Scalar and Tensor Inhomogeneities from Dimensional Decoupling

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Abstract

We discuss some perturbative techniques suitable for the gauge-invariant treatment of the scalar and tensor inhomogeneities of an anisotropic and homogeneous background geometry whose spatial section naturally decomposes into the direct product of two maximally symmetric Euclidean manifolds, describing a general situation of dimensional decoupling in which $d$ external dimensions evolve (in conformal time) with scale factor $a(\eta)$ and $n$ internal dimensions evolve with scale factor $b(\eta)$. We analyze the growing mode problem which typically arises in contracting backgrounds and we focus our attention on the situation where the amplitude of the fluctuations not only depends on the external space-time but also on the internal spatial coordinates. In order to illustrate the possible relevance of this analysis we compute the gravity waves spectrum produced in some highly simplified model of cosmological evolution and we find that the spectral amplitude, whose magnitude can be constrained by the usual bounds applied to the stochastic gravity waves backgrounds, depends on the curvature scale at which the compactification occurs and also on the typical frequency of the internal excitations.

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1 Introduction

The assumption of isotropy and homogeneity of the background manifold permits a consistent theoretical treatment of the space-time evolution of its inhomogeneities which can be classified in scalar, vector and tensor modes with respect to the three dimensional spatial coordinate transformations on the constant time hypersurface [1, 5]. The different modes are decoupled to first order in the amplitude of the fluctuations and this allows the definition of perturbed quantities which are invariant under the gauge group of the infinitesimal coordinates transformations [5, 12, 13]. Within the Bardeen’s gauge-invariant approach, the amplified primordial spectrum of fluctuations can be reliably computed for a wide class of homogeneous and isotropic cosmological models [6, 12] and in particular for the “slow-rolling” scenarios leading to the de-Sitter like inflation [7].

One of the main motivations in order to relax the assumption of isotropy of the background geometry comes from the models of early universe (like superstring theories [8]) describing the unification of gravity with gauge interactions in a higher dimensional manifold [9]. We will then consider a homogeneous and anisotropic manifold which can be written as:

\[ ds^2 = g_{\mu\nu}dx^\mu dx'^\nu = a^2(\eta) d\eta^2 - a^2(\eta) \gamma_{ij}dx^i dx^j - b^2(\eta) \gamma_{ab}dy^a dy^b \]  

(1.1)

(conventions: \(\mu, \nu=1,\ldots, D = d + n + 1; i, j=1,\ldots, d; a, b=d+1,\ldots, n; \eta\) is the conformal time coordinate related, as usual to the cosmic time \( t = \int a(\eta)d\eta \); \( \gamma_{ij}(x), \gamma_{ab}(y) \) are the metric tensors of two maximally symmetric Euclidean manifolds parameterized, respectively, by the “internal” and the “external” coordinates \( \{x^i\} \) and \( \{y^a\} \)). This metric describes the situation in which the external dimensions (evolving with scale factor \( a(\eta) \)) and the internal ones (contracting with scale factor \( b(\eta) \)) are dynamically decoupled from each other. In order to compare the phenomenological consequences of the models formulated with extra dimensions it seems crucial to correctly compute the amplified spectrum of inhomogeneities but, unfortunately, the treatment of the fluctuations in an anisotropic manifold becomes quite cumbersome also because of the natural coupling arising among scalar vector and tensor modes to first order in the amplitude of the fluctuations. The original investigations in the subject [4] stressed that the discussion of the fluctuations in the synchronous gauge is com-
plicated also because of the spurious gauge modes already present [14] in the isotropic case. The problem of the metric fluctuations in an anisotropic background geometry was then addressed within the Bardeen formalism with two different and complementary approaches [15, 16]. It was actually shown that the gauge invariant quantities can be constructed not only, separately, in the external and in the internal manifold [15], but also over the whole manifold of Bianchi-type I [16] (gravitational waves in Bianchi-I universes were also discussed in [17]). Following the first of the two mentioned approaches it is possible to distinguish, from the purely mathematical point of view, the scalar vector and tensor modes in each of the two manifolds. Even though this classification will be technically very useful it does not necessarily coincide with the physical situation (as correctly stressed in [15]), since, for example, the tensor fluctuations polarized along the internal dimensions will be seen by an observer living in the external space as scalar fluctuations. The evolution equations for each type of perturbations were then solved well outside the horizon under the assumption that the Laplacians belonging to the external and internal manifolds were negligible. The anisotropic extension of the scalar Bardeen potentials were shown to grow much faster than the tensor and vector gauge-invariant amplitudes in the vicinity of the collapse of the internal scale factor. The very fast growth of the scalar modes outside the horizon was also discussed in the context of the dilaton-driven solutions in string cosmology [18], where it was found [24] that even though the rate of increase of the scalar fluctuations is much faster than in the usual inflationary models characterized by a quasi de-Sitter spectrum [6] a perturbative treatment is still plausible, at least in the (3+1)-dimensional case with static internal dimensions, by carefully “gauging-down” the scalar growing modes. In spite of these attempts, in an anisotropic background the solution of the evolution equations well outside the horizon does not suffice, by itself, for the calculation of the spectrum of metric perturbations. In order to give a reliable expression for the space-time evolution of the proper amplitude of the fluctuations it can be assumed [21] relying on the particular features of the background evolution, that the only effective dependence of the perturbed quantities from the internal dimensions comes in through the time evolution of the compactification radii. This approximation scheme can be illustrated using the evolution equation of the tensor modes polarized
along the external dimensions (which we will derive in Sec. 3):

\[ h_i^{\mu} + [(d - 1)H + n\mathcal{F}] h_i^\ell - \nabla_\alpha h_i^\alpha - \frac{a^2}{b^2} \nabla_\alpha h_i^\alpha = 0 \]  

(1.2)

\((\ell = \partial/\partial\eta; \ H = (\ln a)', \mathcal{F} = (\ln b)'; \ \nabla_\alpha^e, \text{and} \ \nabla_\alpha^i \text{are, respectively, the external and the internal Laplace-Beltrami operators}). \) In some particular model it can actually happen that \(a \lesssim b,\) at least for scales which went out of the horizon before the compactification was achieved. In this case it is possible to neglect the internal gradients compared to the external ones, and all the dependence from the internal dimensions will be given by \(\mathcal{F}\) which vanishes in the case of static internal scale factors. If, on the contrary \(b \lesssim a,\) the internal Laplacians cannot be neglected especially prior to the dimensional decoupling when the internal and external scale factors were of the same order. Since the tensor modes only couple to the background curvature in order to discuss their evolution we only need to specify the time evolution of the scale factors. The scenario we want to examine consists in general of two phases. A multidimensional phase where the time evolution of the scale factors can be parameterized as

\[ a(\eta) \sim |\eta|^a, \quad b(\eta) \sim |\eta|^b. \]  

(1.3)

This phase can be generally followed by a compactification phase which glues together the multidimensional epoch and the ordinary, isotropic, FRW evolution. There are different compactification scenarios corresponding to the parameterization (1.3). In It is actually possible either to assume that the compactification occurs at some stage after the initial “big-bang” singularity (i.e. for \(\eta > 0\) in (1.3)) as, for instance, in [10, 28]) or at some stage before the “big-bang” singularity (i.e. \(\eta < 0\) in (1.3)) as seems more likely in the pre-big-bang scenarios [18, 19]. Even though our considerations, at this stage, are purely kinematical, it is anyway useful to point out that the two mentioned compactification pictures are dynamically very different since in the pre-big-bang case the end of the higher dimensional phase could coincide, in principle, with the beginning of the ordinary isotropic evolution while in the first picture some other mechanism is required in order to smoothly connect the multidimensional phase (trapped among two singularities) to the FRW universe. An interesting issue is then if in the context of the string inspired models of cosmological evolution the usual problems of the ordinary Kaluza-Klein models (stabilization and isotropization of the internal dimensions,
backreaction effects due to particle production \([42, 43]\)) can be solved (or at least alleviated) by the mechanisms usually proposed \([25]\) in order to regularize the time evolution of the curvature invariants and in order to slow down the dilaton growth. Since at the moment this solution is still unclear not only in more than four dimensions but even in some simplified two-dimensional toy model of cosmological evolution \([27]\) (where the quantum backreaction was taken into account and where the problem of the dilaton seem growth seem still to persist) we will concentrate our attention on the main kinematical features of the dimensional decoupling. Our purpose will only be to stress the regimes of the time evolution of the scale factors where, possibly, the internal Laplacians of eq. (1.2) are leading if compared to the internal ones. More specifically for \(t > 0\) an eventual accelerated expansion in the external space \((\ddot{a} > 0, \dot{a} > 0, i. e. \alpha \leq -1\) in (1.3)) required in order to solve the kinematical difficulties of the Standard Model, together with a simultaneous contracting evolution of the internal dimensions \((\dot{b} < 0, \beta < 0\) in (1.3)), would not forbid in the large \(t\) limit \((|\eta| \to 0\) if \(\alpha \leq -1\) the dominance of the internal Laplacians over the external ones \((\alpha < \beta\) in our parametrization). Similar conclusions can be reached in the limit \(\eta \to 0^-\) if, for \(t < 0\), we consider an accelerated contraction in the external space \((\ddot{a} < 0, \dot{a} < 0, -1 < \alpha < 0\) as suggested by the Einstein frame picture of the string cosmological models \([23, 25]\) in order to solve the flatness and the horizon problems. These qualitative considerations suggest that if a given external Fourier mode \(k\) went out of the horizon \((k\eta \sim 1)\) during an early phase where \(a \gtrsim b\), the contribution of the internal Laplacians have to be seriously considered. At the same time a complete solution of the evolution equations of the scalar and tensor fluctuations depending on the internal and external spatial coordinates was never studied not even in some oversimplified model of background evolution. A reliable computation of the power spectrum for scalar and tensor inhomogeneities in higher dimensional theories is beyond the scope of the present investigation, nonetheless we would like to analyze the evolution and the amplification of the metric fluctuations in some specific toy model with extra dimensions but without assumptions for what concerns the evolution equations of the fluctuations. We would like also to avoid any kind of “slow-rolling” hypothesis in the solution of the background equations which could confuse the analysis of the perturbations. For this reason we shall mainly discuss two classes of exact solutions of the multidimensional Einstein equations
the Kaluza-Klein “vacuum” solutions which can represent a good description in the vicinity of the collapse of the internal dimensions [27] and the multidimensional anisotropic universe filled with scalar field matter. We find quite useful to work from the very beginning with the following scalar-tensor action:

\[ S = S_g + S_m = -\frac{1}{6l_D^2} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \left\{ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi) \right\} \quad (1.4) \]

(where \( l_D = \sqrt{8\pi G_D/3} \); if \( V = 0 \) and \( \varphi = 0 \) the action can describe a vacuum Kaluza-Klein phase and if \( \varphi^2 >> a^2V \) we recover the tree level string theory effective lagrangian [in \( D = 10 \) critical dimensions] for the massless modes of the theory, written in the Einstein frame and in the absence of antisymmetric tensor field).

The plan of the paper is the following. In Sec. 2 we will review the higher dimensional background equations of motion and the particular classes of exact solutions which will be used in the following sections as theoretical laboratory for the analysis of the fluctuations. In Sec. 3 the Bardeen approach for the scalar and tensor perturbations will be discussed. Particular attention, in the case of the scalar fluctuations, will be paid to the possible gauge choices which completely fix the coordinate frame and to the diagonalization of the system of perturbed equations. In Sec. 4 we will focus our study on the evolution of tensor perturbations and we will compute the normalized spectral amplitude for two simplified models of dimensional decoupling. In Sec.5 we will move to the analysis of the scalar inhomogeneities and we will approach the growing mode problem within the formalism discussed in the previous Sections. Sec. 6 contains few concluding remarks.

2 Background models

The variation of the action (1.4) with respect to \( g_{\mu\nu} \) and to \( \varphi \) provide the equations of motion for the background fields:

\[ R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R = 3l_D^2 T^\nu_\mu \quad (2.1) \]

\[ g^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi + \frac{\partial V}{\partial \varphi} = 0 \quad (2.2) \]
If \( T^\nu_\mu = \partial_\mu \varphi \partial^\nu \varphi - g_{\mu\nu}(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi)) \). If we restrict our attention to the case in which the scalar field is homogeneous (\( \varphi = \varphi(\eta) \)) the evolution of the geometry is completely determined by the time evolution of the two scale factors \( a(\eta) \) and \( b(\eta) \).

Using the line element (1.1), eqs. (2.1)-(2.2) become:

\[
\frac{d(d-1)}{2} \mathcal{H}^2 + n(n-1) \frac{a^2}{b^2} \mathcal{K}_b = 3l_D^2 \left( \frac{\varphi'^2}{2} + a^2 V \right), \quad (00)
\]

\[
(d-1) \mathcal{H}' + \frac{(d-1)(d-2)}{2} \mathcal{H}^2 + n \mathcal{F}' + \frac{n(n+1)}{2} \mathcal{F}^2 + n(d-2) \mathcal{H} \mathcal{F}' + \frac{(d-1)(d-2)}{2} \mathcal{K}_a + \frac{n(n-1)}{b^2} a^2 \mathcal{K}_b = 3l_D^2 (a^2 V - \frac{\varphi'^2}{2}), \quad (ii)
\]

\[
(n-1) \mathcal{F}' + d \mathcal{H}' + \frac{d(d-1)}{2} \mathcal{H}^2 + \frac{n(n-1)}{2} \mathcal{F}^2 + (d-1)(n-1) \mathcal{H} \mathcal{F}' + \frac{(n-2)(n-1)}{2} \mathcal{K}_a + \frac{(n-2)(n-1)}{b^2} a^2 \mathcal{K}_b = 3l_D^2 (a^2 V - \frac{\varphi'^2}{2}) \quad (aa)
\]

\[
\varphi'' + [(d-1) \mathcal{H} + n \mathcal{F}] \varphi' + \frac{\partial V}{\partial \varphi} = 0 \quad (\varphi) \quad (2.3)
\]

(\( \mathcal{K}_a \) and \( \mathcal{K}_b \) are respectively the curvatures constants of the external and internal maximally symmetric spaces). In this paper we will generally work in the case of spatially flat internal and external manifold with topology \( M_{d+1} \otimes T_n \) (where \( d+1 \) is the conventional \( (d+1) \)-dimensional flat universe and \( T_n \) is an \( n \)-dimensional torus). Summing and subtracting the previous equations we get two useful relations:

\[
\mathcal{F}' = - \mathcal{F} [n \mathcal{F} + (d-1) \mathcal{H}] + \frac{6l_D^2 a^2 V}{(n+d-1)} \\
\mathcal{H}' = - \mathcal{H} [n \mathcal{F} + (d-1) \mathcal{H}] + \frac{6l_D^2 a^2 V}{(n+d-1)}. \quad (2.4)
\]

If \( V = 0 \) and \( \varphi = 0 \) the solutions of the system (2.3) define the Kaluza-Klein vacuum [27]. A class of exact vacuum solutions can be obtained using the power law ansatz of eq. (1.3) in the equations of motion (2.3)

\[
\alpha = \frac{d \pm \sqrt{d^2 + d(n+d)(n-1)}}{d(n+d-1) \mp \sqrt{d^2 + d(n+d)(n-1)}} \\
\beta = \frac{nd \mp d\sqrt{d^2 + d(n+d)(n-1)}}{nd(n+d-1) \pm n\sqrt{d^2 + d(n+d)(n-1)}} \quad (2.5)
\]
(the exponents of the scale factors in cosmic time ($\tilde{a} \equiv \alpha / (\alpha + 1)$, $\tilde{\beta} \equiv \beta / (\alpha + 1)$) are related by the Kasner sum rules $d\tilde{a} + n\tilde{\beta} = 1$ and $d\tilde{a}^2 + n\tilde{\beta}^2 = 1$). The twofold ambiguity in the sign of the exponents shows that there are two independent solutions for each number of internal and external dimensions. A particularly simple case which will be used in our analysis is the solution with $n = 1$ and $d = 3$. In this case the two solutions are $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = -1$. Since we want to analyze mainly the contribution of the internal dimensions to the evolution of the fluctuations we choose the solution i which the external dimensions are static ($\alpha = 0$, $\beta = 1$) so that all the contribution to the amplification of the fluctuations will come, effectively, from the internal space. The contracting branch ($\beta = +1$, i.e. $\dot{b} = 0$ and $\dot{\beta} < 0$ since $t \sim \eta$) can be matched with a radiation phase

$$
a(\eta) = 1, \quad b(\eta) = -\frac{\eta}{\eta_c}, \quad \eta \leq -\eta_c
\leqno{2.6}
$$

$$
a(\eta) = \left(-\frac{\eta + 2\eta_c}{\eta_c}\right), \quad b(\eta) = 1, \quad \eta \geq -\eta_c.
$$

This toy model is not realistic and somehow artificial since the radiation is not dynamically generated but only assumed. In a more refined treatment the back-reaction effects should be correctly taken into account [42, 45] since the scalar and tensor inhomogeneities amplified during the classical evolution can eventually modify the background dynamics leading to an effective damping of the anisotropy of the background metric (as usually happens in the (3+1)-dimensional anisotropic models of Bianchi type-I [44]). Nonetheless (2.6) shares some essential features of a realistic scenario of dimensional reduction in which the (3+1) external dimensions decouple from the fifth one down to a compactification scale $H_c \sim 1/\eta_c$ and our purpose will be to connect the amplitude of the scalar and tensor fluctuations not only with the curvature scale but also with the typical frequency of the internal oscillations (which can be also constrained, with different arguments, from the present value of the fine structure constant[10]).

