BLM-Resummation and OPE in Heavy Flavor Transitions

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Abstract

An all-order resummation is performed for the effect of the running of the coupling $\alpha_s$, in the zero recoil sum rule for the axial current and for the kinetic operator $\bar{\pi}^2$. The perturbative corrections to well-defined objects of OPE turn out to be very moderate. The renormalization of the kinetic operator is addressed.

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The recent progress in the theoretical description of heavy flavors is based on application of the heavy quark expansion to QCD. Its main elements are nonrelativistic expansion and the Operator Product Expansion (OPE) allowing a treatment of the strong interaction domain in model-independent systematic way. The precision achieved in certain cases is high and requires already a careful field-theoretic definition of effective operators beyond a simple quantum-mechanical (QM) description. Such completely defined operators can be introduced in different ways, however, the common feature is their scale dependence. Likewise, the coefficient functions are always $\mu$-dependent. The general idea of separating two domains and applying different theoretical tools to them was formulated long ago by K. Wilson [1] in the context of problems in statistical mechanics; in the modern language, applied to QCD it is similar to lattice gauge theories. The treatment of essentially Minkowskian quantities has, however, some peculiarities. They were first considered in [2] and then in [3].

In this paper I apply this technique to the zero recoil sum rules [4, 2] to calculate one loop-corrections accounting completely for the effect of running of the strong coupling, which can be called an extended BLM [5] approximation, or, in short, merely BLM approximation. It is thus a direct BLM-generalization of the perturbative calculations of [2]. In this way the BLM-improved perturbative evolution of the kinetic energy operator $\tilde{Q}(iD)^2Q = \tilde{\pi}^2(\mu)$ and the Wilson coefficient function $\xi_A(\mu)$ for the zero recoil sum rule for axial current are obtained.

\section{Zero recoil sum rules}

The heavy quark sum rules and the method to calculate the relevant perturbative effects to them have been discussed in detail in [2]; I will quote here only necessary equations.

The first sum rule for the spatial component of the axial current at zero recoil through terms $1/m_Q^2$ has the form

$$I_0^{(1)A}(\mu) = \xi_a(\mu) - \frac{1}{3} \frac{\mu_G^2(\mu)}{m_c^2} - \frac{\mu_\pi^2(\mu) - \mu_G^2(\mu)}{4} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right) \tag{1}$$

Here $I_0^{(1)A}$ is the zeroth moment of the first structure function $w_1$ of a heavy hadron for the axial current:

$$w_1^A = 2 \frac{1}{3} \text{Im} \ h_\pi^A$$
\[ I^{(1)A}_0(\mu) = \frac{1}{2\pi} \int_0^\mu w_1^A(\epsilon) \, d\epsilon \quad \epsilon = M_B - M_D - q_0 \quad . \]  

Here and in what follows the ultraviolet cutoff in the moments is assumed to be introduced via \( \theta(\epsilon - \mu) \); of course, \( \mu \ll m_Q \) must hold. I use the standard notations for the expectation values

\[ \mu_\pi^2 = \frac{1}{2M_{H_Q}} \langle H_Q | \vec{\pi}^2 | H_Q \rangle \quad , \quad \mu_G^2 = \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} \gamma^\alpha \sigma_{\alpha\beta} G^{\alpha\beta} Q | H_Q \rangle \quad . \]

The sum rule and the lower bound for \( \mu_\pi^2 \) was obtained using the sum rule for the vector structure function \( w_1^V \); for simplicity of notations below I consider the pseudoscalar “weak” current \( J^P = \bar{c} \gamma_5 b \) and the corresponding structure function \( w^P \) (the result for any similar current is the same):

\[ I_0^P(\mu) = \frac{1}{2\pi} \int_0^\mu w_1^P(\epsilon) \, d\epsilon = \xi^\pi^2(\mu) + \left( \frac{1}{m_c} + \frac{1}{m_b} \right)^2 \left( \mu_\pi^2(\mu) - \mu_G^2(\mu) \right) \quad . \]

The Wilson coefficients \( \xi \) depend on the concrete field-theoretic definition of \( \mu_\pi^2 \) and \( \mu_G^2 \) (in what follows terms \( \sim \alpha_s \Lambda_{\overline{\text{QCD}}}^2 / m_Q^2 \) are neglected and, therefore, \( \mu_G^2 \) can be considered \( \mu \)-independent, \( \mu_G^2 \simeq 3(M_B^2 - M_D^2)/4 \)). At tree level one has

\[ \xi_A(\mu) = 1 \quad , \quad \xi^\pi^2(\mu) = 0 \quad . \]

The first – and so far the only – definition of the kinetic operator in field theory was given in [2]: the expectation value was defined to have perturbatively \( \xi^\pi^2 = 0 \) at any \( \mu \). Thus, to calculate \( \mu_\pi^2 \) in an external gluon field one must solve the Dirac (Pauli) equation of motion for nonrelativistic heavy quark, find the spectrum and integrate the spectral density induced by the current \( \vec{\sigma} \vec{\pi} \) from 0 to \( \mu \). For an external soft field with frequencies much below \( \mu \) this yields the classical value of \( (i\vec{D})^2 \) [2], therefore this is a proper definition of the operator \( \vec{\pi}^2 \) in the quantum field theory. In general it clearly depends on \( \mu \). Given the definition, one is able to calculate its \( \mu \)-dependence (mixing with unit operator) and the value of \( \xi_A \) without neglecting terms \( \sim \mu^2 / m_Q^3 \) in the latter.

