we have for $\hat{m}_{ij}\Omega \neq 0$

$$(\hat{m}_{k_0 s_0}^*\hat{m}_{ij}\Omega|\hat{m}_{k_0 s_0}^*\hat{m}_{ij}\Omega) = (\hat{m}_{k_0 s_0}^*\Omega|\hat{m}_{k_0 s_0}^*\hat{Q}_{ij}\Omega) = (\hat{m}_{k_0 s_0}^*\Omega|\hat{m}_{k_0 s_0}^*\Omega), \quad (\text{B.6})$$

because $\hat{Q}_{ij}\Omega = \Omega$ in this case. A similar argument works if we replace $\hat{m}_{ij}$ by $\hat{m}_{ij}^*$ and $\hat{Q}_{ij}$ by $\hat{P}_{ij}$. This means that vanishing of $\hat{m}_{k_0 s_0}^*$ on one of these subspaces would imply vanishing on all of them. $\square$

We conclude that, for $k_0 \in K_-$ and $s_0 \notin L$, also $P_{k_0 s_0}$ does not vanish identically on any of the subspaces $H_{(K_-, L_-)(K_+, L_+)}$. Let us, therefore, choose one of them and let us consider the corresponding restricted subspace

$$H_0 := \hat{P}_{k_0 s_0} H_{(K_-, L_-)(K_+, L_+)} , \quad (\text{B.7})$$

with $k_0 \in K_-$ and $s_0 \notin L$.

Lemma 9 *Let $\Omega \in H_0$. For $k \notin \mathcal{N} \setminus K_-$ and $s \notin L_+$ we have*

$$P_{ks} \Omega = \Omega . \quad (\text{B.8})$$

Proof. 1. For $k \in K_-$ and $s \in L_-$ the assertion follows trivially from the fact that $P_{(K_-, L_-)(K_+, L_+)}$ contains P_{ks} as a factor.

2. It remains to consider the case $k \in K_-$ and $s \notin L$. For that purpose take the following operator

$$\hat{l}_{k_0 k} := [\hat{m}_{kl}^*, \hat{m}_{k_0 l}] , \quad (\text{B.9})$$

with an arbitrary $l \in L_-$. In a first step we show that for any $\Omega \in H_{(K_-, L_-)(K_+, L_+)}$ we have:

$$\hat{l}_{k_0 k} \hat{m}_{k_0 s_0}^* \Omega = \hat{m}_{k s_0}^* \Omega , \quad (\text{B.10})$$

where $k \in K_-$ and $k_0 \in K_-$. Indeed,

$$\begin{aligned} \hat{l}_{k_0 k} \hat{m}_{k_0 s_0}^* \Omega &= (\hat{m}_{kl}^* \hat{m}_{k_0 l} - \hat{m}_{k_0 l} \hat{m}_{kl}^*) \hat{m}_{k_0 s_0}^* \Omega = (\hat{m}_{k_0 s_0} \hat{m}_{k_0 l}^* \hat{m}_{kl} - \hat{m}_{k_0 s_0} \hat{m}_{kl}^* \hat{m}_{k_0 l})^* \Omega \\ &= (\hat{m}_{k s_0} - \hat{m}_{kl} \hat{m}_{k_0 l}^* \hat{m}_{k_0 s_0} + \hat{m}_{k_0 l}^* \hat{m}_{kl} \hat{m}_{k_0 s_0})^* \Omega \\ &= \hat{m}_{k s_0}^* \Omega - \hat{m}_{k_0 s_0}^* \hat{m}_{k_0 l} \hat{m}_{kl}^* \Omega + \hat{m}_{k_0 s_0}^* \hat{m}_{kl}^* \hat{m}_{k_0 l} \Omega . \end{aligned} \quad (\text{B.11})$$

But, $\Omega = \hat{P}_{k_0 l} \Omega = \hat{m}_{k_0 l} \hat{m}_{k_0 l}^* \Omega$. Hence $\hat{m}_{k_0 l} \Omega = 0$ and the last term vanishes. Also the second term vanishes, because we have $\hat{m}_{k_0 l} \hat{m}_{kl}^* \hat{m}_{k_0 l} = 0$ according to (5.12). This proves (B.10).