In the case of negligible potential, using the previous power-law ansatz for the scale factors from eq. (1.2) and a logarithmic ansatz for the scalar field

$$
\varphi \sim \frac{\gamma}{\sqrt{3l_D^2}} \ln |\eta|
\leqno{2.7}
$$

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another class of solutions of eq. (2.3) is given by:

\[
\begin{align*}
\alpha &= -\frac{\mp (1 - d - n) \pm n \mp (d - 1) - \sqrt{d + n - 1}}{(\sqrt{d + n + 1})(d + n - 1)} \\
\beta &= -\frac{\pm (1 - d - n) \pm n \mp (d - 1) - \sqrt{d + n - 1}}{(\sqrt{d + n + 1})(d + n - 1)} \\
\gamma &= -\frac{\sqrt{\frac{2}{d + n - 1}}}{} 
\end{align*}
\] (2.8)

By choosing everywhere the upper sign and in the case of critical dimensions \((D = 10\) with \(d = 3\) and \(n = 6\)) the solution (2.8) is a particular example of the general exact dilaton-driven solutions originally derived [24] in the String frame usually related to the Einstein frame by a conformal rescaling of the metric tensor \((g_{\mu\nu}^{\text{String}} = g_{\mu\nu}^{\text{Einstein}} \exp [(2\varphi - 2\varphi_1)/(d + n - 1)]); the dilaton is only redefined according to \(\varphi^{\text{Einstein}} = \sqrt{2/(d + n - 1)}\varphi^{\text{String}} [6l_D^2 = 1\ so\ that\ \varphi\ is\ dimension-less]).\ We\ will\ discuss,\ as\ a\ particular\ toy\ model,\ a\ 10\text{-dimensional}\ dilaton-driven\ solution\ continuously\ matched\ with\ the\ radiation\ phase:\

\[
\begin{align*}
a(\eta) &= (-\frac{\eta}{\eta_c})^{-1/4}, \quad b(\eta) = (-\frac{\eta}{\eta_c})^{1/4}, \quad \varphi(\eta) = \frac{\sqrt{3}}{8l_D^2} \ln (-\frac{\eta}{\eta_c}), \quad \eta \leq -\eta_c \\
a(\eta) &= (-\frac{\eta + 2\eta_c}{\eta_c}), \quad b(\eta) = 1, \quad \varphi = \text{const.}\ , \quad \eta \geq -\eta_c 
\end{align*}
\] (2.9)

In this model for \(\eta < -\eta_c\) we have an accelerated contraction of the external dimensions supplemented by the a decelerated contraction of the internal ones. We also notice that for \(\eta < -\eta_c\) the scale factor are related by a duality relation \((a = 1/b)\) which more generally holds in the String frame \((a = 1/b)\) provided we choose the upper sign in (2.8). We point out that this model is not realistic for the same reasons mentioned in the case of (2.6) and also because it was shown that in order to have a graceful exit from the dilaton-driven epoch it is crucial to include in the picture a stringy phase during which the background dynamics is driven by the higher order in the string tension expansion [25]. In both the examples (2.6) and (2.9) the internal scale factors are static during the radiation dominated era, while a time dependence could be, in principle, also included in the internal scale factors during the radiation and matter dominated epochs. The time dependence in the internal scale factors would be anyway strongly constrained by nucleosynthesis which would require [31, 32] during the radiation dominated epoch \(b_{ns}/b_0 < 1 + \epsilon\ (b_0\ is\ the\ actual\ value\ of\ b\ and\ \epsilon < 10^{-2})\),
while in the matter dominated epoch the constraint would be instead \( \dot{b}/b \equiv \mathcal{F}/a < 10^{-9}H_0 \) \((H_0 = 1.1 \times 10^{-28}h_{100} \text{ cm}^{-1})\). Moreover the time variation of \( G_D \) is also constrained [33] since \( G_D(\text{nucl})/G_D(\eta_0) = 1 + \epsilon \ (|\epsilon| < 3 \times 10^{-1}) \) in the radiation dominated epoch, and \( |\dot{G}_D/G_D| < 10^{-1}H_0 \) during the matter dominated epoch. In our naive models \( b = 1 \) (for \( \eta > -\eta_c \)) the previous constraints are automatically satisfied.

3 Scalar and tensor fluctuations

The scalar and tensor fluctuations of the geometry (1.1) can be discussed within a generalization [15] of the gauge-invariant formalism [5, 12]. The infinitesimal coordinate transformations preserving the scalar nature of the fluctuations with respect to each maximally symmetric space are:

\[
\begin{align*}
  x^i &\rightarrow \bar{x}^i = x^i + \epsilon^i \\
y^a &\rightarrow \bar{y}^a = y^a + \zeta^a \\
\eta &\rightarrow \bar{\eta} = \eta + \epsilon^0 
\end{align*}
\]

(3.1)

(where \( \epsilon^0 = \epsilon(\vec{x}, \vec{y}, \eta), \zeta^a \equiv \partial^a\zeta(\vec{x}, \vec{y}, \eta), \epsilon^i \equiv \partial^i\epsilon(\vec{x}, \vec{y}, \eta) \)). The perturbed scalar metric can be written in terms of 8 linearly independent scalar quantities

\[
\delta g_{\mu\nu}^{(s)} = \begin{pmatrix}
  2a^2 \phi & -a^2 B_i \\
  -a^2 B_i & 2a^2 \psi \delta_{ij} - 2a^2 E_{ij} \\
  -a^2 C_a & -ab D_{ia} \\
  -ab C_a & 2b^2 \xi \delta_{ab} - 2b^2 G_{ab}
\end{pmatrix} \quad (3.2)
\]

(conventions :\( B_i = B_{[i}, \ E_{ij} = E_{[ij]}, \ C_a = C_{[a}, \ G_{ab} = G_{[ab}, \ D_{ia} = D_{[ia}; \ ) the bar denote the covariant derivative with respect to one of the two internal spatial metrics depending on the index, and it coincides with the ordinary partial derivative if \( K_a = K_b = 0 \)). The fluctuations in the scalar field will be

\[
\varphi(\eta, \vec{x}, \vec{y}) \rightarrow \varphi(\eta) + \chi(\eta, \vec{x}, \vec{y}) \quad .
\]

(3.3)

Under an infinitesimal coordinate transformation (3.1) the perturbed scalar quantities change as follows:

\[
\dot{\phi} \rightarrow \ddot{\phi} = \phi - \mathcal{H}\epsilon^0 - \epsilon^0' 
\]

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\[ \psi \rightarrow \tilde{\psi} = \psi + \mathcal{H} \epsilon^0 \]
\[ \xi \rightarrow \tilde{\xi} = \xi + \mathcal{F} \epsilon^0 \]
\[ E \rightarrow \tilde{E} = E - \epsilon \]
\[ B \rightarrow \tilde{B} = B + \epsilon^0 - \epsilon' \]
\[ C \rightarrow \tilde{C} = C + \frac{a}{b} \epsilon^0 - \frac{b}{a} \zeta' \]
\[ D \rightarrow \tilde{D} = D - \frac{b}{a} \zeta - \frac{a}{b} \epsilon \]
\[ G \rightarrow \tilde{G} = G - \zeta \]
\[ \chi \rightarrow \tilde{\chi} = \chi - \varphi' \epsilon^0 \quad . \]

(3.4)

A possible set of linearly independent gauge-invariant quantities is then:

\[ \Phi = \phi + \frac{1}{a} [(B - E') a]' \]
\[ \Psi = \psi + \mathcal{H}(B - E') \]
\[ \Xi = \xi + \mathcal{F}(B - E') \]
\[ \Omega = \frac{a}{b} \mathcal{F} C - G' \mathcal{F} + \frac{a^2}{b^2} \xi \]
\[ \Theta = D - \frac{b}{a} G - \frac{a}{b} E \]
\[ X = \chi + \varphi'(B - E') \quad . \]

(3.5)

