Dimensional Regularization in Quarkonium Calculations

Eric Braaten and Yu-Qi Chen

Physics Department, Ohio State University, Columbus OH 43210

Abstract

Dimensional regularization is incompatible with the standard covariant projection methods that are used to calculate the short-distance coefficients in inclusive heavy quarkonium production and annihilation rates. A new method is developed that allows dimensional regularization to be used consistently to regularize the infrared and ultraviolet divergences that arise in these perturbative calculations. We illustrate the method by calculating the leading color-octet terms and the leading color-singlet terms in the gluon fragmentation functions for arbitrary quarkonium states. We resolve a discrepancy between two previous calculations of the gluon fragmentation functions for the spin-triplet P-wave quarkonium states.
I. INTRODUCTION

A rigorous theoretical framework for calculating inclusive production cross sections (and inclusive annihilation decay rates) for heavy quarkonium in QCD has recently been developed [1]. The cross section (or decay rate) is expressed as a sum of the product of short-distance coefficients and long-distance matrix elements. The short-distance coefficients can be calculated by perturbation theory to any order in $\alpha_s(m_c)$, the running coupling constant at the energy scale of the heavy quark mass. The long-distance factors are expressed as matrix elements in nonrelativistic QCD (NRQCD), an effective field theory that reproduces QCD to any desired order in the velocity of the heavy quark. The matrix elements scale in a definite way with $v$, the typical relative velocity of the heavy quark in quarkonium. The NRQCD factorization formalism therefore allows the inclusive production cross section to be calculated systematically as a double expansion in $\alpha_s(m_c)$ and $v$.

In calculating the short-distance coefficients beyond leading order in $\alpha_s$, ultraviolet divergences and infrared divergences inevitably arise and need to be regulated. In perturbative calculations, the most convenient method for regulating both ultraviolet and infrared divergences is dimensional regularization. The main advantage of this method is that it preserves many of the symmetries of a field theory, including gauge invariance and Lorentz invariance. It also often leads to simpler expressions for the finite parts of a perturbative calculation. One feature of dimensional regularization that is particularly convenient in a nonrenormalizable effective field theory like NRQCD is that it automatically sets power ultraviolet divergences to zero. This makes it unnecessary to explicitly subtract these divergences as part of the renormalization procedure. Dimensional regularization has been widely used in perturbative calculations of the cross sections and decay rates of elementary particles. It has also been used in some perturbative calculations involving quarkonium, but the consistency of the method has not been carefully examined.

Most calculations of production cross sections and annihilation decay rates for heavy quarkonium have been carried out using the covariant projection method [2]. This method
involves the projection of a $c\bar{c}$ pair onto states with definite total angular momentum $J$. For orbital angular momentum $L = 1$ and higher, the projections are specific to 3 dimensions. In dimensional regularization, momentum integrals are analytically continued to $N = 3 - 2\epsilon$ dimensions. There is therefore a potential inconsistency in combining the covariant projection method with dimensional regularization. Moreover, the projections for spin-singlet states, such as $^1S_0$ and $^1P_1$, involve the $\gamma_5$ matrix, whose extension to $N$ dimensions is problematic.

We have recently developed an alternative method for calculating production cross sections and annihilation decay rates for heavy quarkonium which fully exploits the NRQCD factorization framework [3]. We refer to it as the *threshold expansion method*. In this method, a quantity that is closely related to the cross section for the production of a $c\bar{c}$ pair with total momentum $P$ is calculated using perturbation theory in full QCD and expanded in powers of the relative 3-momentum $q$ of the $c\bar{c}$ pair. Matrix elements of certain 4-fermion operators in NRQCD are also calculated using perturbation theory and expanded around the threshold $q = 0$. The short-distance coefficients in the factorization formula are then determined by matching these perturbative expressions. Finally, the NRQCD matrix elements for quarkonium states are simplified by using rotational symmetry, heavy-quark spin symmetry, and the vacuum saturation approximation.

In this paper, we generalize this method to $N$ spatial dimensions, so that dimensional regularization can be consistently used to regularize infrared and ultraviolet divergences. The perturbative calculation of the QCD side of the matching condition and its expansion around the threshold generalize easily to $N$ dimensions. The perturbative calculation and threshold expansion of the NRQCD matrix elements can also be carried out in $N$ dimensions, provided that the matrix elements are defined in a way that generalizes to $N$ dimensions. By matching the cross section and the matrix elements, we obtain the short-distance coefficients. After renormalization of coupling constants in QCD and NRQCD, the short-distance coefficients may have poles in $N - 3$, which must be removed by renormalization of the 4-fermion operators in NRQCD. One must take care to avoid simplifying the matrix elements of these operators using identities that are specific to 3 dimensions until after these renormalizations.
have been carried out.

In Section II, we review the threshold expansion method of Ref. [3], generalizing it to $N$ spatial dimensions. In Section III, we give a definition of the gluon fragmentation function for quarkonium states that is particularly convenient for low order calculations. We then present a matching prescription for calculating the short-distance coefficients in the factorization formula for these fragmentation functions. In Sections IV and V, we illustrate our calculational method by calculating the color-octet terms in the gluon fragmentation function for a general quarkonium state to order $\alpha_s$ and the color-singlet terms to order $\alpha_s^2$. The color-singlet calculation involves a pole in $N - 3$ which must be removed by the renormalization of an NRQCD matrix element. We apply our general formula for the gluon fragmentation function to S-wave and P-wave states in Section VI. We resolve a discrepancy between two previous calculations of the gluon fragmentation function for $^3P_J$ states.

II. THRESHOLD EXPANSION METHOD

We begin by reviewing the threshold expansion method of Ref. [3], generalizing it to $N$ dimensions. Cartesian vectors and Lorentz vectors are extended to $N$ and $N + 1$ components, respectively. The inclusive cross section for producing the heavy quarkonium $H$ with momentum $P$ can be written in a factorized form [1]:

\[
\sum_X d\sigma(12 \to H(P) + X) = \frac{1}{4E_1E_2v_{12}} \frac{d^N P}{(2\pi)^{N} 2E_P} \sum_{mn} C_{mn} \langle O_{mn}^H \rangle,
\]

(1)

where $E_P = \sqrt{M_H^2 + P^2}$. The coefficients $C_{mn}$ take into account the effects of short distances of order $1/m_c$ or smaller, and therefore can be calculated as perturbation series in the QCD coupling constant $\alpha_s(m_c)$. The matrix elements $\langle O_{mn}^H \rangle$ are expectation values in the NRQCD vacuum of local 4-fermion operators that have the structure

\[
O_{mn}^H = \chi \psi \kappa_{m} \psi \kappa_{n} \chi,
\]

(2)

where $\psi$ and $\chi$ are the field operators for the heavy quark and antiquark in NRQCD, and $\kappa_{n}$ and $\kappa_{m}$ are products of a color matrix (1 or $T^a$), a spin matrix, and a polynomial in the
gauge covariant derivative $D$ in $N$ dimensions. The spin matrix is either the unit matrix or a polynomial in the Pauli matrices $\sigma^i$. The Pauli matrices in $N$ dimensions satisfy the anticommutation relations

$$\{ \sigma_i, \sigma_j \} = 2\delta_{ij}, \quad i, j = 1, \ldots, N. \quad (3)$$

In 3 dimensions, they also satisfy the commutation relations

$$[ \sigma_i, \sigma_j ] = 2i \epsilon_{ijk} \sigma_k, \quad i, j, k = 1, 2, 3. \quad (4)$$

Using both the relations (3) and (4), the spin matrices in $\mathcal{K}_n$ and $\mathcal{K}_m^\dagger$ in (2) can be reduced to a linear combination of 1 and $\sigma^i$. However, the commutation relations (4) can be used to simplify the NRQCD matrix elements only after all poles in $N-3$ have been removed from the short-distance coefficients. Until these poles are removed, we must allow for additional spin matrices in $\mathcal{K}_n$ and $\mathcal{K}_m^\dagger$, such as $[\sigma_i, \sigma_j]$ and $\{[\sigma_i, \sigma_j], \sigma_k\}$. The projection operator $\mathcal{P}_H$ in (1) can be written

$$\mathcal{P}_H = \sum_S \left| H(P = 0) + S \right> \left< H(P = 0) + S \right|,$$

where the sum is over soft hadron states $S$ whose total energy is less than the ultraviolet cutoff $\Lambda$ of NRQCD. This operator projects onto the subspace of states which in the asymptotic feature include the quarkonium state $H$ at rest plus soft hadrons. The standard relativistic normalization of the states in (5) is

$$\left< H(P')\left| H(P) \right> = 2E_{\mathcal{P}} (2\pi)^N \delta^N(P - P'). \quad (6)$$

With this normalization of states, the projection operator $\mathcal{P}_H$ has energy dimension $1 - N$.

The short distance coefficients $C_{mn}$ in (1) can be determined by matching perturbative calculations of the corresponding process in which the quarkonium $H$ is replaced by a $c\bar{c}$ pair. Let $c\bar{c}(P, q, \xi, \eta)$ represent a state that consists of a $c$ and a $\bar{c}$ with total momentum $P$, spatial momenta $\pm q$ in the $c\bar{c}$ rest frame, and spin and color states specified by the spinors $\xi$ and $\eta$. Color and spin indices on these spinors are suppressed. The number of spin components
for these Pauli spinors is that appropriate to Pauli spinors in $N$ spatial dimensions. Using the abbreviated notation $cc(P) \equiv cc(P, q, \xi, \eta)$ and $cc'(P) \equiv cc(P, q', \xi', \eta')$, the matching condition in the threshold expansion method of Ref. [3] is

$$
\sum_X (2\pi)^{N+1} \delta^{N+1}(k_1 + k_2 - P - k_X) \left( T_{cc'(P)+X} \right)^* T_{cc(P)+X} \bigg|_{pQCD} = \sum_{mn} C_{mn} \langle \chi^\dagger K_n^\dagger \psi_{cc}' \psi_{cc} K_n \chi \rangle \bigg|_{pNRQCD}, \tag{7}
$$

where $k_1$ and $k_2$ are the momenta of the incoming particles and $k_X$ is the sum of the momenta of all the outgoing particles except the $c$ and $\bar{c}$. The operator $P_{cc',\bar{c}\bar{c}}$ in the matrix element in (7) is defined by

$$
P_{cc',\bar{c}\bar{c}} = \sum_S \left| c(q_1', \xi')\bar{c}(-q_1', \eta') + S \right\rangle \left\langle c(q_1, \xi)\bar{c}(-q_1, \eta) + S \right|.	ag{8}
$$

The sum is over soft parton states whose total energy is less than the ultraviolet cutoff $\Lambda$ of NRQCD. The standard relativistic normalization is

$$
\langle c(q_1', \xi')\bar{c}(q_2', \eta') \mid c(q_1, \xi)\bar{c}(q_2, \eta) \rangle = 4E_{q_1}E_{q_2} (2\pi)^{2N} \delta^N(q_1 - q_1') \delta^N(q_2 - q_2') \xi^\dagger \xi' \eta^\dagger \eta, \tag{9}
$$

where $E_q = \sqrt{m_c^2 + q^2}$. The spinors are normalized so that $\xi^\dagger \xi = 1$, and similarly for $\eta$, $\xi'$ and $\eta'$. In expressions like $\xi^\dagger \xi'$, both the spin and color indices are contracted. With the normalization (9), the projection operator $P_{cc',\bar{c}\bar{c}}$ has dimension $2 - 2N$. To carry out the matching procedure, the left side of (7) is calculated using perturbation theory in full QCD, and then expanded in powers of $q$ and $q'$. The matrix elements on the right side of (7) are calculated using perturbation theory in NRQCD, and then expanded in powers of $q$ and $q'$. The short-distance coefficients $C_{mn}$ are obtained by matching the terms in the expansions in $q$ and $q'$ order by order in $\alpha_s$.