#### 2 Calculation

The technical method was explained at length in [2, 3]; it reduces to considering the OPE expansion in the perturbation theory. Moreover, \( \mu_G^2 \) vanishes in perturbation theory to leading order in \( 1 / m_Q \); it will be discarded below.
The new element here is the calculation of effects of running $\alpha_s$. In a one loop Euclidean calculation it is done by replacing $\alpha_s$ by $\alpha_s(k^2)$ where $k$ is the gluon momentum. For Minkowski quantities, in particular those that are not obtained by analytic continuation from Euclidean space, one should use the dispersive approach (it was used as long as 20 years ago and recently attracted renewed attention; for the extensive analysis and recent discussion see [6, 7]). It relies on using the dispersion relation for the dressed gluon propagator:

$$\frac{\alpha_s(k^2)}{k^2} \left( \delta_{\mu\nu} - c \frac{k_\mu k_\nu}{k^2} \right) = -\frac{1}{\pi} \text{Im} \int \frac{d\lambda^2}{\lambda^2} \frac{\text{Im} \alpha_s(-\lambda^2)}{k^2 + \lambda^2} \left( \delta_{\mu\nu} - c \frac{k_\mu k_\nu}{k^2} \right).$$  \hspace{1cm} (6)

Then $(\delta_{\mu\nu} - c k_\mu k_\nu/k^2)/(k^2 + \lambda^2)$ is a propagator of a gluon with mass-squared $\lambda^2$ (the longitudinal component does not contribute in the one loop diagrams), and

$$\rho = -\frac{1}{\pi} \text{Im} \alpha_s(-\lambda^2)$$ \hspace{1cm} (7)

plays the role of the weight function for integrating over $\lambda^2$. More details are found in [7] and papers mentioned there.

As the first step a calculation of all quantities of interest in the one loop approximation with a non-zero gluon mass is needed. At $\lambda^2 = 0$ the calculations have been done in [2] and the modifications are straightforward. The perturbative spectral densities in the lhs of the sum rules are given by the “elastic” peak of $b \rightarrow c$ transition with $\epsilon = 0$ and by the transition into the final state $c + g(\bar{k})$ with $\epsilon = \omega(\bar{k}) = \sqrt{\lambda^2 + \bar{k}^2}$ starting $\epsilon_{\text{min}} = \lambda$; The effect of the quark recoil is of higher order in $1/m_Q^2$ and is discarded for $1/m_Q^2$ terms. In the sum rule for $\mu^2$

$$\frac{1}{2\pi} \int_0^\mu w_1^P(\epsilon) \, d\epsilon = \left( \frac{1}{m_c} + \frac{1}{m_b} \right)^2 \mu_\sigma^2(\mu)$$ \hspace{1cm} (8)

the elastic transition is forbidden by parity.

The inelastic transition amplitudes given by Figs. 1 are modified minimally compared to calculation of [2] but require certain care. One should distinguish between $\omega$ and $|\bar{k}|$ and use the proper two-body phase space $|\bar{k}| \, d\omega$; the scalar denominators of the propagators are $\pm 2m_Q \omega$. Finally, the massive gluon has three polarization propagating, instead of two for the massless one. The following trick simplifies the calculation.

3
Since the tree gluon emission amplitude is transverse, the gluon propagator can arbitrarily be redefined in the following way:

\[ \delta_{\mu\nu} \rightarrow \delta_{\mu\nu} - a_{\mu} k_{\nu} - a_{\nu} k_{\mu} \]  

(9)

Choosing \( a = \tilde{a} = (k_0, -\vec{k})/2k_0^2 \equiv \vec{k}/2k_0^2 \) we arrive at the the tensor part of the propagator

\[ \bar{g}_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{2k_0^2} (\vec{k} \mu k_{\nu} + \vec{k} \nu k_{\mu}) = \sum_{i=1,2} e_{\mu\perp}^i e_{\nu\perp}^i + \frac{\lambda^2}{k_0^2} n_\mu n_\nu , \quad n = (0, \vec{k}/|\vec{k}|) . \]  

(10)

In doing so we explicitly excluded the Coulomb quanta. The last term gives the contribution of the longitudinal polarization. In this form the rest of the calculation is simple.