Notice that \( \Psi \) and \( \Phi \) coincides, up to a sign, with the Bardeen’s potentials [4] while \( \Xi, \Omega \) and \( \Theta \) appear only in the anisotropic case. In the homogeneous and isotropic case it is always possible to choose a particular coordinate system by completely fixing, to first order, the arbitrary scalar functions appearing in the transformations (3.1). If the scalar functions are completely fixed (like in the case of the conformally newtonian gauge [5, 12]) the equations of motion of the fluctuations will be second order differential equations, if, on the contrary, the infinitesimal scalar functions are not completely fixed (like in the case of the synchronous gauge[1, 5, 14]) the evolution equations will be of course linear but of higher order. Since we want to make our problem more tractable we completely fix the coordinate system

\[ \bar{D} = 0, \quad \bar{B} = 0, \quad \bar{E} = 0, \quad (3.6) \]
in eq. (3.4) and, as a consequence, $e^0$, $\epsilon$ and $\zeta$ are determined from the same equation. In this gauge the longitudinal fluctuations ($\phi$, $\xi$, $\psi$) coincide with the corresponding gauge-invariant quantities defined in (3.5) and in this sense it represents a generalization to the anisotropic case of the conformally newtonian gauge. By perturbing to first order the Einstein equation (2.1) and the scalar field equation (2.2) we obtain:

$$\delta R^\mu_\mu = -\frac{1}{2} \delta^\nu_\nu (g^{\alpha\beta} \delta R_{\alpha\beta} + \delta g^{\alpha\beta} R_{\alpha\beta}) = 3l_D^2 \delta \Gamma^\nu_\mu$$  

(3.7)

$$\delta g^{\alpha\beta} (\partial_\alpha \partial_\beta \varphi - \Gamma^\gamma_{\alpha\beta} \partial_\gamma \varphi) + g^{\alpha\beta} (\partial_\alpha \partial_\beta \chi + \delta \Gamma^\gamma_{\alpha\beta} \partial_\gamma \varphi + \Gamma^\gamma_{\alpha\beta} \partial_\gamma \chi) - \frac{\partial^2 V}{\partial \varphi^2} \chi = 0$$  

(3.8)

($\delta \Gamma^\gamma_{\alpha\beta}$ and $\delta R_{\mu\nu}$ are the perturbed affine connections and the perturbed Ricci tensors, the indices are raised using always the background metric $g_{\mu\nu}$). By using the background field equations (2.1)-(2.2) we can write down explicitly the evolution equations of the fluctuations given the perturbed form of the metric (3.6) in the gauge (3.4). The $(i \neq j)$ component of eq. (3.7) implies

$$\phi = (d - 2)\psi + n\xi - \nabla^2_G$$  

(3.9)

which allows to eliminate $\phi$ from all the evolution equations of the perturbations. From the $(0i)$, $(0a)$ and $(aj)$ components of (3.7) we get, respectively:

$$(d - 1)\psi' + (d - 2)\psi[(d - 1) \mathcal{H} + \mathcal{F}] + n\xi' + n\xi[(n + 1) \mathcal{F} + (d - 2) \mathcal{H}]$$

$$-(\nabla^2_G \psi)' - [(d - 2) \mathcal{H} + (n + 1) \mathcal{F}] \nabla^2_G \psi + \frac{a}{2b} \nabla^2_C = 3l_D^2 \varphi' \chi$$

$$(0i)$$

$$\frac{-b}{2a} \nabla^2_C + (n - 1) \xi' + d\psi' + [d(d - 1) \mathcal{H} + (d - 2)(n - 1) \mathcal{F}] \psi$$

$$(n\xi - \nabla^2_G)(d\mathcal{H} + (n - 1) \mathcal{F}) = 3l_D^2 \varphi' \chi$$

$$(0a)$$

$$\frac{-b}{2a} [(d - 2) \mathcal{H} + (n + 1) \mathcal{F}] C - \frac{b}{2a} C' + \psi - \xi + \nabla^2_G = 0$$

$$(aj)$$

(3.10)

(3.11)

(3.12)

which are not equations of motion but only constraints connecting the fluctuations to their first time derivative. The repeated use of eq. (3.9) together with the equations of motion
of the background fields allows finally to write down the \((00), (i = j), (a = b), (a \neq b)\) components of the evolution equations

\[
- \frac{1}{2}(\nabla^2 G)'' + \frac{1}{2}(\nabla^2 G)'[(n - 2)\mathcal{F} + (d + 1)\mathcal{H}] - \frac{1}{2} \nabla_x^2 \nabla_y^2 G - \frac{1}{2} \frac{a^2}{b^2} \nabla_x^2 \nabla_y^2 G \\
+ n\nabla^2 \xi - \xi'[nd\mathcal{H} + n(n - 1)\mathcal{F}] + (d - 1)\nabla^2 \psi - \psi'[d(d - 1)\mathcal{H} + nd\mathcal{F}] \\
+ \frac{a}{2b}(\nabla_y^2 C)' - \frac{a}{2b}[d\mathcal{H} + (n - 1)\mathcal{F}]\nabla_y^2 C \\
= 3l_D^2[\varphi'\chi' + 2a^2V((d - 2)\psi + n\xi - \nabla_y^2 G + a^2V'\chi)] \quad (00),
\]

(3.13)

\[
- \frac{1}{2}(\nabla^2 G)'' - \frac{1}{2}(\nabla^2 G)'[(3d - 5)\mathcal{H} + (3n + 2)\mathcal{F}] - \frac{1}{2} \nabla_x^2 \nabla_y^2 G - \frac{1}{2} \frac{a^2}{b^2} \nabla_x^2 \nabla_y^2 G \\
+ n\xi'' + \xi'[2n(d - 1)\mathcal{H} + n(2n + 1)\mathcal{F} - n\mathcal{H}] \\
+ (d - 1)\psi'' + \psi'[(d - 1)(2d - 3)\mathcal{H} + n(2d - 3)\mathcal{F}] \\
+ \frac{a}{2b}(\nabla_y^2 C)' - \frac{a}{2b}(\nabla_y^2 C)\nabla_y^2 C \\
= 3l_D^2[\varphi'\chi' - 2a^2V'\chi - 2a^2V((d - 2)\psi + n\xi - \nabla_y^2 G)] \quad (i = j),
\]

(3.14)

\[
- \frac{1}{2}(\nabla^2 G)'[(3d - 1)\mathcal{H} + (3n - 2)\mathcal{F}] - \frac{1}{2} \nabla_x^2 \nabla_y^2 G - \frac{1}{2} \frac{a^2}{b^2} \nabla_x^2 \nabla_y^2 G \\
- \nabla^2 \psi + d\psi'' + \psi'[2d(d - 1)\mathcal{H} + 2(d - 1)(n - 1)\mathcal{F}] \\
+ \nabla^2 \xi + (n - 1)\xi'' + \xi'[(d(2n - 1) - (n - 1))\mathcal{H} + 2n(n - 1)\mathcal{F}] \\
+ \frac{a}{2b}(\nabla_y^2 C)'[d\mathcal{H} + (n - 1)\mathcal{F}] + \frac{a}{2b}(\nabla_y^2 C)' \\
= 3l_D^2[\varphi'\chi' - a^2V'\chi - 2a^2V((d - 2)\psi + n\xi - \nabla_y^2 G)] \quad (a = b),
\]

(3.15)

\[
G'' + [(d - 1)\mathcal{H} + n\mathcal{F}]G' - \nabla_x^2 G - \frac{a^2}{b^2} \nabla_y^2 G \\
- \frac{a}{b} C' - \frac{a}{b} \frac{1}{d\mathcal{H} + (n - 1)\mathcal{F}} C + 2(\psi - \xi) \frac{a^2}{b^2} = 0 \quad (a \neq b),
\]

(3.16)

The linear system of differential equations with time dependent coefficients formed by the three constraints (3.10), (3.11) and (3.12) and by the equations (3.13), (3.14), (3.15), (3.16) determines the classical space-time evolution of the five fluctuations \(\psi, \xi, G, C\) and \(\chi\). In order to simplify the system we also write the perturbed equation of motion for the scalar field which can be obtained from the combination of the other equations:

\[
- \nabla^2 \chi + \chi'' + [(d - 1)\mathcal{H} + n\mathcal{F}]\chi' - 2\varphi'[(d - 1)\psi' + n\xi'] \\
- V''a^2\chi + 2a^2V((d - 2)\psi + n\xi - \nabla_y^2 G) = 0 \quad (\chi).
\]