In the perturbative calculations of the matching condition (7), infrared and ultraviolet divergences can appear on both sides of the equation. Since NRQCD is constructed to be equivalent to full QCD at low energies, the infrared divergences on both sides must match. They therefore cancel in the short-distance coefficients $C_{mn}$. Any ultraviolet divergences on the left side are eliminated by renormalization of the QCD coupling constant and the

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heavy quark mass. On the right side, some of the ultraviolet divergences are eliminated by renormalization of the gauge coupling constant and other parameters in the NRQCD lagrangian. The remaining ultraviolet divergences are removed by renormalization of the 4-fermion operators of NRQCD.

The matching calculations are particularly simple if dimensional regularization is used to regulate both infrared and ultraviolet divergences. With dimensional regularization, power infrared divergences and power ultraviolet divergences are automatically set equal to zero. The only divergences which remain are logarithmic divergences, which appear as poles in \( \epsilon = (3 - N)/2 \). With dimensional regularization, the NRQCD side of the matching condition is especially simple, because radiative corrections to the matrix elements vanish identically. The reason for this is that one must expand the integrand of the radiative correction using a nonrelativistic expansion in the loop momentum before integrating. Since the integrand is also expanded in powers of \( q \) and \( q' \), there is no momentum scale in the dimensionally regularized integral and it therefore vanishes. The radiative corrections have both infrared and ultraviolet divergences, but the infrared poles in \( \epsilon \) cancel the ultraviolet poles. Thus the only contributions to the NRQCD side of the matching condition are the tree-level contributions of the matrix elements, including those matrix elements that arise from counterterms associated with operator renormalization. The poles in \( \epsilon \) from the coefficients of the counterterm matrix elements will match the infrared poles in \( \epsilon \) on the QCD side of the matching condition. The short-distance coefficients of the renormalized matrix elements in the factorization formula depend on the operator renormalization scheme for NRQCD. The renormalized matrix elements in the minimal subtraction scheme are defined by the condition that the coefficients of the counterterms in the expression for the bare operators in terms of renormalized operators are pure poles in \( \epsilon \). In the more conventional \( \overline{MS} \) renormalization scheme, the coefficients of the counterterms are multiples of \( 1/\epsilon + \log(4\pi) - \gamma \), where \( \gamma \) is Euler’s constant. One must be wary of simplifying the matrix elements using identities that are specific to 3 dimensions until after these renormalizations have been carried out.
To illustrate the use of dimensional regularization in the threshold expansion method, we will calculate the terms of lowest order in $\alpha_s$ in the gluon fragmentation function for an arbitrary quarkonium state. The fragmentation function $D_{i\to H}(z)$ gives the probability for a jet initiated by a high energy parton $i$ to include the hadron $H$ carrying a fraction $z$ of the jet momentum. The factorization formula (1), when applied to the fragmentation function for a gluon into a heavy quarkonium state $H$, yields the factorized form

$$D_{g\to H}(z) = \sum_{mn} d_{mn}(z) \langle O^{H}_{mn} \rangle,$$

where the short-distance coefficients $d_{mn}(z)$ can be calculated using perturbation theory.

Fragmentation functions can be calculated directly from the gauge-invariant field-theoretic definitions [4]. This method is particularly advantageous for higher order calculations. We choose instead to follow as closely as possible the calculational strategy introduced in Ref. [5], in which the fragmentation function is obtained by calculating the decay rate of a virtual parton in the infinite momentum frame and in an appropriate axial gauge. We consider the fragmentation of a gluon into a quarkonium state $H$. We take the gluon momentum to be $\ell = (\ell_0, 0, \ldots, 0, \ell_N)$. The infinite momentum frame is defined by the limit $\ell_0, \ell_N \to \infty$ with $\ell^2 = \ell^2_0 - \ell^2_N$ of order $m_c^2$. We choose an axial gauge with reference vector $n = (1, 0, \ldots, 0, -1)$, so that the gluon propagator is

$$G_{\mu\nu}(\ell) = \frac{1}{\ell^2 + i\epsilon} \left( -g_{\mu\nu} + \frac{n_\mu \ell_\nu + \ell_\mu n_\nu}{n \cdot \ell} \right).$$

We also introduce $\bar{n} = (1, 0, \ldots, 0, +1)$. Let $A^\mu_a g^{\rightarrow H(P)+X}$ be the amplitude for a virtual gluon to decay into a quarkonium state $H$ plus additional final state particles $X$ whose total momentum is $k_X$. The gluon fragmentation function can be defined by

$$D_{g\to H}(z) = \int \frac{d\ell^2}{2\pi} \int \frac{d^N P}{(2\pi)^N 2E_P} \sum_X (2\pi)^{N+1} \delta^{N+1}(\ell - P - k_X) \delta \left( z - \frac{P \cdot n}{\ell \cdot n} \right) \times \frac{1}{8(N - 1)} \frac{1}{(\ell^2)^2} \left( A^\mu_a g^{\rightarrow H(P)+X} \right)^* A^{\mu_a g^{\rightarrow H(P)+X}} \left( -g_{\mu\nu} + \frac{n_\mu \bar{n}_\nu + \bar{n}_\mu n_\nu}{2} \right).$$
The sum over additional final state particles X includes integration over their phase space.

The last factor in (12) is the numerator of the gluon propagator in (11) evaluated at the positive energy pole \( \ell_0 = \ell_N + i\epsilon \). It can also be written in the form

\[
-g_{\mu\nu} + \frac{n_\mu \bar{n}_\nu + \bar{n}_\mu n_\nu}{2} = \sum_{i=1}^{N-1} \epsilon^{(i)}_\mu \epsilon^{(i)*}_\nu,
\]

(13)

where \( \epsilon^{(i)}_\mu, i = 1, \ldots, N - 1 \), are the transverse polarization vectors for a real gluon whose momentum is proportional to \( \bar{n} \). The factor of \( 1/(8(N-1)) \) in (12) comes from averaging over the color and polarization states of the decaying gluon. In calculations beyond leading order, there are infrared poles in \( N-3 \) that arise from the splitting of the gluon into collinear partons. They should be absorbed into the Altarelli-Parisi evolution of the fragmentation functions. This complication will not enter into the leading-order calculations presented in this paper.

The short-distance coefficients \( d_{mn}(z) \) defined by the factorization formula (10) can be obtained by a matching prescription analogous to (7) for the cross section:

\[
\int \frac{d\ell^2}{2\pi} \int \frac{d^N P}{(2\pi)^N 2E_P} \sum_X (2\pi)^{N+1}\delta^{N+1}(\ell - P - k_X) \delta \left( z - \frac{P \cdot n}{\ell \cdot n} \right)
\times \frac{1}{8(N-1)} \left( -g_{\mu\nu} + \frac{n_\mu \bar{n}_\nu + \bar{n}_\mu n_\nu}{2} \right) \left[ A^{\mu a}_{g^* \to c\bar{c}(P)+X} A^{\nu a}_{g^* \to c\bar{c}(P)+X} \right]_{pQCD}
= \sum_{mn} d_{mn}(z) \langle \chi^* K^\dagger_m \psi_p c\bar{c} \psi^\dagger K_n \chi \rangle_{pNRQCD}.
\]

(14)

The left side is calculated using perturbative QCD and expanded around the thresholds \( q = q' = 0 \). The matrix elements on the right side are calculated using perturbative NRQCD and also expanded around the thresholds. Matching the expansions in \( q \) and \( q' \) order by order in \( \alpha_s \), we can determine the short-distance coefficients \( d_{mn}(z) \).

IV. COLOR-OCTET TERMS AT ORDER \( \alpha_s \)

The leading color-octet terms in the gluon fragmentation function \( D_{g\to H}(z) \) have short-distance coefficients of order \( \alpha_s \). These terms come from the decay of the virtual gluon through the process \( g^* \to c\bar{c} \). For this simple case, there are no additional final state particles
$X$ in the left side of the matching condition (14). The energy-momentum-conserving delta function forces $P = \ell$ and the QCD side of the matching condition collapses to

$$
\delta(1 - z) \frac{1}{8(N - 1)(P^2)^2} \left( -g_{\mu\nu} + \frac{n_{\mu}\bar{n}_{\nu} + \bar{n}_{\mu}n_{\nu}}{2} \right) \left( A^{\mu a}_{g^* \to c\bar{c}'(P)} \right)^* A^{\mu a}_{g^* \to c\bar{c}'(P)} \big|_{P_{QCD}}. \tag{15}
$$

At leading order in $\alpha_s$, the amplitude for $g^*(\ell) \to c(p)\bar{c}(\bar{p})$ is given by the Feynman diagram in Fig. 1:

$$
A^{\mu a}_{g^* \to c\bar{c}'(P)} = g s \mu^\epsilon \bar{u}(p)\gamma^\mu T^a v(\bar{p}). \tag{16}
$$

The coupling constant in (16) has been written $g s \mu^\epsilon$, where $\epsilon = (3 - N)/2$ and $\mu$ is the scale of dimensional regularization, so that $g_s$ remains dimensionless in $N$ dimensions. The momenta of the $c$ and $\bar{c}$ are $p = \frac{1}{2} P + Lq$ and $\bar{p} = \frac{1}{2} P - Lq$, where $L_i^\mu$ is the matrix that boosts a spacelike vector in the rest frame of the $c\bar{c}$ pair to the frame in which the pair has total momentum $P$. The components of the boost matrix are given in (A4) of Appendix A. The spinor factor in (16) is expressed in terms of nonrelativistic Pauli spinors in (A9b). The amplitude becomes

$$
A^{\mu a}_{g^* \to c\bar{c}'(P)} = 2m_c g_s \mu^\epsilon L_i^\mu \xi^i \sigma^j T^a \eta. \tag{17}
$$