For the perturbative sum rule (8) we get

\[ \frac{1}{2\pi} w^P(\omega; \lambda) d\omega = \theta(\omega - \lambda) \frac{8\alpha_s}{3\pi} \left( \frac{1}{m_c} + \frac{1}{m_b} \right)^2 \left( 1 + \frac{\lambda^2}{2\omega^2} \right) |\vec{k}| d\omega ; \]  

(11)

the effect of the gluon mass is only kinematic besides the longitudinal contribution \( \lambda^2/2\omega^2 \) in the last bracket.

As in the original analysis [2] one can consider the sum rule for \( \frac{1}{3} \gamma_k \times \gamma_k \) currents:

\[ \frac{1}{2\pi} \int_0^\mu w_1^V(e) de = \frac{\mu_G^2(\mu) - \mu_G^2(\mu_s)}{12} \left[ \left( \frac{1}{m_c} + \frac{1}{m_b} \right)^2 + 2 \left( \frac{1}{m_c} - \frac{1}{m_b} \right)^2 \right] + \frac{\mu_G^2(\mu)}{3m_c^2} . \]  

(12)

The OPE states that all soft physics is given by the rhs; the perturbative \( \mu \)-dependence of the moment is thus given by that of the operators appearing in it. Differentiating over \( \mu \) we would get the perturbative spectral density, which thus must have the same functional form as eq. (11) with the proper dependence on quark masses. A direct computation yields

\[ \frac{1}{2\pi} w_1^V(\omega; \lambda) = \frac{2\alpha_s}{9\pi} \theta(\omega - \lambda) \left[ \left( \frac{1}{m_c} + \frac{1}{m_b} \right)^2 + 2 \left( \frac{1}{m_c} - \frac{1}{m_b} \right)^2 \right] \left( 1 + \frac{\lambda^2}{2\omega^2} \right) |\vec{k}| ; \]  

(13)

the OPE, of course, is not violated in the one loop perturbative calculations.\(^1\)

\(^1\)There were a few explicit checks [8] and even attempts to disprove [9] OPE at this level in the recent years.
A similar calculation for the axial current sum rule (1) yields a more cumbersome result. The elastic peak is present and results in a contribution

$$\frac{1}{2\pi} w^A_{1\text{per}}(\epsilon) = \delta(\epsilon) \cdot \left(1 + \frac{8\alpha_s}{3\pi} r_0(\lambda)\right)$$  \hspace{1cm} (14)$$

where \(r_0(\lambda)\) has been calculated in [10], eq. (B.3). The calculation of the continuum part was detailed above and yields

$$\frac{1}{2\pi} w^A_{1\text{cont}}(\epsilon) = \frac{\alpha_s}{3\pi} \theta(\omega-\lambda) \left[2 \left(\frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_cm_b^2}\right) \frac{\mathbf{k}^2}{\omega^2} + \left(\frac{1}{m_c} - \frac{1}{m_b}\right)^2 \frac{\lambda^2 k^2}{\omega^2 \mathbf{w}^2}\right] \mathbf{k}. \hspace{1cm} (15)$$

### 2.1 Resummation of perturbative corrections

Integrals of the spectral densities eq. (11) and eqs. (14), (15) are readily calculated:

$$\langle \mu^2 \rangle^A_{\text{per}}(\mu; \lambda) = \frac{4\alpha_s}{3\pi} \theta(\mu^2 - \lambda^2) \frac{(\mu^2 - \lambda^2)^{3/2}}{\mu}. \hspace{1cm} (16)$$

Inserting this result into the rhs of the sum rule (1) one gets

$$\xi_A(\mu; \lambda) = 1 + \frac{8\alpha_s}{3\pi} r_0(\lambda) + \frac{\alpha_s}{3\pi} \left\{ \left(\frac{1}{m_c} + \frac{1}{m_b}\right)^2 \left[-2\lambda^2 \log \frac{\mu + \sqrt{\mu^2 - \lambda^2}}{\lambda} + 2\mu \sqrt{\mu^2 - \lambda^2} - \frac{2}{3} \frac{\mu^2 - \lambda^2}{\mu} \right] + \left(\frac{1}{m_c} - \frac{1}{m_b}\right)^2 \left\{(\mu^2 - \lambda^2)^{5/2} + (\mu^2 - \lambda^2)^{3/2}\right\} \right\} \theta(\mu^2 - \lambda^2). \hspace{1cm} (17)$$

Using the explicit expression for \(r_0(\lambda)\) [10] one finds

$$r_0(\lambda) = \frac{3}{4} \left(\frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} - \frac{8}{3}\right) - \frac{1}{8} \lambda^2 \log \frac{\lambda^2}{m_b^2} \left(\frac{1}{m_c} + \frac{1}{m_b}\right)^2 + \mathcal{O}(\lambda^2); \hspace{1cm} (18)$$