(3.17)
Subtracting eq. (3.14) from eq. (3.13) and eq. (3.15) from (3.13) we get respectively:
\[
\Box \lambda + 3[(d-1)\mathcal{H} + n\mathcal{F}]\lambda' - (2(\nabla^2 G) - \frac{a}{b}(\nabla^2 C)) [\mathcal{H} + \frac{n}{d-1}\mathcal{F}] + \frac{6d^2}{d-1}(a^2 V' \chi + 2a^2 V((d-2)\psi + n\xi - \nabla^2 G),
\]
(3.18)
where \(\lambda = \psi + \frac{n}{d-1}\xi\), \(\Box = (\partial/\partial\eta)^2 - \nabla^2 x - \frac{a^2}{b^2} \nabla^2 y\) and
\[
\begin{align*}
&= \Box \xi + \xi' \left[ \frac{\mathcal{H}}{n-1} (3dn - d - n + 1) + 3n\mathcal{F} \right] \\
&+ d(\Box \psi + \psi' \left[ 3(d-1)\mathcal{H} + \frac{\mathcal{F}}{d} (2(d-1)(n-1) + nd) \mathcal{F} \right] \\
&- (2(\nabla^2 G) - \frac{a}{b}(\nabla^2 C)) [d\mathcal{H} + (n-1)\mathcal{F}] \\
&- \frac{6d^2}{d-1} [a^2 V' \chi + 2a^2 V((d-2)\psi + n\xi - \nabla^2 G)] = 0.
\end{align*}
\]
(3.19)
Combining now the constraints (3.12) with eq. (3.16) we get an useful expression which allows to eliminate \(C\) from the other equations
\[
C = \frac{1}{2(\mathcal{H} - \mathcal{F})} [G'' + G'((d-1)\mathcal{H} + n\mathcal{F}) - \nabla^2 G - \frac{a^2}{b^2} \nabla^2 G].
\]
(3.20)
Using (3.20) in (3.18) and (3.19) together with the background equations in the form (2.4) it is possible to show, by linearly combining the obtained relations, that the evolution equations for the longitudinal fluctuations can be written in the gauge (3.6) as
\[
v'' - z_1 \frac{v''}{z_1} - \nabla^2_x v - \frac{a^2}{b^2} \nabla^2_y v = 0,
\]
(3.21)
\[
w'' - \frac{w''}{z_2} - \nabla^2_x w - \frac{a^2}{b^2} \nabla^2_y w = 0,
\]
(3.22)
where
\[
v = z_2 \chi + z_1 \lambda, \quad w = \frac{z_1}{l_D} \left[ \frac{n(n + d - 1)}{6(d-1)} \left( \frac{\mathcal{H}}{\varphi} \xi - \frac{\mathcal{F}}{\varphi} \psi \right) \right]
\]
(3.23)
and
\[
z_1 = \frac{a^{d-1} b^2}{\mathcal{H} + \frac{n}{d-1} \mathcal{F}}, \quad z_2 = a^{d-1} b^2 \equiv \left[ \frac{g_1}{a^2} \right]^{1\over 2}.
\]
(3.24)
Since \(\phi, \psi\) and \(\xi\) coincide, in the gauge (3.4), with the corresponding gauge-invariant quantities listed in eq. (3.5) we can write that
\[
\mathcal{V} = z_2 \chi + z_1 \Lambda, \quad \Lambda = \Psi + \frac{n}{d-1} \Xi, \quad \mathcal{W} = \frac{z_1}{l_D} \left[ \frac{n(n + d - 1)}{6(d-1)} \left( \frac{\mathcal{H}}{\varphi} \Xi - \frac{\mathcal{F}}{\varphi} \Psi \right) \right].
\]
(3.25)
Notice that in the absence of internal dimensions $W$ is zero and $V$ coincides with the scalar normal mode of oscillation which diagonalizes the the action (1.4) perturbed to second order in the amplitude of the fluctuations [12]. By solving the equations for $V$ and $W$ and by using their definition in terms of the gauge-invariant fluctuations it is possible to obtain the time evolution of all the quantities listed in (3.5) (the explicit solution for the longitudinal fluctuations will be discussed in Sec. 5).

The evolution equations for the gauge-invariant tensor modes (only coupled to the scalar curvature and not to the sources of the background) can be obtained without any specific gauge choice. The form of the perturbed metric will be in this case:

$$\delta g_{\mu\nu}^{(T)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a^2 h_{ij} & 0 \\ 0 & 0 & -b^2 H_{ab} \end{pmatrix}$$

with $\nabla_i h_{ij} = 0$ and $\nabla_a H^{ab} = H^a_a = 0$ (since $H^b_a$ and $h^i_i$ are pure tensor modes in each space they are also automatically gauge-invariant with respect to gauge transformations preserving the tensor nature of the fluctuations in the external and internal manifold). The evolution equations can be easily written by perturbing the Ricci tensor. All the components of the Einstein equations (3.7) are zero but $(i, j)$ and $(a, b)$:

$$h'' + [(d - 1) \mathcal{H} + n \mathcal{F}] h' - \nabla^2 h - \frac{a^2}{b^2} \nabla^2 h = 0,$$

$$H'' + [(d - 1) \mathcal{H} + n \mathcal{F}] H' - \nabla^2 H - \frac{a^2}{b^2} \nabla^2 H = 0,$$  

where $h^i_i \equiv h(\eta, \varphi, \bar{\eta}) e^i_j$ and $H^a_a \equiv H(\eta, \varphi, \bar{\eta}) E^b_a$ ($e^i_j$ and $E^b_a$ are, respectively the external and internal polarization tensors). As we already reminded the gravity waves polarized along the internal dimensions will not be able to excite a detector of tensor waves but will be seen in the $(d + 1)$-dimensional “external” world as density fluctuations and this means, from the mathematical point of view, that $h^i_i$ are scalar eigenstates of the internal Laplace operator ($\nabla^2_\varphi$) in the same way as $H^a_a$ are scalar eigenstates of the external Laplace operator ($\nabla^2_\varphi$) (of course $h^i_i$ and $H^a_a$ are also solution of the tensor Helmotz equation, respectively, in the external and internal manifold). By defining $\mu = (1/24l_D)a^{\frac{d-1}{2}} b^{\frac{d}{2}} h$ and $\mathcal{M} = (1/24l_D)a^{\frac{d-1}{2}} b^{\frac{d}{2}} H$, equations (3.27) can be easily rewritten as:

$$\mu'' - \nabla^2_\varphi \mu - \nabla^2_\varphi \mu - \frac{\mu}{\varphi^2} \mu = 0,$$

$$\mathcal{M}'' - \nabla^2_\varphi \mathcal{M} - \nabla^2_\varphi \mathcal{M} - \frac{\mathcal{M}}{\varphi^2} \mathcal{M} = 0$$

(3.28)
We notice that in the absence of the internal dimensions $\mu$ coincide with the amplitude of the tensor normal modes of oscillation of the action (1.4) [12]. The solution of the coupled systems of differential equations which describe the evolution of the scalar and tensor inhomogeneities was previously investigated [15, 24] well outside the horizon (where the internal and external Laplace operators are subleading) in the case of the Kaluza-Klein “vacuum” solutions (2.5) and in the case of the dilaton-driven solutions (2.8). Provided we neglect the internal and external Laplacians it can be actually shown quite easily that the solution to eq. (3.28) is:

$$
\mu(\eta) = a^\frac{d-1}{2} b^{-\frac{d}{2}} (A_1 + B_1 \int \frac{ad\eta}{a^d b^d}), \quad \mathcal{M}(\eta) = a^\frac{d-1}{2} b^{-\frac{d}{2}} (A_2 + B_2 \int \frac{ad\eta}{a^d b^d}), \quad \text{(3.29)}
$$

where $A_i, B_i$ are the integration constants. Since from (2.5) and (2.8) $a^\frac{d-1}{2} b^{-\frac{d}{2}} \equiv \sqrt{\eta}$ for both the backgrounds in an arbitrary number of dimensions, we will have (from eq. (3.29)) that outside the horizon the gravity wave amplitude diverges at most logarithmically:

$$
\mu \simeq \sqrt{\eta} (A_1 + B_1 \ln |\eta|), \quad \mathcal{M} \simeq \sqrt{\eta} (A_2 + B_2 \ln |\eta|). \quad \text{(3.30)}
$$

In the case of the scalar inhomogeneities it can be shown instead (for example from eq. (3.18) neglecting the Laplacians and for $V = 0$) that the longitudinal fluctuations diverge like a power outside the horizon for both the background solutions (2.5)-(2.8) and typically we will have:

$$
\Lambda \simeq c_1 + \frac{c_2}{\eta^2} \quad \text{(3.31)}
$$

(a similar behaviour can be deduced also for $\Psi$ and $\Xi$). If growing solutions are present it is in general not possible to keep the amplitude of the fluctuations small all the time and at some point the perturbative approach will unavoidably break down leading to the so called “growing-mode problem” (which will be addressed in Sec. 5). In order to reliably compute the spectrum of the scalar and tensor fluctuations it seems important to consider explicitly the contribution of the internal and external Laplacians and since this could be quite difficult for a generic multidimensional background we will limit our attentions to the models described in eq. (2.6) and (2.9) which will be analyzed in the following two Sections.
4 Graviton production from extra spatial dimensions