Inserting this into the QCD side of the matching condition (15), it reduces to

$$
\delta(1 - z) \frac{\pi \alpha_s \mu^{2\epsilon}}{8(N - 1)m_c^2} \left( -g_{\mu\nu} + \frac{n_{\mu}\bar{n}_{\nu} + \bar{n}_{\mu}n_{\nu}}{2} \right) L_j^\nu L_i^\mu \eta^\dagger \sigma^j T^a \xi^i \sigma^i T^a \eta. \tag{18}
$$

Using the explicit expression for the boost matrices in (A4), the contraction of $L_j^\nu L_i^\mu$ with $n_{\nu}\bar{n}_{\mu}$ reduces to $(n \cdot L)_j (\bar{n} \cdot L)_i = -\hat{z}^i \hat{z}^j$, where $\hat{z}$ is the unit vector in the $N'$th coordinate direction. Using also the identity (A5a), the QCD side of the matching condition reduces to

$$
\delta(1 - z) \frac{\pi \alpha_s \mu^{2\epsilon}}{8(N - 1)m_c^2} \left( \delta^{ij} - \hat{z}^i \hat{z}^j \right) \eta^\dagger \sigma^j T^a \xi^i \sigma^i T^a \eta. \tag{19}
$$

We now consider the NRQCD side of the matching condition (14). The spinor factor in (19) can be identified as the expansion to leading order in $\alpha_s$ and to linear order in $q$ and $q'$ of the following NRQCD matrix element:
\begin{align*}
\langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{\bar{c}c',c\bar{c}} \psi^\dagger \sigma^i T^a \chi \rangle \bigg|_{p_{N\text{RQCD}}} & \approx 4 m_c^2 \eta^i \sigma^j T^a \xi^\dagger \xi^\dagger \sigma^i T^a \eta. \tag{20}
\end{align*}

The tree-level expression for the matrix element in (20) is represented diagramatically in Fig. 2. The dot represents the operators \(\psi^\dagger \sigma^i T^a \chi\) and \(\chi^\dagger \sigma^j T^a \psi\), which create and annihilate \(c\bar{c}\) pairs from the vacuum. The 2 lines emerging from the right side of the diagram represent the \(c\) and \(\bar{c}\) in the bra of the projection operator \(\mathcal{P}_{\bar{c}c',c\bar{c}}\) defined in (8). The 2 lines entering the left side of the diagram represent the \(c\) and \(\bar{c}\) in the ket of the projection operator. The radiative corrections to the matrix element include diagrams in which virtual gluons are exchanged between the \(c\) and \(\bar{c}\) on the right or between the \(c\) and \(\bar{c}\) on the left. They also include diagrams in which real gluons enter the diagram on the left side and emerge on the right side with the same momenta and in the same color and spin states. These contributions come from higher Fock states in the projection operator \(\mathcal{P}_{\bar{c}c',c\bar{c}}\) defined in (8). The sum over soft states \(S\) in the definition of \(\mathcal{P}_{\bar{c}c',c\bar{c}}\) includes integrals over the momenta of the real gluons and sums over their spin and color quantum numbers. Since \(\langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{\bar{c}c',c\bar{c}} \psi^\dagger \sigma^i T^a \chi \rangle\) is a vacuum matrix element, the only diagrams that are allowed are those for which no propagators are cut by a vertical line through the dot.

The coefficient of the factor (20) in (19) is the short-distance coefficient for the matrix element. The same short-distance coefficient will hold for operators defined using the projection \(\mathcal{P}_{H(\lambda)}\), where \(\lambda\) specifies the polarization of the quarkonium state \(H\). The color-octet term in the gluon fragmentation functions at leading order in \(\alpha_s\) is therefore

\begin{align*}
D_{g \rightarrow H(\lambda)}(z) &= \delta(1 - z) \frac{\pi \alpha_s \mu^{2\epsilon}}{32(N - 1)m_c^2} \left( \delta^{ij} - \hat{z}^i \hat{z}^j \right) \langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{H(\lambda)} \psi^\dagger \sigma^i T^a \chi \rangle. \tag{21}
\end{align*}

The indices \(i\) and \(j\) range from 1 to \(N\). We will find that, in order to obtain ultraviolet finite results at order \(\alpha_s^2\), the matrix element in (21) will require renormalization. After that renormalization has been carried out, we can take the limit \(N \rightarrow 3\). Our final result for the order-\(\alpha_s\) term in the gluon fragmentation function is then

\begin{align*}
D_{g \rightarrow H(\lambda)}(z) &= \delta(1 - z) \frac{\pi \alpha_s (m_c)}{64m_c^2} \left( \delta^{ij} - \hat{z}^i \hat{z}^j \right) \langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{H(\lambda)} \psi^\dagger \sigma^i T^a \chi \rangle^{(\mu)}. \tag{22}
\end{align*}
The indices $i$ and $j$ now range from 1 to 3. We have set the scale of the running coupling constant in the short-distance coefficient equal to $m_c$, since that coefficient is only sensitive to momenta on the order of $m_c$ or larger. The superscript $(\mu)$ on the matrix element is a reminder that it is a renormalized matrix element that may depend on a renormalization scale $\mu$. For a specific quarkonium state, it may be possible to simplify the matrix element in (22) by using rotational symmetry, heavy-quark spin symmetry, and the vacuum saturation approximation. For example, if we sum over polarizations, we can use rotational symmetry to set

$$\sum_\lambda \langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{H(\lambda)} \psi^\dagger \sigma^j T^a \chi \rangle (\mu) = \frac{1}{3} \delta^{ij} \sum_\lambda \langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_{H(\lambda)} \psi^\dagger \sigma^k T^a \chi \rangle (\mu).$$

For unpolarized quarkonium states, the calculation of the fragmentation function can be simplified by using rotational symmetry at an earlier stage of the calculation. Under the rotation group in $N$ dimensions, $q$ and $q'$ transform as vectors and $\xi$, $\eta$, $\xi'$, and $\eta'$ transform as spinors. Denoting the action of an element $R$ of the rotation group by $R.q$, $R.\xi$, etc., the average over the rotation group of a function of these vectors and spinors can be defined by

$$\overline{f(q, \xi, \eta, q', \xi', \eta')} \equiv \int dR f(R.q, R.\xi, R.\eta, R.q', R.\xi', R.\eta'),$$

where $dR$ is the invariant integration element on the group, normalized so that $\int dR = 1$. We can average the factor $(A_\nu^a g^{* \rightarrow c\bar{c}(P)+X} A_\mu^a g^{* \rightarrow c\bar{c}(P)+X})^*$ on the right side of (14) over rotations if we also average the projection operator $\mathcal{P}_{c\bar{c},c\bar{c}}$ on the left side. Alternatively, we can leave $\mathcal{P}_{c\bar{c},c\bar{c}}$ unchanged and simply restrict the sum on the left side to matrix elements for which $\langle \chi^\dagger K_m^I \psi^\dagger K_n \chi \rangle$ is a scalar.

Applying this rotational average to the QCD side of the matching condition in (18), it becomes

$$\delta(1 - z) \frac{\pi \alpha_s \mu^2 e}{8(N - 1)m_c^2} \left( -g_{\mu\nu} + \frac{n_\mu n_\nu + \bar{n}_\mu \bar{n}_\nu}{2} \right) L^\nu_j L^\mu_i \eta^\dagger \sigma^j T^a \xi^\dagger \sigma^i T^a \eta.$$

The average over rotations of the spinor factor is

$$\overline{\eta^\dagger \sigma^j T^a \xi^\dagger \sigma^i T^a \eta} = \frac{1}{N} \delta^{ij} \eta^\dagger \sigma^k T^a \xi^\dagger \sigma^k T^a \eta.$$
Using the identity (A5b), the QCD side of the matching condition in (25) immediately collapses to
\[
\frac{\pi \alpha_s \mu^2}{8 N m_c^2} \delta(1 - z) \eta^\dagger \sigma^k T^a \xi \eta \eta^\dagger \sigma^k T^a \xi. \tag{27}
\]
The spinor factor is the lowest order expression for the matrix element
\[
\langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_{c \bar{c}, \bar{c}c} \psi^\dagger \sigma^k T^a \chi \rangle_{\perp NRQCD} \approx \frac{4}{N m_c^2} \delta(1 - z) \eta^\dagger \sigma^k T^a \xi \eta \eta^\dagger \sigma^k T^a \xi. \tag{28}
\]
Thus the QCD side of the matching condition can be written
\[
\frac{\pi \alpha_s \mu^2}{32 N m_c^2} \delta(1 - z) \langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_{c \bar{c}, \bar{c}c} \psi^\dagger \sigma^k T^a \chi \rangle_{\perp NRQCD}. \tag{29}
\]
The fragmentation function for an unpolarized quarkonium state \( H \) is therefore
\[
D_{g \to H}(z) = d(\xi, \lambda)(z) \langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_H \psi^\dagger \sigma^k T^a \chi \rangle^{(\mu)}, \tag{30}
\]
where the short-distance coefficient is
\[
d(\xi, \lambda)(z) = \frac{\pi \alpha_s (m_c)}{96 m_c^4} \delta(1 - z). \tag{31}
\]
In the short-distance coefficient, we have set \( N = 3 \) and set the scale of the running coupling constant equal to \( m_c \). The superscript \((\mu)\) on the matrix element in (30) indicates that it is a renormalized matrix element. The projection operator \( \mathcal{P}_H \) in the matrix element in (30) is the sum over helicities of the projection operators defined in (5): \( \mathcal{P}_H = \sum_\lambda \mathcal{P}_{H(\lambda)} \). The result (30) can also be obtained by using the identity (23) in the unpolarized fragmentation function (22).