\(\xi_A(\lambda)\) does not contain nonanalytic in \(\lambda^2\) terms through order \(1/m_b^2\) thus showing the absence of \(1/m_b^2\) infrared (IR) renormalon singularity in the BLM calculation. This, of course, is ensured by OPE since the infrared contribution below \(\mu\) is peeled off from \(\xi_A;^2\) nonanalytic terms, on the other hand, can appear only in the infrared –

\[\text{The residual nonanalytic terms } \sim \lambda^3/m_b^3 \text{ and smaller remain since the way to calculate } \xi_A \text{ is accurate up to such terms; they can be arbitrarily added or removed unless the operators of } D = 6 \text{ and higher are incorporated. Their inclusion in the similar calculation would kill the next nonanalytic terms as well.}\]
elsewhere the propagator $1/(k^2 + \lambda^2)$ is analytic in $\lambda^2$. The cancellation thus provides an independent check of the cumbersome expression for $r_0(\lambda)$.

The absence of these nonanalytic terms ensures the absence of the corresponding IR renormalon in the resummation procedure; in other words, purely perturbatively the IR renormalons match in the sum rule. The concrete analysis thus disagrees with the claims of [9] of a discrepancy; basing on it the sum rules were declared erroneous in [9, 11]. The criterion as applied in those papers does not make sense in general, and there is no point to address it here.

One also notes the absence of nonanalytic terms in $\langle \mu_1^2 \rangle_{\text{loop}}$, in spite of non-trivial operator mixing. It signifies the absence of the IR renormalon in $\bar{Q}(i\bar{D})Q$ in the BLM approximation [8] (see also [3]). The reason will be touched upon later.

To calculate the BLM-type resummed result one is to fix a particular form of the strong coupling according to one’s preference. Then, for an observable $A$ having the perturbative expansion through order $\alpha_s$,

$$A(\lambda^2) = 1 + \frac{\alpha_s}{\pi} A_1(\lambda^2)$$

one has

$$A_{\text{resum}} = 1 + \int \frac{d\lambda^2}{\lambda^2} A_1(\lambda^2) \rho(\lambda^2) .$$

For calculating $\xi_A$ the exact infrared behavior of the coupling is not essential since the IR domain has been removed.

The most popular choice for “all-order” resummation is the literal one-loop $\alpha_s$:

$$\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 + \frac{b}{4\pi} \log \frac{k^2}{Q^2}} = \frac{4\pi}{b \log \frac{k^2}{\Lambda_v^2}}$$

with

$$\rho^{\text{BLM}}(\lambda^2) = \frac{4}{b} \left( \Lambda_v^2 \delta(\lambda^2 + \Lambda_v^2) - \frac{1}{\log \frac{\lambda^2}{\Lambda_v^2} + \pi^2} \right) .$$

$\Lambda_v$ is $\Lambda_{\text{QCD}}$ in the $V$-scheme [5], $\Lambda_v = e^{5/6} \Lambda_{\text{QCD}}^{\text{MS}} \approx 2.3 \Lambda_{\text{QCD}}^{\text{MS}}$; here and throughout the paper $\alpha_s$ is the $V$-scheme coupling $\alpha_s(k) = \alpha_s^{\text{MS}}(e^{-5/6}k)$ neglecting higher-order effects in the $\beta$-function.

In this case one has

$$A^{\text{BLM}} = 1 + \frac{4}{b} \left[ A_1(-\Lambda_v^2) - \int \frac{d\lambda^2}{\lambda^2} A_1(\lambda^2) \frac{1}{\log \frac{\lambda^2}{\Lambda_v^2} + \pi^2} \right] .$$
Using the explicit expression eq. (17) the all-order BLM result for $\xi_A$ is obtained by simple integration.

The $\mu$-dependence of $\mu^2_\pi$ and of the coefficient function $\xi_A$

$$\frac{d\pi^2(\mu)}{d\mu} = 2 c_\pi(\alpha_s(\mu)) \mu \cdot 1$$

$$\frac{d\xi_A(\mu)}{d\mu} = -2 \alpha_A(\alpha_s(\mu)) \mu$$

in this approximation is given by integrating directly the continuum parts of $u^\text{pert}$:

$$c_\pi = \frac{8}{3} \int \frac{d\lambda^2}{\lambda^2} \rho^\text{BLM}(\lambda^2) \left(1 + \frac{1}{2} \frac{\lambda^2}{\mu^2}\right) \sqrt{\mu^2 - \lambda^2} \theta(\mu^2 - \lambda^2)$$

$$\alpha = \frac{\alpha_s}{3\pi} \int \frac{d\lambda^2}{\lambda^2} \rho^\text{BLM}(\lambda^2) \left\{4 \left(\frac{1}{m_c} + \frac{1}{m_b}\right)^2 \left(1 + \frac{\lambda^2}{3\mu^2}\right) + \right.$$}