Changing the role of k_0 and k , we have:

$$l_{k k_0} \hat{m}_{k s_0}^* \Omega = \hat{m}_{k_0 s_0}^* \Omega . \quad (\text{B.12})$$

Using $l_{k k_0} = l_{k_0 k}^*$ we have

$$\| \hat{m}_{k s_0}^* \Omega \|^2 = (\hat{l}_{k_0 k} \hat{m}_{k_0 s_0}^* \Omega | \hat{m}_{k s_0}^* \Omega) = (\hat{m}_{k_0 s_0}^* \Omega | l_{k k_0} \hat{m}_{k s_0}^* \Omega) = \| \hat{m}_{k_0 s_0}^* \Omega \|^2 . \quad (\text{B.13})$$

Next, take the following operator

$$\hat{r}_{s s_0} := [\hat{m}_{i s_0}, \hat{m}_{i s}^*] , \quad (\text{B.14})$$

with an arbitrary $i \notin K$. We prove that for any $\Omega \in H_{(K_-, L_-)(K_+, L_+)}$ we have:

$$-\hat{r}_{ss_0} \hat{m}_{ks_0}^* \Omega = \hat{m}_{ks}^* \Omega, \quad (\text{B.15})$$

where $s \notin L$ and $s_0 \notin L$. Indeed,

$$\begin{aligned} -\hat{r}_{ss_0} \hat{m}_{ks_0}^* \Omega &= (\hat{m}_{is}^* \hat{m}_{is_0} - \hat{m}_{is_0} \hat{m}_{is}^*) \hat{m}_{ks_0}^* \Omega = (\hat{m}_{ks_0} \hat{m}_{is_0}^* \hat{m}_{is} - \hat{m}_{ks_0} \hat{m}_{is} \hat{m}_{is_0}^*)^* \Omega \\ &= (\hat{m}_{ks} - \hat{m}_{is} \hat{m}_{is_0}^* \hat{m}_{ks_0} + \hat{m}_{is_0}^* \hat{m}_{is} \hat{m}_{ks_0})^* \Omega \\ &= \hat{m}_{ks}^* \Omega - \hat{m}_{ks_0}^* \hat{m}_{is_0} \hat{m}_{is}^* \Omega + \hat{m}_{ks_0}^* \hat{m}_{is}^* \hat{m}_{is_0} \Omega = \hat{m}_{ks}^* \Omega, \end{aligned} \quad (\text{B.16})$$

where we have used the fact that the operators $\hat{m}_{is}$ and $\hat{m}_{is}^*$ with external indices ($i \notin K$ and $s \notin L$) annihilate our subspaces $H_{(K_-, L_-)(K_+, L_+)}$.

Again, changing the role of s and s_0 we have

$$-\hat{r}_{s_0s} \hat{m}_{ks}^* \Omega = \hat{m}_{ks_0}^* \Omega. \quad (\text{B.17})$$

Using $\hat{r}_{s_0s}^* = \hat{r}_{ss_0}$ we get $\|\hat{m}_{ks}^* \Omega\|^2 = \|\hat{m}_{ks_0}^* \Omega\|^2$ for any $k \in K_-$ and $s \notin L$.

We conclude that for every $k \in K_-$ and $s \notin L$, the identity

$$\|\hat{m}_{ks}^* \Omega\|^2 = \|\hat{m}_{ks_0}^* \Omega\|^2 \quad (\text{B.18})$$

or, equivalently,

$$(\Omega | \hat{P}_{ks} \Omega) = (\Omega | \hat{P}_{ks_0} \Omega), \quad (\text{B.19})$$

holds, with $\Omega \in H_{(K_-, L_-)(K_+, L_+)}$. Hence, for vectors fulfilling $\hat{P}_{ks_0} \Omega = \Omega$ we have $\hat{P}_{ks} \Omega = \Omega$. $\square$

Lemma 10 *Let there be given a pair $(k, l) \in (K, L)$ and let us denote in accordance with (B.7)*

$$H_1 := \hat{P}_{k_0 s_0} H_{(K_- \setminus \{k\}, L_- \setminus \{l\})(K_+ \cup \{k\}, L_+ \cup \{l\})}, \quad (\text{B.20})$$

for $(k, l) \in (K_-, L_-)$, and

$$H_2 := \hat{P}_{k_0 s_0} H_{(K_- \cup \{k\}, L_- \cup \{l\})(K_+ \setminus \{k\}, L_+ \setminus \{l\})}, \quad (\text{B.21})$$

for $(k, l) \in (K_+, L_+)$.

1. The operators $\hat{m}_{kl}^*$ map H_0 isomorphically onto H_1 if $k \in K_-$ and $l \in L_-$, and vanish otherwise.
2. The operators $\hat{m}_{kl}$ map H_0 isomorphically onto H_2 if $k \in K_+$ and $l \in L_+$, and vanish otherwise.