**V. LEADING COLOR-SINGLET TERMS AT ORDER \( \alpha_s^2 \)**

The leading color-singlet terms in the gluon fragmentation function \( D_{g \to H}(z) \) have short-distance coefficients of order \( \alpha_s^2 \). These terms come from the decay of the virtual gluon...
through the process $g^* \to c\bar{c}g$. The phase space integrals and the energy-momentum-conserving delta function in the matching condition (14) reduce to

$$\int \frac{d^N P}{(2\pi)^N 2E_P} \int \frac{d^N k}{(2\pi)^N 2|k|} (2\pi)^{N+1} \delta^{N+1}(\ell - P - k)$$

$$= \frac{1}{2} \int_0^1 \frac{dz}{z(1-z)} \int \frac{d^{N-1} P_\perp}{(2\pi)^{N-1}} \delta \left( \ell^2 - \frac{P_\perp^2 + P^2}{z} - \frac{P_\perp^2}{1-z} \right).$$

(32)

If we only require fragmentation functions that are summed over the polarizations of the quarkonium states, the calculation can be simplified by averaging both sides of the matching condition over rotations of the vectors and spinors that specify the state of the $c\bar{c}$ pair in its rest frame. The integrand on the QCD side of the matching condition (14) then reduces to a function of $P_\perp^2$, and the integration over $\mathbf{P}_\perp$ can be carried out using the remaining delta function in (32). The QCD side of the matching condition in (14) reduces to

$$\left( \frac{\sqrt{4\pi}}{16 \Gamma^{(N+1)/2}} \right) \int_{4m_c^2/|s|}^\infty \frac{ds}{s^2} P_N^{-3} \left( -g_{\mu\nu} + \frac{n_\mu n_\nu + \bar{n}_\mu \bar{n}_\nu}{2} \right) \left( A_{g^* \to c\bar{c}(P)+g}^{\mu a} \right) \left| A_{g^* \to c\bar{c}(P)+g}^{\mu a} \right|_p \eta_{QCD},$$

(33)

where $s = \ell^2$ and $P_\perp^2 = (1-z)(z \ell^2 - 4E_q^2)$. At leading order in $\alpha_s$, the amplitude for the process $g^*(l) \to c(p) \bar{c}(\bar{p}) g(k)$, with the $c\bar{c}$ pair in a color-singlet state, is the sum of the Feynman diagrams in Fig. 3:

$$A_{g^* \to c\bar{c}(P)+g}^{\mu a} = \frac{g_s \mu^{2\epsilon}}{6} e^{\epsilon a}(k) \bar{u}(p) \left[ \frac{\gamma^\mu (\not\bar{p} + \not k - m_c) \gamma^\nu}{2\not p \cdot \not k} - \frac{\gamma^\nu (\not\bar{p} + \not k + m_c) \gamma^\mu}{2\not p \cdot \not k} \right] v(\bar{p}).$$

(34)

Using the identities for carrying out the nonrelativistic expansion of spinor factors given in Appendix A, we expand (34) to linear order in $q$:

$$A_{g^* \to c\bar{c}(P)+g}^{\mu a} = \frac{g_s \mu^{2\epsilon}}{6P \cdot k} e^{\epsilon a}(k) \left\{ m_c L^\mu_i L^\nu_j (k \cdot L)_k \xi^i \{[\sigma^i, \sigma^j], \sigma^k\} \eta + \frac{2}{m_c} \left[ \frac{4m_c^2}{P \cdot k} (k \cdot L)_i \left( k^\mu L^\nu_j + L^\mu_j \ell^\nu - (k \cdot L)_j g^{\mu\nu} \right) \right. \right.$$

$$- (k \cdot L)_i (P^\mu L^\nu_j - L^\mu_j P^\nu) + (k \cdot L)_j (P^\mu L^\nu_i - L^\mu_i P^\nu)$$

$$+ \left. P \cdot L^\mu_i L^\nu_j - P \cdot \ell L^\mu_j L^\nu_i \right\} \xi^i \eta^j \sigma^l.$$

(35)

In $N > 3$ dimensions, the spin matrix $\{[\sigma^i, \sigma^j], \sigma^k\}$, which is totally antisymmetric in its three indices, is linearly independent of 1 and $\sigma^i, i = 1, \ldots, N$. In 3 dimensions, it reduces to the unit matrix multiplied by $-4i\epsilon^{ijk}$. 

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After inserting the amplitude \((35)\) into the QCD side of the matching condition \((33)\), the spinor factors can be simplified by averaging over the rotation group. The factor \(\eta^\dagger q^k \sigma^i \xi^\dagger q^j \sigma^j \eta\) can be reduced to a linear combination of three independent rotationally-invariant spinor factors:

\[
\begin{align*}
\xi^\dagger q^k \sigma^i \eta^\dagger q^j \sigma^j \xi &= \frac{1}{N(N-1)(N+2)} \times \left[ ((N+1)\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \eta^\dagger \mathbf{q} \cdot \sigma \xi^\dagger \mathbf{q} \cdot \sigma \eta \\
&+ ((N+1)\delta^{ik}\delta^{jl} - \delta^{ij}\delta^{kl} - \delta^{il}\delta^{jk}) \eta^\dagger q^m \sigma^n \xi^\dagger q^m \sigma^n \eta \\
&+ ((N+1)\delta^{il}\delta^{jk} - \delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl}) \eta^\dagger q^m \sigma^n \xi^\dagger q^m \sigma^n \eta \right].
\end{align*}
\] (36)

The spinor factors \(\eta^\dagger q^k \sigma^m \xi^\dagger \{[\sigma^i, \sigma^j], \sigma^k\} \eta\) and \(\eta^\dagger \{[\sigma^i, \sigma^j], \sigma^m\} \xi^\dagger q^i \sigma^j \eta\) average to zero, while \(\eta^\dagger \{[\sigma^i, \sigma^j], \sigma^k\} \xi^\dagger \{[\sigma^i, \sigma^j], \sigma^k\} \eta\) reduces to a single rotationally-invariant spinor factor:

\[
\begin{align*}
\eta^\dagger \{[\sigma^i, \sigma^m], \sigma^n\} \xi^\dagger \{[\sigma^i, \sigma^j], \sigma^k\} \eta &= \frac{1}{N(N-1)(N-2)} \left| \begin{array}{ccc}
\delta^{il} & \delta^{jl} & \delta^{kl} \\
\delta^{im} & \delta^{jm} & \delta^{km} \\
\delta^{in} & \delta^{jn} & \delta^{kn}
\end{array} \right| \eta^\dagger \{[\sigma^r, \sigma^s], \sigma^t\} \xi^\dagger \{[\sigma^r, \sigma^s], \sigma^t\} \eta.
\end{align*}
\] (37)

We will find that there are no poles in \(N-3\) multiplying this spinor factor. It can therefore be simplified by using the commutation relations (4) for \(N=3\):

\[
\eta^\dagger \{[\sigma^r, \sigma^s], \sigma^t\} \xi^\dagger \{[\sigma^r, \sigma^s], \sigma^t\} \eta = -96 \eta^\dagger \xi^\dagger \eta, \quad N=3.
\] (38)

After averaging the spinor factors over the rotation group using (36) and (37), all the Cartesian indices of the boost tensors \(L^\mu_i\) are contracted and they can be simplified using the identity (A5b) of Appendix A. The factor \(\epsilon^{\nu a}(k)^* \epsilon^{\mu a}(k)\), summed over colors and polarizations of the real gluon, can be replaced by \(8(-g^{\nu \nu})\). After simplifying all the Lorentz algebra, the QCD side of the matching condition \((33)\) reduces to integrals of scalar quantities:

\[
\frac{(\sqrt{\pi})^{3-N} \alpha_s^2 \mu^{4\epsilon}}{72 \Gamma \left( \frac{N+1}{2} \right) N(N-1)(N+2) m_c^{4+2\epsilon}} \int_{(1-z)/z}^\infty dx \frac{dx}{x^4 (1+x)^2} \xi^{(N-3)/2}
\]
\[
\times \left( m_c^2 W_0(x, z) \left( -\frac{1}{96} \eta \{ [\sigma^r, \sigma^s], \sigma^t \} \xi \xi^\dagger \{ [\sigma^r, \sigma^s], \sigma^t \} \eta \right) \\
+ W_1(x, z) \eta \eta^{q'} \cdot \sigma \xi \xi^\dagger q \cdot \sigma \eta \\
+ W_2(x, z) \eta \eta^{q''} \sigma^m \sigma^\eta \xi \xi^\dagger q^m \sigma^m \eta \\
+ W_3(x, z) \eta \eta^{q''} \sigma^m \sigma^\eta \xi \xi^\dagger q^m \sigma^m \eta \right),
\]

where

\[
x = (s - 4m_c^2)/(4m_c^2), \ t = (1 - z)(zx + z - 1), \text{ and}
\]

\[
W_0(x, z) = 6 (N + 2) x^2 \left[ 4t (1 + x) + (N - 1)x^2 \right],
\]

\[
W_1(x, z) = -4t \left[ 4N + 4(N - 2)x + (N - 9)x^2 + (N - 1)x^3 \right] \\
+ (N - 1)x^2 \left[ 4 - 4(N + 1)x + (N - 1)x^2 \right],
\]

\[
W_2(x, z) = -4t \left[ -8 - 16x + (2N^2 + N - 17)x^2 + (2N^2 + N - 9)x^3 \right] \\
+ (N - 1)x^2 \left[ 4(N^2 - 3) + 4(N^2 - 3)x + (2N^2 + N - 9)x^2 \right],
\]

\[
W_3(x, z) = -4t \left[ 4N - 4(N^2 - 2N - 4)x - (6N^2 - 5N - 23)x^2 - (2N^2 - N - 7)x^3 \right] \\
+ (N - 1)x^2 \left[ 4 - 4(N^2 - 5)x - (2N^2 - N - 7)x^2 \right].
\]

Upon integrating (39) over \( x \), we obtain distributions in \( z \). Some of these distributions are infrared divergent in the sense that the integrals over \( z \) diverge as \( N \to 3 \). The infrared divergences arise from the following integrals:

\[
\int_{(1-z)/z}^\infty dx \frac{t^{-\epsilon}}{x^2} = \Gamma(1 + \epsilon)\Gamma(1 - \epsilon) \frac{z}{(1 - z)^{1-2\epsilon}},
\]

\[
\int_{(1-z)/z}^\infty dx \frac{t^{1-\epsilon}}{x^4} = \frac{\Gamma(2 + \epsilon)\Gamma(2 - \epsilon)}{6} z^3 (1 - z)^{-1-2\epsilon}.
\]

The divergences as \( \epsilon \to 0 \) can be made explicit by using the expansion

\[
(1 - z)^{-1-2\epsilon} = -\frac{1}{2\epsilon} \delta(1 - z) + \frac{1}{(1 - z)_+} + \ldots.
\]

There is no infrared divergence in the \( W_0 \) term. The divergences in the \( W_1 \) and \( W_3 \) terms are multiplied by factors of \( \epsilon \), and they give \( \delta(1 - z) \) terms. There is an infrared divergence that survives in the \( W_2 \) term.