$$+ \left(\frac{1}{m_c} - \frac{1}{m_b}\right)^2 \left(2 + \frac{2\lambda^2}{3\mu^2} + \frac{\lambda^2(\mu^2 - \lambda^2)}{\mu^4}\right)\left\} \sqrt{\mu^2 - \lambda^2} \theta(\mu^2 - \lambda^2)$$

$$\Delta^\text{BLM}_{\mu^2}(\mu) \equiv \langle \mu^2_\pi \rangle^\text{pert}(\mu) = \frac{4}{3} \int \frac{d\lambda^2}{\lambda^2} \rho^\text{BLM}(\lambda^2) \left(\frac{\mu^2 - \lambda^2}{3\mu^2}\right)^{3/2} \theta(\mu^2 - \lambda^2).$$

Although the integrals here formally run over the “infrared domain” of $\lambda^2$ as well, they actually do not include infrared effects due to specific constraints on the moments of $\rho(\lambda^2)$ [12] with the weights in eqs. (25) analytic below $\mu^2$.

In a similar way one could have tried to resum the perturbative series in $\mu^2_\pi(\mu)$ itself:

$$\Delta^\text{BLM}_{\mu^2}(\mu) \equiv \langle \mu^2_\pi \rangle^\text{pert}(\mu) = \frac{4}{3} \int \frac{d\lambda^2}{\lambda^2} \rho^\text{BLM}(\lambda^2) \left(\frac{\mu^2 - \lambda^2}{\mu}\right)^{3/2} \theta(\mu^2 - \lambda^2).$$

However, this procedure is physically senseless, no such object can be defined and any correction beyond BLM makes it ill defined. It can be convenient, however, as an intermediate result to relate the particular field-theoretic definition of $\pi^2$ to another one if the similar calculations in the latter are made. The absence of the IR renormalon in the one-loop approximation allows at least formal assigning a definite value to $\Delta^\text{BLM}_{\mu^2}(\mu)$. For this reason its anatomy will be addressed below.
3 Analysis

In Figs. 2 I draw the plots of $\Delta_{\pi^2}^{\text{BLM}}(\mu)$; Fig. 2a shows the dimensionless ratio $\tau_{\pi^2} = \Delta_{\pi^2}^{\text{BLM}}(\mu)/\Lambda_{\text{LO}}^2$, as a function of $\mu/\Lambda_{V}$; Fig. 2b illustrates the absolute value for three choices of $\mu$.

The value of

$$
\eta_A(\mu) \equiv \left( \xi_A(\mu) \right)^{1/2}
$$

is shown in Fig. 3; $\eta_A(\mu)$ must be used to calculate the physical formfactors in the OPE approach instead of $\eta_A$ of HQET in the model estimates. Numerical evaluation was performed assuming $m_b = 4.8 \, \text{GeV}$ and $z = m_c/m_b = 0.3$.

For reasonable values of $\mu$ and $\Lambda_V$ natural for low-energy physics (the corresponding analysis [13] suggests $\Lambda_V \simeq 300 \, \text{MeV}$) the “perturbative” BLM-piece of $\mu^2$ is clearly moderate, $\simeq 0.2 \, \text{GeV}^2$, and the latter does not depend too strongly on the renormalization point. Nevertheless, the BLM series, having a finite radius of convergence, is still divergent: the radius of convergence is given by

$$
\left( \frac{\alpha_s(\mu)}{\pi} \right)_{\text{max}} \equiv \frac{\alpha_{\text{sh}}}{\pi} = \frac{4}{\pi b} ;
$$

the series is sign-varying and asymptotically behaves like

$$
\frac{\alpha_s}{\pi} \frac{\pi^2}{\pi^2} \cos \left( \frac{\pi n}{2} - \frac{3\pi}{4} \right) \left( \frac{\pi b}{4} \right)^n \left( \frac{\alpha_s(\mu)}{\pi} \right)^n .
$$

Even at $\Lambda_V = 300 \, \text{MeV}$ the scale $\mu_{\text{sh}}$ where the series starts to converge, $\alpha_s(\mu_{\text{sh}}) = 4/b$, constitutes 1.44 GeV. The first ten terms are written below:

$$
\Delta_{\pi^2}^{\text{BLM}}(\mu) \simeq \frac{4\alpha_s(\mu)}{3\pi} \cdot \mu^2 \cdot \left[ 1 + 2.88x - 2.51x^2 - 44.8x^3 - 6.4x^4 + 980x^5 + 924x^6 - 
-25900x^7 - 40900x^8 + 769400x^9 + 162000x^{10} + \ldots \right] ; \quad x = \frac{\alpha_s(\mu)}{\pi} \quad (29)
$$

(this series is written in terms of the $V$-scheme $\alpha_s(\mu)$). It is clear that the first BLM correction cannot represent numerically the complete result for any $\mu$ relevant for the heavy quark expansion in charm, as soon as the effects of the running of $\alpha_s$ in this domain is taken into account.