The spectral energy density of the cosmic gravitons produced thanks to the time evolution of an homogeneous and isotropic cosmological model is an important source of dynamical informations [2, 3] since the slopes of the spectra versus the frequency offer a snapshot of the early history of the hubble parameter [34, 35]. Moreover the graviton spectra can indirectly constrain the homogeneous and isotropic inflationary models [36, 20]. If we relax the assumption of isotropy of the background manifold it is unclear how to perform the calculation of the spectral amplitudes which will have to be eventually compared to the phenomenological bounds (see [20] and [37] for two reviews concerning the stochastic gravity-waves backgrounds and their detectability). Our purpose is to calculate the graviton spectra produced in the two oversimplified models of dimensional decoupling presented in Sec. 2 in order to get the feeling of what could happen in a realistic situations. We will see that thanks to the coupling among the scalar and the tensor modes the gravity wave evolution equation will get a “massive” contribution which might also be relevant in the case of more motivated background geometries. In terms of the eigenstates of the Laplace operators

\[ \nabla^2 h_i^j(k, q) = -k^2 h_i^j(k, q) \]
\[ \nabla^2 h_i^j(q, k) = -q^2 h_i^j(k, q), \]  

eq. (3.28) will be

\[ \mu'' + \left[k^2 + q^2 \frac{a^2}{q^2} - \frac{z_2''}{z_2}\right] \mu = 0. \]  

We consider first of all the model (2.6). For \( \eta \leq -\eta_c \) eq. (4.2) becomes:

\[ \mu'' + \left[k^2 + \frac{1}{4\eta^2} + \frac{q^2 \eta_c^2}{\eta^2}\right] \mu = 0, \]  

whereas for \( \eta \geq -\eta_c \) the same equation will be

\[ \mu'' + \left[k^2 + \frac{q^2 \eta_c^2}{\eta_c^2}(\eta + 2\eta_c)^2\right] \mu = 0. \]  

For \( \eta < -\eta_c \) an exact solution of eq. (4.3) can be written in term of the Hankel functions:

\[ \mu(k\eta, q) = \frac{1}{\sqrt{k}} \sqrt{k\eta} H_{\nu}^{(2)}(k\eta), \quad \nu = i(q\eta_c) \]  

16
(we have chosen the positive frequency mode which corresponds [for $\eta \to -\infty$] to the Bunch-Davies vacuum). In the absence of the internal Laplacians (or, equivalently, if we would keep only the lowest mode of the internal excitations, $q = 0$) instead of (4.5) we would get a completely different solution:

$$\mu(k\eta) = \frac{1}{\sqrt{k}} \sqrt{\kappa \eta H_0^{(2)}(k\eta)}$$  \hspace{1cm} (4.6)$$

whose limit for small arguments holding when the given mode is well outside the horizon ($k\eta << 1$) gives:

$$\mu(k\eta) = \sqrt{\eta}(1 - \frac{2i}{\pi} \ln k\eta)$$  \hspace{1cm} (4.7)$$

which is clearly consistent with the evolution of the gravity waves outside the horizon in an arbitrary number of dimensions (and for generic initial conditions) obtained in (3.30). For $\eta > -\eta_c$ the solution of eq. (4.4) can be written in terms of parabolic cylinder functions (defining a new variable $z = \sqrt{2q/\eta_c}(\eta + 2\eta_c)$ eq. (4.3) becomes exactly the parabolic cylinder equation expressed in its standard form [52, 53]):

$$\mu(k\eta, q) = \left(\frac{2q}{\eta_c}\right)^{1/4}(c_- E(a, z) + c_+ E^*(a, z))$$  \hspace{1cm} (4.8)$$

$(a = -k^2\eta_c/2q$ and $E(a, z), E^*(a, z)$ are complex conjugated solutions of the parabolic cylinder equation [52, 53]). If $a > z^2/4$ namely if $k^2 > q^2((\eta + 2\eta_c)/\eta_c)^2$ we have that the solution (4.8) becomes:

$$\mu(k\eta, q) \to \frac{1}{\sqrt{k}}(c_- e^{-ik(\eta + 2\eta_c)} + c_+ e^{ik(\eta + 2\eta_c)}).$$  \hspace{1cm} (4.9)$$

In the opposite limit ($k^2 < q^2((\eta + 2\eta_c)/\eta_c)^2$) solution (4.8) becomes instead

$$\mu(k\eta, q) \to \sqrt{\frac{\eta_c}{q(\eta + 2\eta_c)}}(c_- e^{i(q(\eta + 2\eta_c))/\eta_c^2} + c_+ e^{i(q(\eta + 2\eta_c))/\eta_c^2})$$  \hspace{1cm} (4.10)$$

(as can be directly obtained by solving eq. (4.3) for a negligible $k^2$). The last solution is identical to the evolution equation of a minimally coupled scalar fields in the radiation dominated era with mass $m \sim q$ so that the effect of the internal Laplacians on the evolution of an externally polarized gravity wave evolving during the radiation dominated era can be described with an effective mass term whose magnitude depends on the magnitude of the
excitations belonging to the internal space. In the Schrödinger-like equation (4.2)-(4.3) the mass term modifies the potential barrier whose maximum is now \(1/4\eta_c^2 + q^2\). This effective potential barrier leads to wave amplification [2, 29], or, equivalently, to particle production [30]. Actually the positive frequency modes (for \(\eta \to -\infty\)) in eq. (4.5) will be in general a linear combination of modes which are of positive or negative frequency with respect to the vacuum to the right (\(\eta \to +\infty\)). The coefficients of the Bogoliubov transformation (\(\epsilon_\pm, |\epsilon_+|^2 - |\epsilon_-|^2 = 1\)) connecting the left and right vacuum and fixed by matching, in \(\eta = -\eta_c\), each solution and its first derivative will determine the spectral density of the produced gravitons. We can now compute the amplification coefficient \(c_-\) in the sudden approximation [42], namely for \((k\eta_c)^2 < 1 + (q\eta_c)^2\) (for \((k\eta_c)^2 > 1 + (q\eta_c)^2\) there is no wave amplification and the Bogoliubov coefficient \(c_-\) is exponentially suppressed). We will have in general a two branches amplification coefficient depending if the mode \(k\) is “non-relativistic” \((k^2 < q^2(\frac{\eta_{+2\eta_c}}{\eta_c})^2)\) or “ultrarelativistic” \((k^2 > q^2(\frac{\eta_{+2\eta_c}}{\eta_c})^2)\). So matching the solutions (4.5) and (4.9) in \(\eta = -\eta_c\) we obtain:

\[
e^{ik\eta_c c_-} \simeq \frac{1}{2\sqrt{2}}[1 - \nu] \frac{\Gamma(\nu)}{\pi} \left(\frac{k\eta_c}{2}\right)^{-\frac{1}{2}} - i \frac{\Gamma(\nu)}{\pi} \left(\frac{k\eta_c}{2}\right)^{-\frac{1}{2}} (\nu + 1 \nu \Gamma(\nu)),
\]

(4.11) for \(k^2 > q^2(\frac{\eta_{+2\eta_c}}{\eta_c})^2\), and

\[
e^{iq\frac{2\eta_c}{2}} c_- \simeq \frac{\Gamma(\nu)}{2\pi} [\sqrt{q\eta_c} + (1 - \nu)(q\eta_c)^{-\frac{1}{2}}(\frac{k\eta_c}{2})^{-\nu}]
\]

(4.12) for \(k^2 < q^2(\frac{\eta_{+2\eta_c}}{\eta_c})^2\). The typical amplitude of gravity waves over scales \(k^{-1}\) is \(\delta h(k, q, \eta) = l_D k^{d/2} q^{n/2} |h(k, q, \eta)|\) (where \(l_D = M_P^{-\frac{d-1}{2}}\)) and can be easily computed using the definition of \(\mu\) in terms of \(h\) (from equations (3.27) and (3.28))

\[
\delta h(\omega) = \left(\frac{\omega}{\omega_c}\right) \left(\frac{\omega}{\omega_c}\right)^{-\frac{1}{2}} \left(\frac{\omega}{M_P}\right)^{-\frac{1}{2}} \left(\frac{\omega}{\omega_c}\right)^{-\frac{1}{2}} \sqrt{\frac{\omega}{M_P}} |c_-|
\]

(4.13)