Integrating over \( x \) in (39) and taking the limit \( \epsilon \to 0 \), the QCD side of the matching condition reduces to
The remaining three functions in (44) can be identified as the tree-level expressions for NRQCD matrix elements, expanded to linear order in \( q \) and \( q' \). The first matrix element is

\[
\langle \chi^\dagger \psi \mathcal{P}_{c\bar{c},c\bar{c}} \psi^\dagger \chi \rangle |_{p_{\text{NRQCD}}} \approx 4m_c^2 \eta^\dagger \xi^\dagger \xi \eta,
\]

and the remaining three are

\[
\langle \chi^\dagger (-\frac{i}{2} \mathbf{D} \cdot \sigma) \psi \mathcal{P}_{c\bar{c},c\bar{c}} \psi^\dagger (-\frac{i}{2} \mathbf{D} \cdot \sigma) \chi \rangle |_{p_{\text{NRQCD}}} \approx 4m_c^2 \eta^\dagger q^\dagger \cdot \sigma \xi^\dagger q \cdot \sigma \eta,
\]

\[
\langle \chi^\dagger (-\frac{i}{2} \mathbf{D})^m \sigma^n \psi \mathcal{P}_{c\bar{c},c\bar{c}} \psi^\dagger (-\frac{i}{2} \mathbf{D})^m \sigma^n \chi \rangle |_{p_{\text{NRQCD}}} \approx 4m_c^2 \eta^\dagger q^\dagger m \sigma^n \xi^\dagger q^m \sigma^n \eta,
\]

\[
\langle \chi^\dagger (-\frac{i}{2} \mathbf{D})^m \sigma^n \psi \mathcal{P}_{c\bar{c},c\bar{c}} \psi^\dagger (-\frac{i}{2} \mathbf{D})^n \sigma^m \chi \rangle |_{p_{\text{NRQCD}}} \approx 4m_c^2 \eta^\dagger q^\dagger m \sigma^n \xi^\dagger q^m \sigma^n \eta.
\]
For the matrix elements (47), (48a), and (48c), we can immediately read off the short-distance coefficients from (44). They are the functions of $z$ given in (45), (46a), and (46c). However the coefficient of $\eta^\dagger q^m \sigma^n \xi^\dagger q^m \sigma^n \eta$ in (44) contains an infrared pole in $\epsilon$, indicating that it is sensitive to long-distance effects. Since an infrared divergence cannot appear in a short-distance coefficient, that divergence must be matched by an infrared divergence from some matrix element on the NRQCD side of the matching condition. Infrared divergences in NRQCD matrix elements can arise only from radiative corrections. Since the divergence in (44) has a coefficient of order $\alpha_s^2$, the infrared-divergent NRQCD matrix element must have a short-distance coefficient of order $\alpha_s$. The only such scalar matrix element is the one whose short-distance coefficient has already been determined in (29). Thus the infrared divergence on the NRQCD side of the matching condition must come from that term.

If the $c\bar{c}$ pairs are in color-singlet states, the tree level expression (28) for the matrix element vanishes and the leading contribution comes instead from radiative corrections. Specifically, it comes from the $c\bar{c}g$ term in the projection operator $\mathcal{P}_{c\bar{c},c\bar{c}}$ defined in (8):

$$\sum_{\lambda b} \int \frac{d^N k}{(2\pi)^N 2k} \left[ c(q', \xi') \bar{c}(-q', \eta') + g(k, \lambda, b) \right] \times \left[ c(q, \xi) \bar{c}(-q, \eta) + g(k, \lambda, b) \right].$$  \hspace{1cm} (49)

The sum is over the $N - 1$ physical polarizations and the 8 color states of the real gluon. The leading contributions to the matrix element are represented by the diagrams in Figure 4. The expression for diagram 4a is

$$8\pi \alpha_s \mu^{2s} \eta^\dagger \sigma^n T^a T^b \xi^\dagger \sigma^n T^b T^a \eta \int \frac{d^N k}{(2\pi)^N 2k} \frac{1}{E_q + k - (q + k)^2/(2m_c) + i\epsilon} \frac{1}{E_q + k - (q' + k)^2/(2m_c) + i\epsilon},$$  \hspace{1cm} (50)

where $E_q = q^2/(2m_c) = (q')^2/(2m_c)$. As discussed in Appendix B of Ref. [1], the proper way to evaluate the diagram is to first expand out the denominators in powers of $q/m_c$, $q'/m_c$, and $k/m_c$, and then integrate over $k$. Keeping only terms up to linear order in $q/m_c$ and $q'/m_c$, the diagram reduces to

$$8\pi \alpha_s \mu^{2s} \eta^\dagger \sigma^n T^a T^b \xi^\dagger \sigma^n T^b T^a \eta \int \frac{d^N k}{(2\pi)^N} \frac{q \cdot q' - q \cdot \hat{k} \hat{k} \cdot q'}{k^3}.$$  \hspace{1cm} (51)
The integral is both ultraviolet and infrared divergent. It vanishes in dimensional regularization due to a cancellation between an ultraviolet pole in $\epsilon$ and an infrared pole. Making these poles explicit, the diagram can be written
\[ \frac{4\alpha_s}{3\pi} \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \eta\tilde{q}^m \sigma^n T^a T^b \xi\tilde{\xi}^\dagger \eta. \] (52)

The subscripts $UV$ and $IR$ on $\epsilon$ indicate whether the pole is of ultraviolet or infrared origin. We have set $N = 3$ in the prefactor, since any finite terms obtained by expanding the prefactor in powers of $\epsilon$ will cancel. The effect of the other 3 diagrams is simply to symmetrize both of the products of color matrices $T^a T^b$. Since the projector $\mathcal{P}_{c\bar{c},c\bar{c}}$ requires the asymptotic $c\bar{c}$ pairs to be in color-singlet states, we can replace $\{T^a, T^b\}$ by $\delta^{ab}/3$. The final result for the matrix element is
\[ \langle \chi\tilde{\psi} \rangle_{p_{NRQCD}} = \frac{32\alpha_s}{27\pi} \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \eta\tilde{q}^m \sigma^n \xi\tilde{\xi}^\dagger \eta. \] (53)

After multiplying by the short-distance coefficient in (29), we find that the infrared pole in $\epsilon$ matches the one on the QCD side of the matching condition, which is given in (44).

After taking into account the matrix element (53) on the NRQCD side of the matching condition, the short-distance coefficient of the matrix element (48b) can be read off from (44). The net effect of taking into account the matrix element (53) is simply to change the infrared pole in $\epsilon$ into an ultraviolet pole. The short-distance coefficient of the matrix element (48b) therefore contains an ultraviolet divergence. This divergence must be removed by operator renormalization in NRQCD. As is evident from the ultraviolet pole in $\epsilon$ in (53), it is the matrix element $\langle \chi\tilde{\psi} \rangle$ that requires renormalization. In the $\overline{MS}$ renormalization scheme, the relation between the matrix element of the bare operator and matrix elements of renormalized operators is
\[ \langle \chi\tilde{\psi} \rangle = \mu^{-4\epsilon} \left( \langle \chi\tilde{\psi} \rangle \right)^{(\mu)} + \frac{8\alpha_s}{27\pi m_c^2} \left( \frac{1}{\epsilon_{UV}} + \ln(4\pi) - \gamma \right) \langle \tilde{q}(-i\frac{\tilde{D}}{2})^m \sigma^n \tilde{\psi} \rangle \left( \tilde{q}(-i\frac{\tilde{D}}{2})^m \sigma^n \chi \right)^{(\mu)}. \] (54)

The superscripts $(\mu)$ on the matrix elements on the right side indicate that they are renormalized matrix elements with renormalization scale $\mu$. We will suppress this superscript on
P-wave matrix elements, since they do not require any renormalization at this order in \( \alpha_s \).
The fermion field operators in the bare matrix element on the left side of (54) have dimension \( N/2 \). The fermion field operators in the renormalized matrix elements on the right side have dimension \( 3/2 \). The factor of \( \mu^{-4\epsilon} \) on the right side of (54) compensates for the difference between the dimensions of the two sides. Solving (54) for \( \langle \chi^\dagger \sigma^k T^a \psi P c\bar{c}, c\bar{c} \psi^\dagger \sigma^k T^a \chi \rangle^{(\mu)} \) and using (53) and (48b), we find that the renormalized matrix element, with dimensional regularization as the infrared cutoff, is

\[
\langle \chi^\dagger \sigma^k T^a \psi P c\bar{c}, c\bar{c} \psi^\dagger \sigma^k T^a \chi \rangle^{(\mu)} \bigg|_{p_{NRQCD}} = -\frac{32\alpha_s}{27\pi} \left( \frac{1}{\epsilon_{IR}} + \ln(4\pi) - \gamma \right) \eta^\dagger q^m \sigma^n \xi^\dagger q^m \sigma^n \eta. \tag{55}
\]

Multiplying (54) by the short-distance coefficient in (29), we find that the contribution from the renormalized matrix element to the NRQCD side of the matching condition is

\[
\frac{\pi\alpha_s \mu^{2\epsilon}}{32N m_c^4} \delta(1 - z) \langle \chi^\dagger \sigma^k T^a \psi P c\bar{c}, c\bar{c} \psi^\dagger \sigma^k T^a \chi \rangle^{(\mu)} \bigg|_{p_{NRQCD}} = -\frac{\alpha_s}{324 m_c^6} \left( \frac{1}{\epsilon_{UV}} + \ln(4\pi \mu^2) - \gamma + \frac{2}{3} \right) \delta(1 - z) 4m_c^2 \eta^\dagger q^m \sigma^n \xi^\dagger q^m \sigma^n \eta. \tag{56}
\]

This term on the NRQCD side of the matching condition matches the part of the \( \eta^\dagger q^m \sigma^n \xi^\dagger q^m \sigma^n \eta \) term on the QCD side that contains the pole in \( \epsilon \). Subtracting this term from (44), we find that the short-distance coefficient of the matrix element (48b) is the function \( d_2(z, \mu) \) given in (46b).

Note that the calculation of the radiative correction to the NRQCD matrix element in (53) was necessary only to determine the coefficient of the counterterm in (54). The radiative correction itself vanishes if we identify \( \epsilon_{UV} = \epsilon_{IR} \). Its only effect is to transform the ultraviolet pole in the coefficient of the counterterm into an infrared pole on the NRQCD side of the matching condition.