\footnote{The notations used here follow [14] where their meaning is clarified.}
The analytic properties and the expansion are most conveniently obtained using the representation

$$
\Delta_{\pi^2}^{BLM}(\mu) = \frac{4\alpha_s(\mu)}{3\pi} \frac{1}{2\pi i} \oint \frac{d\lambda^2}{\lambda^2} \frac{(\mu^2 - \lambda^2)^{3/2}}{\mu} \frac{\alpha_s(\mu^2)}{1 + \frac{b}{4\pi\alpha_s(\mu^2)} \log \frac{\lambda^2}{\mu^2}}
$$

with the contour shown in Fig. 4.

The results for $\xi_A$ are shown only in the limited range $\mu < 0.8$ GeV; irrespectively of the actual onset of duality, values of $\mu$ above 0.7 GeV are not appropriate for OPE in charm: the power expansion runs in powers of $\mu/m_c$. With the running mass $m_c(\mu) < 1.15$ GeV one cannot sensibly adopt larger $\mu$ if the running of $\alpha_s$ is included: in the BLM approximation the perturbative corrections are not small in this domain and the condition $\mu \ll m_c$ must be carefully respected.

Assuming $\alpha_s$ running one cannot allow larger values of $\Lambda_{QCD}$ either; it is enough to recall that the “standard” value of $\Lambda_{QCD}^{MS} \simeq 250$ MeV implies that the usual three-loop coupling hits the Landau singularity as early as at the gluon momentum $\sim 900$ MeV. These arguments having a more general practical relevance will be discussed in more detail in a separate publication.

A clear illustration is merely the impact of $1/m_Q^3$ and higher IR renormalons in $\xi_A$; as shown below already at $\Lambda_{QCD}^{MS} = 220$ MeV one has $\delta_{1/m_Q^2}(\eta^2_A) \simeq 0.05$, thus making the uncertainty irreducible without an account for $1/m_c^3$ effect larger than the perturbative corrections themselves.

The quantity $\xi_A(\mu^2; \lambda)$ calculated through terms $\mu^2/m_c^2$ still contains nonanalytic terms $\sim \lambda^3, \lambda^4 \log \lambda^2$ etc.:

$$
\pi \frac{11 + 5z + 5z^2 + 11z^3}{16} \left( \frac{\lambda^2}{m_c^2} \right)^{3/2} \frac{\alpha_s}{\pi} \\
\frac{9 + 2z + 2z^2 + 2z^3 + 9z^4}{18} \left( \frac{\lambda^4}{m_c^4} \right) \log \frac{\lambda^2}{m_c^2} \frac{\alpha_s}{\pi}.
$$

The results shown in Fig. 3 correspond to a linear extrapolation of $\xi_A(\mu^2; \lambda)$ in $\lambda^2$ from $\lambda^2 = 0$ to $-\Lambda_V^2$ which almost coincides with the principal value prescription; all arbitrariness is an effect of $\Lambda_{QCD}/m^3$ and higher terms. The corresponding uncertainties (defined as the formal imaginary part)

$$
\delta_{1/m_Q^3}(\eta^2_A) \simeq 0.0105 \left( \frac{\Lambda_V}{300 \text{ MeV}} \right)^3 \quad \delta_{1/m_Q^4}(\eta^2_A) \simeq 0.0015 \left( \frac{\Lambda_V}{300 \text{ MeV}} \right)^4
$$

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are still small at $\Lambda_V = 300$ MeV but too significant already at $\Lambda_V = 500$ MeV.

Again, the radius of convergence if the BLM series for the Wilson coefficient (discarding in an arbitrary way $1/m_{Q}^2$ and higher terms) is given by $a_1^{(V)}(\mu) = 4/b \simeq 0.45$ corresponding to $\bar{\sigma}_\pi(\mu) \simeq 0.29$.

4 Once more about the kinetic operator

Certain confusion exists in the literature about the expectation value of the kinetic operator $\mu_\pi^2$. In the HQET a similar quantity is denoted $-\lambda_1$; they coincide on the classical level but generally differ in the quantum field theory (QFT). However, a similar field-theoretic definition of $-\lambda_1$ has never been given.

On one hand, $-\lambda_1$ is often defined as $\mu_\pi^2(\mu)$ from which some calculated perturbative corrections are subtracted (different in different approaches) [15]; on the other hand, the values deduced for such $-\lambda_1$ are also claimed for $\mu_\pi^2$ [15, 16] – which certainly cannot be true simultaneously.