Proof. Let $\Omega \in H_0$. We know already that

$$\hat{m}_{kl}^* \Omega \in H_{(K_- \setminus \{k\}, L_- \setminus \{l\})(K_+ \cup \{k\}, L_+ \cup \{l\})}. \quad (\text{B.22})$$

Using the fact that $\hat{m}_{kl}^*$ commutes with $\hat{P}_{is}$, for $i \neq k$ and $s \notin L$, we get $\hat{P}_{is} \hat{m}_{kl}^* \Omega = \hat{m}_{kl}^* \Omega$ for $i \in K_- \setminus \{k\}$ and $l \in L_- \setminus \{l\}$.

Analogously, we have

$$\hat{m}_{kl} \Omega \in H_{(K_- \cup \{k\}, L_- \cup \{l\})(K_+ \setminus \{k\}, L_+ \setminus \{l\})}. \quad (\text{B.23})$$

But

$$\hat{P}_{ks} \hat{m}_{kl} \Omega = \hat{m}_{ks} \hat{m}_{ks}^* \hat{m}_{kl} \Omega = (\hat{m}_{kl} - \hat{m}_{kl} \hat{m}_{ks}^* \hat{m}_{ks}) \Omega = \hat{m}_{kl} \Omega, \quad (\text{B.24})$$

because choosing any $i \in K_-$ we have $\hat{m}_{ks} \Omega = \hat{m}_{ks} \hat{P}_{is} \Omega = 0$.

Obviously, $\hat{m}_{kl}$ and $\hat{m}_{kl}^*$ are mutually inverse isometries, because $\hat{m}_{kl} \hat{m}_{kl}^* = \hat{P}_{kl}$ reduces to the identity in the first part and $\hat{m}_{kl}^* \hat{m}_{kl} = \hat{Q}_{kl}$ reduces to the identity in the second part of the Lemma. $\square$

Lemma 11 *Let $\Omega \in H_0$. For $k \in \mathcal{N} \setminus K_-$ or $s \in L_+$ we have*

$$P_{ks} \Omega = 0. \quad (\text{B.25})$$

Proof. 1. For $k \in K_+$ and s arbitrary the proof is trivial, because $P_{(K_-, L_-)(K_+, L_+)}$ contains a factor Q_{kr} , which is annihilated by P_{ks} , due to (6.8).

2. For k arbitrary and $s \in L_+$ a similar argument applies, because $P_{(K_-, L_-)(K_+, L_+)}$ contains a factor Q_{js} , which is annihilated by P_{ks} , due to (6.9).

3. It remains to consider the case $k \notin K$ and s arbitrary. Since $\hat{P}_{ks} = \hat{m}_{ks} \hat{m}_{ks}^*$, it is sufficient to prove that the operators $\hat{m}_{ks}^*$, with $k \notin K$, annihilate H_0 : Indeed, this statement is trivially true if $s \in L_+$ or $s \notin L$. For the case $s \in L_-$ we have

$$\hat{m}_{ks}^* = [\hat{m}_{kl}, \hat{m}_{rl}^*, \hat{m}_{rs}]^* = \hat{m}_{rs}^* \hat{m}_{rl} \hat{m}_{kl}^* + \hat{m}_{kl}^* \hat{m}_{rl} \hat{m}_{rs}^* - \hat{m}_{rl} \hat{m}_{rs}^* \hat{m}_{kl}^* - \hat{m}_{rs}^* \hat{m}_{kl}^* \hat{m}_{rl}, \quad (\text{B.26})$$

for any $k \in K_-$ and arbitrary l . But the first and the third operator on the right-hand side annihilate every $\Omega \in H_{(K_-, L_-)(K_+, L_+)}$, because both indices of $\hat{m}_{kl}^*$ are external with respect to (K, L) . This is also true for the second and the last term, because $\hat{m}_{rl}$ annihilates H_0 . $\square$

Lemma 12 *Let $\Omega \in H_0$. For $k \in \mathcal{N} \setminus K_-$ and $s \in L_+$ we have*

$$Q_{ks} \Omega = \Omega. \quad (\text{B.27})$$

Proof. 1. For $k \in K_+$ and $s \in L_+$ the assertion follows trivially from the fact that $P_{(K_-, L_-)(K_+, L_+)}$ contains Q_{ks} as a factor.