Combining the color-octet term in (30) with the color-singlet terms determined above, we obtain a general expression for the gluon fragmentation function of an unpolarized quarkonium state \( H \):
\[ D_{g \to H}(z) = d(\xi^3 S_1)(z) \langle \chi^\dagger \sigma^n T^a \psi \mathcal{P}_H \psi^\dagger \sigma^n T^a \chi \rangle^{(\mu)} + d(\xi^3 S_0)(z) \langle \chi^\dagger \psi \mathcal{P}_H \psi^\dagger \chi \rangle \]
\[ + d_1(z) \langle \chi^\dagger (-\frac{i}{2} \vec{D} \cdot \sigma) \psi \mathcal{P}_H \psi^\dagger (-\frac{i}{2} \vec{D} \cdot \sigma) \chi \rangle \]
\[ + d_2(z, \mu) \langle \chi^\dagger (-\frac{i}{2} \vec{D})^n \sigma^n \psi \mathcal{P}_H \psi^\dagger (-\frac{i}{2} \vec{D})^n \sigma^n \chi \rangle \]
\[ + d_3(z) \langle \chi^\dagger (-\frac{i}{2} \vec{D})^m \sigma^m \psi \mathcal{P}_H \psi^\dagger (-\frac{i}{2} \vec{D})^m \sigma^m \chi \rangle, \quad (57) \]

where the short-distance coefficients are given in (31), (45), and (46). This expression for the fragmentation function includes all those color-octet matrix elements with short-distance coefficient of order \( \alpha_s \) and all those color-singlet matrix elements with short-distance coefficients of order \( \alpha_s^2 \) that involve at most one derivative acting on \( \chi^\dagger \) and \( \psi \) and at most one derivative acting on \( \psi^\dagger \) and \( \chi \).

The \( \mu \)-dependence of the coefficient \( d_2(z, \mu) \) in (57) is cancelled by the \( \mu \)-dependence of the color-octet matrix element. The renormalization group equation that determines the \( \mu \)-dependence of that matrix element is obtained by differentiating (54) with respect to \( \mu \).

We use the fact that the bare matrix element on the left side is independent of \( \mu \) and that the bare coupling constant, which at this order is \( \alpha_s^2 \mu^2 c \), is also independent of \( \mu \). Replacing \( \mathcal{P}_{c\bar{c}, c\bar{c}} \) by the projector \( \mathcal{P}_H \), we find

\[ \frac{\mu}{d\mu} \langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_H \psi^\dagger \sigma^k T^a \chi \rangle^{(\mu)} = \frac{16}{27 \pi m_c^3} \alpha_s(\mu) \langle \chi^\dagger (-\frac{i}{2} \vec{D})^m \sigma^n \psi \mathcal{P}_H \psi^\dagger (-\frac{i}{2} \vec{D})^m \sigma^n \chi \rangle. \quad (58) \]

The solution to this renormalization group equation at leading order in \( \alpha_s \) is

\[ \langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_H \psi^\dagger \sigma^k T^a \chi \rangle^{(\mu)} = \langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_H \psi^\dagger \sigma^k T^a \chi \rangle^{(\mu_0)} \]
\[ - \frac{32}{9(33 - 2n_f)m_c^2} \log \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right) \langle \chi^\dagger (-\frac{i}{2} \vec{D})^m \sigma^n \psi \mathcal{P}_H \psi^\dagger (-\frac{i}{2} \vec{D})^m \sigma^n \chi \rangle, \quad (59) \]

where \( n_f \) is the number of flavors of light quarks (\( n_f = 3 \) for charmonium).

**VI. GLUON FRAGMENTATION FUNCTIONS**

The relative importance of the various terms in the fragmentation function (57) depends on the quarkonium state. The magnitude of a particular term is determined by the order in
\( \alpha_s \) of its short-distance coefficient and by the scaling of the matrix element with \( v \), which is given by the velocity-scaling rules of NRQCD [1]. Below, we apply this general fragmentation formula to specific S-wave states and P-wave states.

**A. Spin-singlet S-wave states**

The dominant Fock state of the \( \eta_c \) consists of a \( c\bar{c} \) pair in a color-singlet \( ^1S_0 \) state. The dominant matrix element is therefore \( \langle \chi^\dagger \psi \mathcal{P}_{\eta_c} \psi^\dagger \chi \rangle \), which scales as \( v^3 \). It has a short-distance coefficient of order \( \alpha_s^2 \), and therefore contributes to the gluon fragmentation function at order \( \alpha_s v^3 \). Since the dominant Fock state can be reached from a color-octet \( ^3S_1 \) state through a chromomagnetic dipole transition, the matrix element \( \langle \chi^\dagger \sigma^k T^a \psi \mathcal{P}_{\eta_c} \psi^\dagger \sigma^k T^a \chi \rangle \) is suppressed by \( v^3 \) relative to the dominant matrix element. It contributes to \( D_{g\to \psi} (z) \) at order \( \alpha_s v^6 \). The color-singlet P-wave matrix elements in (57) are all suppressed by \( v^7 \). Their contributions are therefore of order \( \alpha_s^2 v^{10} \), and are expected to be negligible.

The most important term in the gluon fragmentation function for the \( \eta_c \) is the color-singlet \( ^1S_0 \) term. Keeping only this term, the fragmentation function reduces to

\[
D_{g\to \eta_c} (z) = \frac{\alpha_s^2 (m_c)}{144 m_c^4} \left[ 3z - 2z^2 + 2(1-z) \ln(1-z) \right] \langle \chi^\dagger \psi \mathcal{P}_{\eta_c} \psi^\dagger \chi \rangle. \tag{60}
\]

Since the short-distance coefficient is dominated by contributions from large momenta of order \( m_c \) or larger, we have set the scale of the running coupling constant equal to \( m_c \). The standard NRQCD matrix elements \( \langle O_n (2S+1L_J) \rangle \) introduced in Ref. [1] were defined using a projection operator analogous to \( \mathcal{P}_H \) in (5), except that the states have the standard nonrelativistic normalization. The relation between our matrix elements and those defined in Ref. [1] is discussed in Appendix B of [3]. At leading order in \( v^2 \), they differ simply by a normalization factor. For the matrix element in (60), the relation is

\[
\langle \chi^\dagger \psi \mathcal{P}_{\eta_c} \psi^\dagger \chi \rangle \approx 4m_c \langle O_1^{\eta_c} (1 S_0) \rangle. \tag{61}
\]

The term (60) in the fragmentation function was first calculated by Braaten and Yuan [5]. The next most important term should be the color-octet \( ^3S_1 \) term in (57).
B. Spin-triplet S-wave states

The dominant Fock state of the $J/\psi$ consists of a $c\bar{c}$ pair in a color-singlet $^3S_1$ state. The dominant matrix element is therefore $\langle \chi^\dagger \sigma^k \psi P_\psi \psi^\dagger \sigma^k \chi \rangle$, which scales as $v^3$. Since the dominant Fock state can be reached from a color-octet $^3S_1$ state through a double chromoelectric dipole transition, the matrix element $\langle \chi^\dagger \sigma^k T^a \psi P_\psi \psi^\dagger \sigma^k T^a \chi \rangle$ is suppressed by $v^4$ relative to the dominant matrix element. The matrix element $\langle \chi^\dagger \psi P_\psi \psi^\dagger \chi \rangle$ is suppressed by $v^7$ and the color-singlet P-wave matrix elements in (57) are all suppressed by $v^8$. Their contributions are therefore expected to be negligible.

Of the terms that we have calculated, the most important is the color-octet $^3S_1$ term. Keeping only this term, the fragmentation function reduces to

$$D_{g \to \psi}(z) = \delta(1-z) \frac{\pi \alpha_s(m_c)}{96m_c^4} \langle \chi^\dagger \sigma^i T^a \psi P_\psi \psi^\dagger \sigma^i T^a \chi \rangle. \quad (62)$$

Since the short-distance coefficient is dominated by contributions from large momenta of order $m_c$ or larger, we have set the scale of the running coupling constant equal to $m_c$. Up to corrections of relative order $v^2$, the matrix element in (62) differs from the standard NRQCD matrix element introduced in Ref. [1] only by a normalization factor:

$$\langle \chi^\dagger \sigma^i T^a \psi P_\psi \psi^\dagger \sigma^i T^a \chi \rangle \approx 4m_c \langle \mathcal{O}_8^\psi (^3S_1) \rangle. \quad (63)$$

The result (62) was first given by Braaten and Fleming [6]. This contribution to the fragmentation function is of order $\alpha_s v^7$. If we assume that the effect of a suppression factor of $\alpha_s$ is comparable to that of a suppression factor of $v^2$, then there is one other term in the fragmentation function that could be equally important. The color-singlet $^3S_1$ term contributes at order $\alpha_s^3 v^3$. It has been calculated in Ref. [7].

C. Spin-singlet P-wave states

The dominant Fock state of the $h_c$ consists of a $c\bar{c}$ pair in a color-singlet $^1P_1$ state. According to the velocity-scaling rules, the largest matrix elements are
\[ \langle \chi^\dagger (-\frac{i}{2} \vec{D})^m \psi_P \chi^\dagger (-\frac{i}{2} \vec{D})^m \chi \rangle \text{ and } \langle \chi^\dagger T^a \psi_P \psi^\dagger T^a \chi \rangle, \] both of which scale as \( v^5 \). The reason that they are comparable in magnitude is that one is suppressed by a factor of \( v^2 \) from the covariant derivatives, while the other is suppressed by \( v^2 \) because of the chromoelectric dipole transition required to reach the dominant Fock state from a color-octet \( 1S_0 \) state. The color-octet \( 3S_1 \) and color-singlet \( 1S_0 \) matrix elements in (57) contribute at orders \( \alpha_s v^8 \) and \( \alpha_s^2 v^9 \), respectively. The color-singlet P-wave matrix elements in (57) contribute at order \( \alpha_s^2 v^{12} \) and should be negligible. If we assume that the effect of a suppression factor of \( \alpha_s \) is comparable to that of a suppression factor of \( v^2 \), then the most important term is the color-octet \( 1S_0 \) term, which contributes at order \( \alpha_s^2 v^5 \). This term has not yet been calculated.

### D. Spin-triplet P-wave states

The dominant Fock state of the \( \chi_{cJ} \) consists of a \( c\bar{c} \) pair in a color-singlet \( 3P_J \) state. According to the velocity-scaling rules, the largest matrix elements are \( \langle \chi^\dagger \sigma^i T^a \psi_P \chi^\dagger \sigma^i T^a \chi \rangle \) and the color-singlet P-wave matrix elements in (57), all of which scale as \( v^5 \). The color-octet \( 3S_1 \) matrix element contributes to the gluon fragmentation function at order \( \alpha_s v^5 \), while the color-singlet P-wave matrix elements contribute at order \( \alpha_s^2 v^5 \). The matrix element \( \langle \chi^\dagger \psi_P \chi^\dagger \chi \rangle \) contributes at order \( \alpha_s^2 v^8 \) and should be negligible. We will therefore consider only the color-octet \( 3S_1 \) term and the color-singlet P-wave terms in the fragmentation function.