(\(\omega = k/a, \omega = q/a, \omega_c = H_c a_c/a\) is the maximal amplified frequency and \(\omega_0 = 3.1 \times 10^{-18}\) Hz is the present value of the Hubble parameter; we used that, in our case, \(a(\eta_c) \simeq b(\eta_c) \simeq 1\). Keeping only the leading terms for \(k\eta_c < 1\) in eq. (4.11) and (4.12) the power spectrum (4.13) is:

\[
|\delta h(\omega)| \simeq z \epsilon q^{-\frac{1}{2}} (\frac{\omega}{\omega_c})^{-\frac{1}{2}} \left(\frac{H_c}{M_P}\right) \left(\frac{\omega_0}{M_P}\right)^{\frac{1}{2}} (\sinh \pi (\frac{\omega}{\omega_c}))^{-\frac{1}{2}}, \quad \omega > \omega_q
\]

\[
|\delta h(\omega)| \simeq z \epsilon q^{-\frac{1}{2}} (\frac{\omega}{\omega_c}) \left(\frac{H_c}{M_P}\right) \left(\frac{\omega_0}{M_P}\right)^{\frac{1}{2}} (\sinh \pi (\frac{\omega}{\omega_c}))^{-\frac{1}{2}}, \quad \omega < \omega_q
\]

(4.14)
(where we used that $\Gamma(i\eta_e)\Gamma(-i\eta_e) = \pi/(q\eta_e\sinh q\eta_e)$ [52]). Since we assumed that the evolution starts dominating suddenly after $-\eta_e$, we can estimate that $\omega_c = 10^{11}\sqrt{H_c/M_P}\ Hz$, assuming that the evolution is adiabatic (if the evolution is not adiabatic and entropy is produced at some stage this result could be slightly modified but for our illustrative purposes it is not crucial [37] [see however [46] for a more quantitative analysis, in a more specific four dimensional model]). In order to compare the power spectrum with the phenomenological bounds which could constrain the parameter space of our naive model it is useful to work with the fraction of critical density stored in the gravity wave background per logarithmic interval of $\omega$:

$$\Omega_{GW}(\omega) = \frac{1}{\rho_c} \frac{d\rho_{GW}}{d\ln \omega} = z_{\text{dec}}^{-1} \left( \frac{\omega_q}{\omega_c} \right)^3 \alpha_c^2 \simeq \left( \frac{\omega}{\omega_0} \right)^2 |\delta h(\omega)|^2, \quad \rho_c \sim l_D^2 H_c^2 (a_c/a)^4. \quad (4.15)$$

Using again the explicit expression for the Bogoliubov coefficients (4.11) - (4.12) in the two different regimes we get:

$$\Omega_{GW}(\omega) \simeq z_{\text{dec}}^{-1} \left( \frac{\omega}{\omega_c} \right)^3 \left( \frac{H_c}{M_P} \right)^3 (\sinh \pi (\omega_q/\omega_c)^{-1}, \quad \omega > \omega_q$$

$$\Omega_{GW}(\omega) \simeq z_{\text{dec}}^{-1} \left( \frac{\omega}{\omega_c} \right)^4 (\omega_q/\omega_c)^{-1} \left( \frac{H_c}{M_P} \right)^3 (\sinh \pi (\omega_q/\omega_c)^{-1}, \quad \omega < \omega_q \quad (4.16)$$

($H_c/M_P$ measures how far from the Planck scale the compactification occurs and $\omega_q/\omega_c$ estimates the typical frequency of the internal excitations $\omega_q$ evaluated at the beginning of the radiation epoch ($\eta = -\eta_e$) with respect to the maximal amplified frequency $\omega_c$; notice also that since $\omega_c$ is the maximal amplified frequency $|\omega_q/\omega_c| < 1$). While the amplitude of the spectra are characterized by the two dimensionless quantities $x = \log_{10} (\omega_q/\omega_c)$ and $y = \log_{10} (H_c/M_P)$, the spectral slope is instead fixed by the background evolution and can be also more difficult to estimate in a different model of dimensional decoupling. In our case the parameter space can be constrained by the observations and since the spectrum is increasing in frequency we would expect that the bounds coming from the large scales like the COBE bound [47] ($\Omega_{GW}(\omega) < 7.1 \times 10^{-11}$ for $\omega_0 < \omega < 30\omega_0$) and the pulsar bound [48] ($\Omega_{GW} < 10^{-8}$ at $\omega \sim 10^{-8}\ Hz$) will be less constraining than the bounds arising from nucleosynthesis [49] ($\int d\ln \omega \Omega_{GW}(\omega, \eta_0) < 0.2 \Omega_\gamma(\eta_0) \sim 10^{-5}$, $[\Omega_\gamma(\eta)$ is the fraction of critical energy density present in form of radiation, at a given observation time $\eta]$) or from
the critical energy density \((\Omega(\omega) < 1\) for all the frequencies). In particular from (4.16) we can find that the COBE bound is satisfied either if \(y \leq 0.6 x + 35 - 2 \log_{10} h_{100}\) (provided \(\omega_{\text{COBE}} \simeq 10 \omega_0 < \omega_q\)) or if \(y \leq 0.6 x + 8 - 3 \log_{10} h_{100}\) (for \(\omega_{\text{COBE}} > \omega_q\)); the pulsar bounds are satisfied either if \(y \leq 0.3 x + 35\) (for \(\omega < \omega_p\)) or if \(y \leq 2 x + 72\) (for \(\omega > \omega_p\)). The COBE and the pulsar bounds are less constraining, while the critical density and the nucleosynthesis bound combined together give \(y \leq -0.6 x - 0.6\) (for \(\omega < \omega_q\)) and \(y \leq 0.3 x - 0.6\) (for \(\omega > \omega_q\)) which is compatible with \(H_c < 10^{-1} M_P\) and \(\omega_q < \omega_c\).

A similar analysis can be performed in the case of the model (2.9). The evolution equation (4.2) is then (for \(\eta < -\eta_c\)):

\[
\mu'' + \left[ -\frac{1}{4} + \frac{a}{z} + \frac{1}{4z^2} \right] \mu = 0, \quad z = 2ik(\eta - \eta_c), \quad a = i \frac{q^2 \eta_c^2}{k} \tag{4.17}
\]

which is formally equivalent to the radial Schroedinger equation for the problem of the Coulomb diffusion and which can be easily solved in terms of Confluent Hypergeometric functions

\[
\mu(k\eta, q) = e^{-ik(\eta - \eta_c)} \sqrt{2k(\eta - \eta_c)} U \left( \frac{1}{2} - i \frac{q^2 \eta_c}{k}, 1, 2ik(\eta - \eta_c) \right), \quad \eta < \eta_c \tag{4.18}
\]

\((U)\) is the Kummer function defined with the conventions of [52]; for \(\eta \rightarrow -\infty\) the solution behaves like a positive frequency mode, but does not define, asimptotically, a Bunch-Davies adiabatic vacuum). As in the previous example we have to match the solution (4.18) valid for \(\eta < -\eta_c\) with the solutions (4.9) and (4.10) valid for \(\eta > -\eta_c\). The result of this procedure will give the Bogoliubov coefficients describing the mixing positive and negative frequency modes:

\[
\begin{align*}
c_- &= e^{-2ik\eta_c} \sqrt{\frac{i}{4\sqrt{2k\eta_c}}} U \left( \frac{1}{2} - \frac{(q\eta_c)^2}{k\eta_c}, 1, -4ik\eta_c \right) \\
&+ \sqrt{\frac{i}{2k\eta_c}} U \left( \frac{1}{2} - \frac{(q\eta_c)^2}{k\eta_c}, 1, -4ik\eta_c \right), \quad k^2 > q^2 \left( \frac{\eta + 2\eta_c}{\eta_c} \right)^2 \\
&
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
c_- &= e^{-i\frac{q\eta_c}{k}} \left[ \sqrt{\frac{2q\eta_c}{\sqrt{2k\eta_c}}} - \sqrt{2k\eta_c} U \left( \frac{1}{2} - \frac{(q\eta_c)^2}{k\eta_c}, 1, -4ik\eta_c \right) \right] \\
&+ \sqrt{\frac{k}{q}} \sqrt{\frac{2q\eta_c}{\sqrt{2k\eta_c}}} U \left( \frac{1}{2} - \frac{(q\eta_c)^2}{k\eta_c}, 1, -4ik\eta_c \right), \quad k^2 < q^2 \left( \frac{\eta + 2\eta_c}{\eta_c} \right)^2 \tag{4.19}
\end{align*}
\]