The difference between possible definitions of the renormalized operator is seen already at order-$\alpha_s$ perturbative corrections. They are given by two diagrams in Fig. 5; both diverge quadratically and must be cut off. A direct computation shows 4 that if one averages over directions of $k_\alpha$ before integrating over $k^2$, these diagrams exactly cancel each other. Therefore, if a cutoff is introduced by any function depending on $k^2$, the one-loop renormalization vanishes. Clearly, one does not get renormalization in BLM resummation either.

The operator $\bar{\pi}^2$, however, is defined differently in [2]. It is easy to see that at the one loop level it corresponds to introducing $\theta(\mu - |\vec{k}|)$ with no constraints on $k_0$. In such a case the renormalization is obviously present, and is given by $c_\pi = 4\alpha_s/3\pi$ [2].

The ‘Lorentz-invariant’ cutoff at $k^2 = \mu^2$ at first sight seems more convenient. However, the advantage is purely technical and applies exclusively to the perturbative calculations. It is possible to work with such definition in Euclidean theories and statistical mechanics.

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4This simple calculation has been checked in discussions with M. Shifman and A. Vainshtein in late 1994 to clarify the observation [8] about the disappearance of the renormalon in the kinetic operator in one loop.
On the other hand, theory of heavy quarks is essentially Minkowskian, and a 
Lorentz-invariant cutoff generally does not allow to formulate theory on the nonper-
turbative level. Imposing constraints on the timelike components of momenta leads
to non-locality in time, and the standard Hamiltonian approach is not applicable.
For example, a heavy hadron state \( |H_Q\rangle \) over which one wants to average \( \hat{Q}(i\hat{D})^2Q \)
is not completely defined. Using another language, the Euclidean amplitudes either
cannot be continued to the physical domain, or possess wrong analytic properties
there. These deep theoretical questions will be discussed in subsequent publications.

Practically, as soon as a proper particular definition of an operator is given, one
can relate its expectation value to that in other schemes. For example, in routine
calculations in the QCD sum rules [17] the cutoff is extrapolated to zero in the one-
loop approximation. Such a value of \( -\lambda_1 \) is less than \( \mu_\pi^2 \) by some small amount
calculated in [2], therefore, strictly speaking, in some practical applications \( \mu_\pi^2 \) must
be taken even larger than estimates quoted in [17]. On the other hand, for purely
technical reason a definition with the covariant cutoff was applied in the calculation
of the semileptonic widths in [3], therefore the literal numbers of [17] were used
there, which are somewhat smaller than \( \mu_\pi^2 \) discussed in [2] and in the present paper.
The numerical analysis above easily allows one to use a more convenient calculation
scheme in each particular case.

The advantage of the “physical” definition of \( \mu_\pi^2 \) is that it allows to derive an
important lower bound
\[
\mu_\pi^2 > \mu_G^2
\]  
(33)
and still leads to sensible – and well-controlled – perturbative corrections in the sum
rules for observable quantities.

It is important to note that the relations between properly given QFT definitions
of the operators is done perturbatively and excludes the IR domain. For example, if
there is no renormalon in the perturbative series for them it in a certain approximation
in one scheme, the same holds in any other scheme as well. This explains the absence
of the renormalon in the BLM series for \( \mu_\pi^2(\mu) \) noted in Sect. 2.1: with the cutoff over
\( k^2 \) all terms merely vanish.

It does not contradict the presence of mixing:
\[
\frac{d\mu_\pi^2(\mu)}{d\mu^2} = c_\pi(\alpha_s(\mu)) .
\]  
(34)
Even in the large-\( n_f \) approximation \( c_\pi \) is a whole series in \( \alpha_s(\mu) \), see eq. (25). It is
easy to understand this fact: calculating \( \mu_\pi^2(\mu) \) perturbatively one integrates over the
‘cylinder’ domain

\[-\infty < k_0 < \infty , \ |\vec{k}\| < \mu .\]

On the other hand, the integral over the spherical domain \(k^2 < \mu^2\) vanishes. For this reason the integral can be reduced to the domain

\[ k_0 > \sqrt{\mu^2 - \vec{k}^2} , \ |\vec{k}\| < \mu \]

(see Fig. 6) where \(k^2 > \mu^2\) always holds and the coupling remains small.

It means that in eq. (34) the coefficient \(c_\pi\) is given by an effective coupling that never grows to infinity at finite \(\mu\) in the BLM approximation. These subtleties were missed in [18] where mixing eq. (34) was equated to the presence of renormalons.

The situation is completely different in the case of the pole mass: the leading IR contribution \(\delta m_Q \sim \mu\) comes from \(k_0 \lesssim \frac{k^2}{m^2_Q} \lesssim \frac{\mu^2}{m^2_Q} \simeq 0\) [19], therefore \(k^2 = -\vec{k}^2\) and the corresponding coupling is always \(\alpha_s(\mu)\) in the BLM approximation.