2. It remains to consider the case $k \notin K$ and $s \in L_+$. Choosing any $r \in K_+$ and $l \notin L$, we obtain

$$\hat{m}_{ks}\Omega = [\hat{m}_{kl}, \hat{m}_{rl}^*, \hat{m}_{rs}] \Omega = \hat{m}_{kl}\hat{m}_{rl}^*\hat{m}_{rs}\Omega, \quad (\text{B.28})$$

because the remaining terms coming from the double commutator annihilate Ω . Hence, denoting $\tilde{\Omega} := \hat{m}_{rs}\Omega$, we get

$$\|\hat{m}_{ks}\Omega\|^2 = (\tilde{\Omega}|\hat{m}_{rl}\hat{m}_{kl}^*\hat{m}_{kl}\hat{m}_{rl}^*\tilde{\Omega}) . \quad (\text{B.29})$$

But

$$\hat{m}_{rl}\hat{m}_{kl}^*\hat{m}_{kl}\hat{m}_{rl}^*\tilde{\Omega} = \hat{m}_{rl}(\hat{m}_{rl}^* - \hat{m}_{rl}^*\hat{m}_{kl}\hat{m}_{kl}^*)\tilde{\Omega} = \hat{m}_{rl}\hat{m}_{rl}^*\tilde{\Omega}, \quad (\text{B.30})$$

because both k and l are external indices. Due to Lemma 10, $\hat{m}_{rl}^*$ preserves the length of $\hat{m}_{rs}\Omega = \tilde{\Omega}$ and $\hat{m}_{rs}$ preserves the length of Ω . Hence,

$$\|\hat{m}_{ks}\Omega\|^2 = \|\hat{m}_{rl}^*\tilde{\Omega}\|^2 = \|\tilde{\Omega}\|^2 = \|\Omega\|^2 . \quad (\text{B.31})$$

On the other hand,

$$\|\hat{m}_{ks}\Omega\|^2 = (\Omega|\hat{Q}_{rl}\Omega), \quad (\text{B.32})$$

which implies (B.27). $\square$

Lemma 13 *Let $\Omega \in H_0$. For $k \notin \mathcal{N} \setminus K_-$ or $s \notin L_+$ we have*

$$Q_{ks}\Omega = 0. \quad (\text{B.33})$$

Proof. 1. For $k \in K_-$ and s arbitrary the proof is trivial, because $P_{(K_-, L_-)(K_+, L_+)}$ contains a factor P_{kr} , which is annihilated by Q_{ks} , due to (6.8).

2. For k arbitrary and $s \in L_-$ a similar argument applies, because $P_{(K_-, L_-)(K_+, L_+)}$ contains a factor P_{js} , which is annihilated by Q_{ks} , due to (6.9).

3. It remains to consider the case k arbitrary and $s \notin L$. Since $\hat{Q}_{ks} = \hat{m}_{ks}^*\hat{m}_{ks}$, it is sufficient to prove that the operators $\hat{m}_{ks}$ annihilate all the subspaces $H_{(K_-, L_-)(K_+, L_+)}$. This is obviously the case for $k \notin K_+$. Thus, it remains to consider the case $k \in K_+$ only. Choosing any $l \in L_+$, we have

$$\|\hat{m}_{ks}\Omega\|^2 = (\Omega|\hat{Q}_{ks}\Omega) = (\Omega|\hat{Q}_{ks}\hat{Q}_{kl}\Omega) . \quad (\text{B.34})$$

But

$$\hat{Q}_{ks}\hat{Q}_{kl} = \hat{m}_{ks}^*\hat{m}_{ks}\hat{m}_{kl}^*\hat{m}_{kl} = (\hat{m}_{kl}^* - \hat{m}_{kl}^*\hat{m}_{ks}\hat{m}_{ks}^*)\hat{m}_{kl} . \quad (\text{B.35})$$

Since $\hat{m}_{ks}^*$ preserves the length of $\hat{m}_{kl}\Omega$, we get

$$\|\hat{m}_{ks}\Omega\|^2 = \|\hat{m}_{kl}\Omega\|^2 - \|\hat{m}_{ks}^*\hat{m}_{kl}\Omega\|^2 = 0 . \quad (\text{B.36})$$

$\square$

Equations (B.8), (B.25), (B.27) and (B.33) prove that the subspace H_0 , defined by (B.7), satisfies conditions (7.8) – (7.11), if we put $I = \mathcal{N} \setminus K_-$ and $J = L_+$.