The symmetries of NRQCD can be used to reduce the three color-octet S-wave matrix elements \( \langle \chi^\dagger \sigma^i T^a \psi_P \chi^\dagger \sigma^i T^a \chi \rangle \) to one independent matrix element, which we take to be the \( J = 0 \) one. These symmetries are rotational symmetry and heavy-quark spin symmetry. Spin symmetry is an approximate symmetry with corrections that are of relative order \( v^2 \). It implies that the state \( \chi_{cJ}(\lambda) \), which is an eigenstate of \( J^2 \) and \( J_z \), can be expressed in the form

\[
\chi_{cJ}(\lambda) \approx \sum_{l_z s_z} \langle 1l_z; 1s_z | J\lambda \rangle \chi_{c}(l_z s_z),
\]

(64)
where $\chi_c(l_z s_z)$ is an eigenstate of $L_z$ and $S_z$. Using the definition (5) of the projection operator $P_H$, the color-singlet $^3S_1$ matrix element can be written

$$
\langle \chi^\dagger \sigma^i T^a \psi P_{c,i,j} \psi^\dagger \sigma^i T^a \chi \rangle \approx \sum_\lambda \sum_{l_z s_z l_z' s_z'} \sum_{1l_z; 1s_z; J\lambda} \langle J\lambda|1l_z'; 1s_z' \rangle 
\times \sum_S \langle 0|\chi^\dagger \sigma^i T^a \psi |\chi_c(l_z s_z) + S \rangle \langle \chi_c(l_z' s_z') + S |\psi^\dagger \sigma^i T^a \chi|0 \rangle. \quad (65)
$$

Spin symmetry also implies that the two matrix elements in (65) are proportional to $U_{sz}$ and $U_{iz}$, where $U_{mi}$ is the unitary $3 \times 3$ matrix that transforms vectors from the Cartesian basis to the spherical basis. Finally, rotational symmetry implies that the product of the two matrix elements in (65) summed over soft states $S$ must be proportional to $\delta_{l_z l_z'}$. Using the orthogonality relations of the Clebsch-Gordan coefficients, the equation (65) can be reduced to

$$
\langle \chi^\dagger \sigma^i T^a \psi P_{c,i,j} \psi^\dagger \sigma^i T^a \chi \rangle \approx (2J + 1) \langle \chi^\dagger \sigma^i T^a \psi P_{c,0} \psi^\dagger \sigma^i T^a \chi \rangle. \quad (66)
$$

These relations hold up to corrections of relative order $v^2$.

The color-singlet P-wave matrix elements in (57) can also be reduced to a single independent matrix element, which we choose to be $\langle \chi^\dagger (-\frac{i}{2} \hat{D} \cdot \sigma) \psi P_{c,i,j} \psi^\dagger (-\frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle$. Using the expression (64) for the $\chi_{c,j}$ states and the definition (5), we can write

$$
\langle \chi^\dagger (-\frac{i}{2} \hat{D})^i \sigma^j \psi P_{c,i,j} \psi^\dagger (-\frac{i}{2} \hat{D})^m \sigma^n \chi \rangle \approx \sum_\lambda \sum_{l_z s_z l_z' s_z'} \sum_{1l_z; 1s_z; J\lambda} \langle J\lambda|1l_z'; 1s_z' \rangle 
\times \sum_S \langle 0|\chi^\dagger (-\frac{i}{2} \hat{D})^i \sigma^j \psi |\chi_c(l_z s_z) + S \rangle \langle \chi_c(l_z' s_z') + S |\psi^\dagger (-\frac{i}{2} \hat{D})^m \sigma^n \chi|0 \rangle. \quad (67)
$$

Spin symmetry implies that the two matrix elements in (67) are proportional to $U_{sz}$ and $U_{iz}$. Since the operators create and annihilate $c\bar{c}$ pairs in the dominant Fock state of $\chi_c$, we can use the vacuum-saturation approximation, which is accurate up to corrections of relative order $v^4$. Keeping only the vacuum term in the sum over $S$, the matrix elements reduce to $\langle 0|\chi^\dagger (-\frac{i}{2} \hat{D})^i \sigma^j \psi |\chi_c(l_z s_z) \rangle$ and $\langle \chi_c(l_z' s_z') |\psi^\dagger (-\frac{i}{2} \hat{D})^m \sigma^n \chi|0 \rangle$. By rotational symmetry, these must be proportional to $U_{l_z}$ and $U_{m_l'}$. Thus the tensorial structure of the matrix element (67) is completely determined. The proportionality constant can be deduced by taking the special case $J = 0$, $i = j$ and $m = n$. The resulting formula is
\[
\langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle \approx \frac{1}{3} \langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle 
\]

\[
\times \sum \sum \sum \langle U_i \chi \sigma \rangle \langle J \lambda \rangle \langle J \lambda \rangle \langle l s I \rangle U_{s s I} U_{m s} U_{l i} U_{m l} . \tag{68}
\]

The scalar combinations of these matrix elements can be simplified by using the orthogonality relations of Clebsch-Gordan coefficients together with the identity

\[
\left( U U^\dagger \right)_{m_1 m_2} = - \sqrt{3} \langle 1 m_1 ; 1 m_2 | 0 0 \rangle . \tag{69}
\]

The resulting formulas are

\[
\langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle \approx \delta_{J 0} \langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle , \tag{70a}
\]

\[
\langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma)^m \sigma^n \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma)^m \sigma^n \chi \rangle \approx \frac{2 J + 1}{3} \langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle , \tag{70b}
\]

\[
\langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma)^m \sigma^n \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma)^m \sigma^n \chi \rangle 
\approx (-1)^J \frac{2 J + 1}{3} \langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle . \tag{70c}
\]

Using the relations (66) and (70), the dominant terms in the gluon fragmentation function for the \( \chi_{\sigma} \) reduce to

\[
D_{g \rightarrow \chi_{\sigma}} (z) = (2 J + 1) d^{(8,3)} (z) \langle \chi^\dagger \sigma^n T^a \mathcal{P}_{\chi \sigma} \psi^\dagger \sigma^n T^a \chi \rangle^{(\mu)}
\]

\[
+ d^{(2,3)} (Z, \mu) \langle \chi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi \sigma} \psi^\dagger (\mp \frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle , \tag{71}
\]

where \( d^{(8,3)} (z) \) is given in (31) and \( d^{(2,3)} (Z, \mu) \) is a linear combination of the functions in (46):

\[
d^{(2,3)} (Z, \mu) = \delta_{J 0} d_1 (z) + \frac{2 J + 1}{3} d_2 (Z, \mu) + (-1)^J \frac{2 J + 1}{3} d_3 (Z) . \tag{72}
\]

More explicitly, these coefficients are

\[
d^{(2,3)} (Z, \mu) = \frac{a^2 (m_c)}{486 m_c^3} \left[ (2 J + 1) \frac{Z}{(1 - Z)} + \left( Q_J - (2 J + 1) \log \frac{\mu}{2 m_c} \right) \delta (1 - Z) + P_J (Z) \right] , \tag{73}
\]

where the numbers \( Q_J \) are
\[ Q_0 = \frac{1}{4}, \quad Q_1 = \frac{3}{8}, \quad Q_2 = \frac{7}{8}, \quad (74) \]

and the functions \( P_J(z) \) are

\[ P_0(z) = \frac{z(85 - 26z)}{8} + \frac{9(5 - 3z)}{4} \log(1 - z), \quad (75) \]
\[ P_1(z) = -\frac{3z(1 + 4z)}{4}, \quad (76) \]
\[ P_2(z) = \frac{5z(11 - 4z)}{4} + 9(2 - z) \log(1 - z). \quad (77) \]

The matrix elements in (71) are related to the standard matrix elements defined in Ref. [1] by

\[ \langle \chi^\dagger \sigma^i T^a \psi \mathcal{P}_{\chi_0} \psi^\dagger \sigma^i T^a \chi \rangle \approx 4m_c \mathcal{O}_8^{\chi_0}(^3S_1), \quad (78) \]
\[ \langle \chi^\dagger (-\frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi_0} \psi^\dagger (-\frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle \approx 12m_c \mathcal{O}_1^{\chi_0}(^3P_0). \quad (79) \]

We now compare our results for the gluon fragmentation functions of the \( \chi_{cJ} \) states with previous calculations. These fragmentation functions were first calculated by Braaten and Yuan [8]. They regularized the infrared divergence from the process \( g^* \to c\bar{c}g \) by imposing a cut-off \( |k| > \Lambda \) on the momentum of the real gluon in the final state, and they used the covariant projection method to isolate the contributions to the fragmentation functions for \( \chi_{c0}, \chi_{c1}, \) and \( \chi_{c2} \). They also assumed implicitly that the renormalization of the NRQCD matrix element \( \langle \chi^\dagger \sigma^i T^a \psi \mathcal{P}_{c\bar{c} \to c\bar{c}'} \psi^\dagger \sigma^i T^a \chi \rangle \) was carried out in such a way that the net effect of terms on the NRQCD side of the matching condition was simply to transform the infrared cutoff \( \Lambda \) into an ultraviolet cutoff on NRQCD. The fragmentation functions calculated in Ref. [8] have been checked by Cho, Wise, and Trivedi [9], who extended the calculation to the polarized fragmentation functions of \( g \to \chi_J \). Our results differ from those in Ref. [8] only in the coefficients of the \( \delta(1 - z) \) terms in (73). Braaten and Yuan obtained \( Q_J' - (2J + 1) \ln(\Lambda/m_c) \), where \( Q_0' = \frac{13}{12}, Q_1' = \frac{23}{8}, \) and \( Q_2' = \frac{121}{24} \). The numbers \( Q_J \) and \( Q_J' \) differ by \( \frac{5}{6}(2J + 1) \). Since the difference between the coefficients of \( \delta(1 - z) \) is proportional to \( 2J + 1 \), it can be absorbed into a shift in the value of the color-octet matrix element \( \langle \chi^\dagger \sigma^i T^a \psi \mathcal{P}_{\chi_0} \psi^\dagger \sigma^i T^a \chi \rangle \). The relation between the matrix elements in the two calculations is
\[
\langle \chi^\dagger \sigma^i T^a \psi \mathcal{P}_{\chi_0} \psi^\dagger \sigma^i T^a \chi \rangle (\mu) \bigg|_{\text{dim.reg.}} = \langle \chi^\dagger \sigma^i T^a \psi \mathcal{P}_{\chi_0} \psi^\dagger \sigma^i T^a \chi \rangle (\Lambda) \bigg|_{\text{cutoff}} + \frac{16\alpha_s}{81\pi m_c^2} \left( \ln \frac{\mu}{2\Lambda} + \frac{5}{6} \right) \langle \chi^\dagger (-\frac{i}{2} \hat{D} \cdot \sigma) \psi \mathcal{P}_{\chi_0} \psi^\dagger (-\frac{i}{2} \hat{D} \cdot \sigma) \chi \rangle. \quad (80)
\]

Thus our calculation is consistent with that of Braaten and Yuan. The differences in the short-distance coefficients are due simply to different definitions of the color-octet matrix element.