## 5 Conclusions

All-order BLM effects are calculated for the zero recoil sum rules within proper OPE approach. Contrary to existing claims they have quite moderate impact. In particular, for the available field-theoretic definition of the kinetic operator \(\vec{\pi}^2\) its expectation value obeys the inequality [4, 2]

\[ \mu^2_{\pi}(\mu) > \mu^2_G \]

for any normalization point \(\mu\); the normalization-point dependence is very moderate.

The conclusion differs from a recent paper [15] where the first nontrivial BLM terms \(\mathcal{O}(b_0a_s^2)\) were addressed; those represent the first term of the expansion of the full function calculated in the present paper. It is seen that the series for this functions are still divergent and the calculated terms already grow in magnitude; in this case the estimates based on the first term \(\mathcal{O}(b_0a_s^2)\) are numerically misleading. The analysis shows that for the scale governed by charm effects of running of the coupling – if addressed at all – must include the whole resummation, made in the OPE-consistent way.

\[ ^5 \text{The phenomenological relevance of such theoretical improvement is far from obvious; it will be discussed elsewhere.} \]
Moreover, the calculation of $O(b_0^2(\mu))$ terms in [15] missed the similar contribution from $\eta_A^2$ and this is erroneous. It is only the whole set of corrections that eliminates the infrared domain from the perturbative factor. As a result of this omission the major part of the corrections came from the domain near and below the Landau singularity and therefore seems irrelevant.

It is worth mentioning that the smearing procedure considered in [15] practically does not improve the convergence: the critical value of $\alpha_s$ for the weight $\mu^{2n}/(\mu^{2n} + \epsilon^{2n})$ decreases by only a factor of $1 - 1/(2n)$ for the price of increasing the actual cutoff scale while the room for it is severely limited by the small value of $m_c$.

The value of $\eta_A(\mu) = (\xi_A(\mu))^{1/2}$ at a representative choice $\mu \simeq 0.5$ GeV, $\Lambda_V^{[V]} = 300$ MeV is

$$
\eta_A(\mu) = \begin{cases} 
1 & \text{tree level} \\
0.975 & \text{one loop} \\
0.99 & \text{all-order BLM}
\end{cases}
$$

Clearly, the perturbative corrections including BLM effects are very moderate and differ minimally from the estimate 0.98 used in the original analysis [4, 2]. The difference falls well below the effect of $1/m_c^3$ corrections not addressed so far.

The nonperturbative corrections, on the other hand, do have a grave numerical impact on the model analysis [11, 21] (see also [9]) where it was postulated to use ill-defined $\eta_A^{[HQET]}(\mu) = \eta_A(0)$ according to the routine practice of HQET [22]. In all later publications the perturbative factor was used about 0.95 which is not supported by the analysis. Moreover, such an approach results in double-counting of the soft domain contributions.

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Figure Captions

Fig. 1: Perturbative diagrams leading to nontrivial inelastic structure functions.

Fig. 2: The perturbative part of the kinetic operator in the resummed BLM approximation, $\Delta_{\pi^2}^{BLM}(\mu)$.
   a): The dimensionless ratio $\tau_{\pi^2} = \Delta_{\pi^2}^{BLM}(\mu)/\Lambda_V^2$ as a function of the dimensionless scale parameter $\mu/\Lambda_V$.
   b): The absolute value of $\Delta_{\pi^2}^{BLM}(\mu)$ as a function of $\Lambda_Q^{(V)}$ for $\mu = 0.5$ GeV, $\mu = 0.75$ GeV and $\mu = 1$ GeV.

Fig. 3: The value of the Wilson coefficient $\eta_A(\mu) = (\xi_A(\mu))^{1/2}$ for $\Lambda_{QCD}^{(V)} = 200$ MeV, $\Lambda_{QCD}^{(V)} = 300$ MeV and $\Lambda_{QCD}^{(V)} = 400$ MeV. The value of $\eta_A(\mu)$ must be used in the QCD-based calculations of the exclusive zero-recoil $B \to D^*$ formfactor when nonperturbative effects are addressed. The shaded bars show the purely perturbative uncertainty irreducible without the OPE account for $1/m_3^2$ and $1/m_4^2$ effects; they thus represent the lower bound for the corresponding actual nonperturbative corrections.

Fig. 4: The contour of integration over $\lambda^2$ allowing a straightforward perturbative expansion of $\Delta_{\pi^2}^{BLM}(\mu)$.

Fig. 5: Feynman diagrams contributing to the one-loop renormalization of the operator $\bar{Q}(i\tilde{D})^2 Q$. Dashed line is the gauge boson (gluon); dark box represents the operator. In the diagram a) only the Coulomb quanta ($A_0$) propagation contributes whereas in b) the gluon is spacelike.

Fig. 6: The domain of integration over the gluon momentum $k$; the cylinder $-\mu < |\tilde{k}| < \mu$ gives the renormalization of $\mu^2_\pi$ in one loop. The integral over the shaded disk (sphere) vanishes.