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References

- [1] J. v. Neumann, Math. Ann., bd. 104 (1931) 570
- [2] P. Jordan and E. P. Wigner, Zeitschr. f. Phys. 47 (1928) 631
- [3] L. Gårding and A. S. Wightman, Proc. of the Nat. Acad. of Sc. 40 (1954) 617
- [4] J. Kijowski, G. Rudolph, Lett. Math. Phys. 29 (1993) 103
- [5] J. Kijowski, M. Rudolph and G. Rudolph, Lett. Math. Phys. 33 (1995) 139
- [6] T. Takabayashi, Suppl. Progr. Theor. Phys., No. 4 (1957) 1
- [7] J. Mickelson, Czech J. Phys. B 32 (1982) 521
- [8] S. G. Mikhov and D. T. Stoyanov, preprint E2-12865 Dubna (1979)
- [9] A. Jakubiec, J. Kijowski, Lett. Math. Phys. 9 (1985) 1 ,
A. Jakubiec, Lett. Math. Phys. 9 (1985) 171
- [10] J. Kijowski and G. Rudolph, Nucl. Phys. B325 (1989) 211
- [11] L. D. Faddeev and V. N. Popov, Phys. Lett B25 (1967) 30
- [12] S. Mandelstam, Ann. Phys. 19 (1962) 1
Phys. Rev. D, Vol. 175, N. 5 (1968) 1580
- [13] J. Kijowski and G. Rudolph, Lett. Math. Phys. 16 (1988) 27
- [14] J. Kijowski and G. Rudolph, Phys. Rev. D31 (1985) 856 ,
G. Rudolph, Annalen der Physik, 7. Folge, Bd. 47, 2/3 (1990) 211
- [15] J. Kijowski, A. Thielmann, J. of Geom. and Phys. 19 (1996) 173
- [16] J. Kijowski, G. Rudolph and M. Rudolph, "Effective Bosonic Degrees of Freedom for One-flavour Chromodynamics", to appear in Ann. Inst. H. Poincaré
- [17] F. A. Berezin, *The Method of Second Quantization*, Academic Press, New York and London 1966

- [18] R. Haag and D. Kastler, *J. Math. Phys.* 5 (1964) 848
- [19] S. Doplicher, R. Haag and J. Roberts, *Commun. Math. Phys.* 23 (1971) 199
- [20] F. Strocchi and A. Wightman, *J. Math. Phys.* 15 (1974) 2198
- [21] F. Strocchi, *Commun. Math. Phys.* 56 (1977) 57
- [22] F. Strocchi, *Phys. Rev. D* 17 (1978) 2010
- [23] J. Fröhlich, *Commun. Math. Phys.* 66 (1979), 223
- [24] D. Buchholz, *Commun. Math. Phys.* 85 (1982) 49
D. Buchholz, *Phys. Lett. B* 174 (1986) 331
- [25] K. Fredenhagen and M. Marcu, *Commun. Math. Phys.* 92 (1983) 81
- [26] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Phys., vol. 159, Springer (1982)
E. Seiler, "Constructive Quantum Field Theory: Fermions", in *Gauge Theories: Fundamental Interactions and Rigorous Results*, eds. P. Dita, V. Georgescu, R. Purice
- [27] K. Kuchař, *Phys. Rev. D* 34 (1986) 3031; *Phys. Rev. D* 34 (1986) 3044
- [28] I. Białynicki-Birula, *The Hamiltonian of Quantum Dynamics*, in "QED and Quantum Optics", Ed. A. O. Barut, Plenum Press, N. Y. 1984
- [29] W. H. Greub, *Linear Algebra*, Springer-Verlag 1967
- [30] S. L. Woronowicz, *Rev. Math. Phys.* 7 (1995) 481
- [31] S. Baaĵ and P. Jungl, *C. R. Acad. Sci. Paris, Série I*, 296 (1983) 875
- [32] J. Dixmier, *Les C^* -Algebres et leurs Representations*, Paris Gauthier-Villars Editeur 1969
- [33] A. Ashtekar and J. Lewandowski, *J. of Geom. Phys.* 17 (1995) 191
- [34] J. Kijowski, *Rep. Math. Phys.* 11 (1977) 97
S. Zakrzewski, *On the lattice approximation and Feynman path integrals for gauge fields*, Thesis, Warsaw 1980, unpublished
S. Zakrzewski, *Unitary relations*, *Publicações de Física Matemática* 4 (1985), Coimbra