The gluon fragmentation functions for \( \chi_{cJ} \) have also been calculated by Ma [10]. Ma used dimensional regularization to cut off the infrared divergence from the process \( g^* \to c\bar{c}g \). He also used the covariant projection method to isolate the contributions to the fragmentation functions for \( \chi_{c1}, \chi_{c1}, \) and \( \chi_{c2} \). His results differ from ours only in the numbers \( Q_J \) multiplying the \( \delta(1-z) \) terms in (73). He obtained \( Q_0 = \frac{1}{4}, Q_1 = \frac{15}{8}, \) and \( Q_2 = \frac{19}{8} \). The differences between his values of \( Q_J \) and ours are not proportional to \( 2J+1 \). Therefore, they cannot be absorbed into a redefinition of the color-octet matrix element. We attribute the discrepancy to an inconsistency in Ma’s calculation. Dimensional regularization of the infrared divergence involves analytically continuing integrals to \( 3 - 2\epsilon \) dimensions, while the standard covariant projection method uses projection matrices that are specific to 3 dimensions.

VII. CONCLUSIONS

Dimensional regularization is the most convenient method for regularizing the infrared and ultraviolet divergences that arise in calculations of quarkonium production and annihilation rates beyond leading order. The standard covariant projection method is incompatible with dimensional regularization. An alternative method for calculating the short-distance coefficients in the NRQCD factorization formulas is the threshold expansion method developed in Ref. [3]. In this paper, we have generalized this method to \( N \) spatial dimensions, so that dimensional regularization can be used consistently to regularize the infrared and ultraviolet divergences that arise in perturbative calculations. We illustrated the method by calculating the color-octet terms of order \( \alpha_s \) and the color-singlet terms of order \( \alpha_s^2 \) in the
gluon fragmentation functions for arbitrary quarkonium states. We resolved a discrepancy between two previous calculations of the gluon fragmentation functions for the spin-triplet P-wave quarkonium states. This general and systematic method should be very useful in extending calculations of the production and decay rates of heavy quarkonium states to higher orders in perturbation theory.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy, Division of High Energy Physics, under Grant DE-FG02-91-ER40684.

APPENDIX A: NONRELATIVISTIC EXPANSION OF SPINORS

This appendix is identical to Appendix A of Ref. [3], except that it includes only those formulas that generalize to $N$ spacial dimensions. Formulas involving Levi-Civita tensors that cannot be easily generalized are omitted. We give the nonrelativistic expansions for the spinors of a heavy quark $c$ and antiquark $\bar{c}$ for arbitrary momentum $P = (P_0, P_1, \ldots, P_N)$. We assume that the relative momentum $q = (q_1, \ldots, q_N)$ of the $c$ in the center-of-momentum (CM) frame of the $c\bar{c}$ pair is small compared to the quark mass $m_c$. The momenta $p$ and $\bar{p}$ of the $c$ and $\bar{c}$ can be written

\begin{align}
    p &= \frac{1}{2}P + Lq, \\
    \bar{p} &= \frac{1}{2}P - Lq, 
\end{align}

where $P$ is the total momentum and $L$ is a Lorentz boost matrix. From the mass-shell conditions, $p^2 = \bar{p}^2 = m_c^2$, we have $P \cdot Lq = 0$ and $P^2 = 4E_q^2$, where $E_q = \sqrt{m_c^2 + q^2}$. The components of the momenta $P$ and $Lq$ in the CM frame of the pair are

\begin{align}
    P_{\mu}^{\text{CM}} &= (2E_q, \ 0), \\
    (Lq)^\mu_{\text{CM}} &= (0, \ q). 
\end{align}
When boosted to an arbitrary frame in which the pair has total spacial momentum $P$, these momenta are
\[ P^\mu = \left( \sqrt{4E_q^2 + P^2}, \ P \right), \quad (A3a) \]
\[ (Lq)^\mu = L_j^\mu q^j. \quad (A3b) \]

The boost matrix $L_j^\mu$, which has one Lorentz index and one Cartesian index, has components
\[ L_0^\mu = \frac{1}{2E_q} P_j^\mu, \quad (A4a) \]
\[ L_i^\mu = \delta^{ij} - \frac{P^i P^j}{P^2} + \frac{P^0}{2E_q} \frac{P^i P^j}{P^2}. \quad (A4b) \]

The contraction of the boost tensor $L_i^\mu$ with the Lorentz vector $P$ vanishes: $P_\mu L_j^\mu = 0$.

The contractions of two boost matrices in their Lorentz indices or in their Cartesian indices have simple forms:
\[ g_{\mu\nu} L_i^\mu L_j^\nu = -\delta^{ij}, \quad (A5a) \]
\[ L_i^\mu L_i^\nu = -g^{\mu\nu} + \frac{P^\mu P^\nu}{P^2}. \quad (A5b) \]

The representation for gamma matrices that is most convenient for carrying out the nonrelativistic expansion of a spinor is the Dirac representation:
\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (A6) \]

In the CM frame of the $c\bar{c}$ pair, the spinors for the $c$ and the $\bar{c}$ are
\[ u(p) \bigg|_{CM} = \frac{1}{\sqrt{E_q + m_c}} \begin{pmatrix} (E_q + m_c) \xi \\ q \cdot \sigma \xi \end{pmatrix}, \quad (A7a) \]
\[ v(\bar{p}) \bigg|_{CM} = \frac{1}{\sqrt{E_q + m_c}} \begin{pmatrix} -q \cdot \sigma \eta \\ (E_q + m_c) \eta \end{pmatrix}. \quad (A7b) \]
Color and spin quantum numbers on the Dirac spinors and on the Pauli spinors $\xi$ and $\eta$ are suppressed. When boosted to a frame in which the pair has total spacial momentum $P$, the spinors for the $c$ and $\bar{c}$ are

$$u(p) = \frac{1}{\sqrt{4E_q(P_0 + 2E_q)(E_q + m_c)}} (2E_q + P\gamma_0) \begin{pmatrix} (E_q + m_c) \xi \\ q \cdot \sigma \xi \end{pmatrix}, \quad (A8a)$$

$$v(\bar{p}) = \frac{1}{\sqrt{4E_q(P_0 + 2E_q)(E_q + m_c)}} (2E_q + P\gamma_0) \begin{pmatrix} -q \cdot \sigma \eta \\ (E_q + m_c) \eta \end{pmatrix}. \quad (A8b)$$

These spinors are normalized so that $\bar{u}u = -\bar{v}v = 2m_c$ if the Pauli spinors are normalized so that $\xi^\dagger \xi = \eta^\dagger \eta = 1$.

The independent quantities that can be formed by sandwiching 3 or fewer Dirac matrices between $\bar{u}(p)$ and $v(\bar{p})$ are

$$\bar{u}(p)v(\bar{p}) = -2 \xi^\dagger (q \cdot \sigma) \eta, \quad (A9a)$$

$$\bar{u}(p)\gamma^\mu v(\bar{p}) = L^\mu_j \left( 2E_q \xi^\dagger \sigma^j \eta - \frac{2}{E_q + m_c} q^j \xi^\dagger (q \cdot \sigma) \eta \right), \quad (A9b)$$

$$\bar{u}(p)(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) v(\bar{p}) = (P^\mu L^\nu_j - P^\nu L^\mu_j) \left( \frac{2m_c}{E_q} \xi^\dagger \sigma^j \eta + \frac{2}{E_q(E_q + m_c)} q^j \xi^\dagger (q \cdot \sigma) \eta \right) + L^\mu_j L^\nu_k \xi^\dagger \{[\sigma^j, \sigma^k], q \cdot \sigma \} \eta, \quad (A9c)$$

$$\bar{u}(p)(\gamma^\mu \gamma^\nu \gamma^\lambda - \gamma^\lambda \gamma^\nu \gamma^\mu) v(\bar{p}) = L^\mu_i L^\nu_j L^\lambda_k \left( -E_q \xi^\dagger \{[\sigma^i, \sigma^j], \sigma^k \} \eta + \frac{q^j}{E_q + m_c} \xi^\dagger \{[\sigma^i, \sigma^j], q \cdot \sigma \} \eta \right. \right.$$ \nonumber

$$\left. + \frac{q^j}{E_q + m_c} \xi^\dagger \{[\sigma^i, \sigma^j], q \cdot \sigma \} \eta + \frac{q^k}{E_q + m_c} \xi^\dagger \{[\sigma^i, \sigma^j], q \cdot \sigma \} \eta \right)$$ \nonumber

$$- \frac{2}{E_q} \left( P^\mu L^\nu_j L^\lambda_i + L^\mu_i L^\nu_j P^\lambda + L^\mu_j P^\nu L^\lambda_i \right) \left( \xi^\dagger q^i \sigma^j \eta - \xi^\dagger q^j \sigma^i \eta \right). \quad (A9d)$$

The simplest way to derive these formulas is to use the identities

$$\left( 2E_q + \gamma_0 P \right) \left( 2E_q + P \gamma_0 \right) = 4E_q(P_0 + 2E_q), \quad (A10a)$$

$$\left( 2E_q + \gamma_0 P \right) \gamma^\mu \left( 2E_q + P \gamma_0 \right) = 4E_q(P_0 + 2E_q) \left( \frac{P^\mu}{2E_q} \gamma_0 + L^\mu_j \gamma^j \right). \quad (A10b)$$

Using the expressions for the spinor factors given in (A9), it is easy to carry out the nonrelativistic expansions in powers of $q$. 

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REFERENCES


Fig. 1. The lowest-order Feynman diagram in QCD for $g^*(l) \to c(p) + \bar{c}(\bar{p})$.

Fig. 2. The lowest-order Feynman diagram for the NRQCD matrix element $\langle \chi^\dagger \sigma^k T^a \psi P_{\bar{c}c',c\bar{c}} \bar{\psi} \sigma^k T^a \chi \rangle$. The lines entering at the left and leaving at the right come from a term of the form $|\bar{c}c'\rangle \langle c\bar{c}|$ in the projection operator $P_{\bar{c}c',c\bar{c}}$.

Fig. 3. The lowest-order Feynman diagrams in QCD for $g^*(l) \to c(p) + \bar{c}(\bar{p}) + g(k)$ when the $c\bar{c}$ pair is in a color-singlet state.

Fig. 4. The lowest-order Feynman diagrams for the NRQCD matrix element $\langle \chi^\dagger \sigma^k T^a \psi P_{\bar{c}c',c\bar{c}} \bar{\psi} \sigma^k T^a \chi \rangle$ when the projection operator $P_{\bar{c}c',c\bar{c}}$ requires the $c\bar{c}$ pairs to be in color-singlet states. The lines entering at the left and leaving at the right come from a term of the form $|c\bar{c}'g\rangle \langle c\bar{c}g|$ in the projection operator.