Using the small argument limit of the Kummer functions we can compute the normalized
The spectral amplitude:

\[
\delta h(\omega) \approx z_{eq}^{-\frac{1}{2}} \left( \frac{\omega_q}{\omega_c} \right)^3 \left( \frac{H_c}{M_P} \right)^{\frac{7}{2}} \left( \frac{\omega}{\omega_c} \right)^{\frac{3}{2}}, \quad \omega > \omega_q
\]

\[
\delta h(\omega) \approx z_{eq}^{-\frac{1}{2}} (\sinh \pi \frac{\omega_q}{\omega_c})^{-\frac{1}{2}} \left( \frac{H_c}{M_P} \right)^{\frac{7}{2}} \left( \frac{\omega_0}{\omega_c} \right)^{\frac{1}{2}} \left( \frac{\omega}{\omega_c} \right)^{\frac{1}{2}}, \quad \omega < \omega_q
\]

and the spectral energy density distribution in critical units:

\[
\Omega_{GW} \approx z_{dec}^{-1} \left( \frac{\omega_q}{\omega_c} \right)^6 \left( \frac{H_c}{M_P} \right)^{8} \left( \frac{\omega}{\omega_c} \right)^{3} \ln \left( \frac{\omega}{\omega_c} \right) \quad \omega > \omega_q
\]

\[
\Omega_{GW} \approx z_{dec}^{-1} \left( \frac{\omega_q}{\omega_c} \right)^6 \left( \frac{H_c}{M_P} \right)^{8} (\sinh \pi \frac{\omega_q}{\omega_c})^{-1} \left( \frac{\omega}{\omega_c} \right)^{4} \quad \omega < \omega_q
\]

For \( \omega > \omega_q \) the slopes of the spectral energy density distribution (and of the related spectral amplitudes) agree with the result previously obtained for the spectra of gravity waves produced during a dilaton driven phase [26], neglecting the internal Laplace-Beltrami operators. The amplitudes are instead different due to the presence of \( \omega_q \). Our partial conclusion is that to neglect the internal Laplacians is a good approximation for the slopes of the spectra (for \( \omega > \omega_q \)) but not for their amplitude. For \( \omega < \omega_q \) both the slopes and the amplitudes of the spectra are affected by the presence of the internal Laplacians which cannot be overlooked. Since the spectra are increasing in frequency we will keep the most stringent bound which comes from nucleosynthesis and which gives, if applied separately in each of the two branches, \( y \lesssim -0.7 \times -0.2 \) (for \( \omega > \omega_q \)), and \( y \lesssim -1.1 \times -0.25 \) (for \( \omega < \omega_q \)).

For \( \omega > \omega_q \) the slopes of the spectral energy density distribution (and of the related spectral amplitudes) agree with the result previously obtained for the spectra of gravity waves produced during a dilaton driven phase [26], neglecting the internal Laplace-Beltrami operators. The amplitudes are instead different due to the presence of \( \omega_q \). Our partial conclusion is that to neglect the internal Laplacians is a good approximation for the slopes of the spectra (for \( \omega > \omega_q \)) but not for their amplitude. For \( \omega < \omega_q \) both the slopes and the amplitudes of the spectra are affected by the presence of the internal Laplacians which cannot be overlooked. Since the spectra are increasing in frequency we will keep the most stringent bound which comes from nucleosynthesis and which gives, if applied separately in each of the two branches, \( y \lesssim -0.7 \times -0.2 \) (for \( \omega > \omega_q \)), and \( y \lesssim -1.1 \times -0.25 \) (for \( \omega < \omega_q \)).

The obtained spectral amplitudes (4.14)-(4.20) are quite different since they are produced by two different background geometries, but the spectral slopes are exactly equal in spite of the differences in the solutions (2.5)-(2.8). More specifically we obtained “violet” type of spectra \( \delta h \sim \omega/\omega_c \) for \( \omega > \omega_q \) and \( \delta h \sim (\omega/\omega_c)^{1/2} \) for \( \omega < \omega_q \) which are a common feature of the contracting backgrounds also in the isotropic case [39]. This apparent puzzle is due to the fact that \( a^{d-1} b^n \sim \sqrt{\eta/\eta_c} \) for (2.5) and (2.8) in arbitrary number of internal and external dimensions.

We would like finally to point out that the graviton production due to the transition from the radiation dominated stage to the matter dominated stage should also be included. This further amplification will modify the low frequency tail of the spectrum \( (10^{-18} H \eta < \omega < 10^{-16} \) Hz). The qualitative aspects of our analysis show that the presence of the internal
gradients introduces in the spectral amplitude a new parameter related to the frequency of the internal excitation which can be interestingly constrained by the bounds usually analyzed in the context of the stochastic gravity-wave backgrounds.

5 Growing solutions for the scalar modes

The tensor inhomogeneities can be treated perturbatively keeping track of the internal Laplacians also because they evolve logarithmically outside the horizon. The situation changes in the case of the scalar inhomogeneities because, as we explicitly pointed out at the end of Sec. 2, the growing solution increases much faster than a logarithm and a perturbative treatment become inappropriate after some time. The hope would be in this case that the growing mode appearing in the multidimensional case could be consistently gauged down as happens for the dilaton driven solutions in the \((3+1)\)-dimensional case [24]. In this section we do not want to address specifically the problem of the growing modes in an anisotropic manifold but we want to show how the problem can be consistently formulated in the presence of the internal Laplacians and for this purpose we will study the 10-dimensional dilaton-driven solutions (2.8). Using the all set of equations (3.13)-(3.16) and equations (3.21), (3.22), (3.23) the Fourier modes of the longitudinal fluctuations \(\Psi, \Lambda\) can be expressed in terms of the Fourier modes of \(\mathcal{V}\) and \(\mathcal{W}\)

\[
(k^2 + \frac{a^2}{b^2}q^2)\Psi(k, q, \eta) = \frac{n(n + d - 1)\mathcal{H}\varphi'}{[(d - 1)\mathcal{H} + n\mathcal{F}]^2} \left[ \frac{6\rho^2_D(d - 1)}{n(n + d - 1)} \right]^{1/2} \left( \frac{\mathcal{W}(k, q, \eta)}{z_1} \right)' - \\
- \frac{3\rho^2_D\varphi'\mathcal{H}}{[(d - 1)\mathcal{H} + n\mathcal{F}]^2} \left( \frac{\mathcal{V}(k, q, \eta)}{a^{d-1}b^{d\tau}} \right)' - \\
- \frac{n\varphi'}{[(d - 1)\mathcal{H} + n\mathcal{F}]^2} \left[ \frac{6\rho^2_D(d - 1)}{n(n + d - 1)} \right]^{1/2} \left( \frac{\mathcal{W}(k, q, \eta)}{z_1} \right)\right), \tag{5.1}
\]

\[
(k^2 + \frac{a^2}{b^2}q^2)\Lambda(k, q, \eta) = \frac{n\mathcal{F}}{d - 1} \left[ \frac{(n + d - 1)\varphi'}{[(d - 1)\mathcal{H} + n\mathcal{F}]^2} \left[ \frac{6\rho^2_D(d - 1)}{n(n + d - 1)} \right]^{1/2} \left( \frac{\mathcal{W}(k, q, \eta)}{z_1} \right)' - \\
- \frac{3\rho^2_D\varphi'}{(d - 1)} \left( \frac{\mathcal{V}}{a^{d-1}b^{d\tau}} \right)' \right)\right). \tag{5.2}
\]

where \(q, k\) have to be considered both scalar eigenvalues (while in the previous section \(k\) was labeling the eigenvalues of the tensor Helmotz equation). If \(k^2 > q^2a^2/b^2\) the evolution
equation for \( \mathcal{V}(k, q, \eta) \) and \( \mathcal{W}(k, q, \eta) \) will be (from (3.21), (3.22) through (3.23))

\[
\mathcal{V}'' + \left[ k^2 + \frac{1}{4(\eta c - \eta)^2} \right] \mathcal{V} = 0, \quad \mathcal{W}'' + \left[ k^2 + \frac{1}{4(\eta c - \eta)^2} \right] \mathcal{W} = 0 \tag{5.3}
\]

whose solution is exactly identical to (4.7), provided we choose, for \( \eta \to -\infty \), the Bunch-Davies vacuum as initial condition. In the limit \( k^2 < q^2a^2/b^2 \) eq. (3.21)-(3.22) and (3.23) will give instead

\[
\mathcal{V}'' + \left[ \frac{\eta_cq^2}{(\eta_c - \eta)} + \frac{1}{4(\eta c - \eta)^2} \right] \mathcal{V} = 0, \quad \mathcal{W}'' + \left[ \frac{\eta_cq^2}{(\eta_c - \eta)} + \frac{1}{4(\eta c - \eta)^2} \right] \mathcal{W} = 0 \tag{5.4}
\]

with solution [52]

\[
\mathcal{V}(k, q, \eta) = \sqrt{\eta_c - \eta} H_0^{(2)}(q\sqrt{\eta_c - \eta}), \quad \mathcal{W}(k, q, \eta) = \sqrt{\eta_c - \eta} H_0^{(2)}(q\sqrt{\eta_c - \eta}), \tag{5.5}
\]

The typical amplitude of the longitudinal fluctuations over a length scale \( k^{-1} \) is \(|\delta\Psi(k, q, \eta)| = l_Pk^{\frac{4z_1}{3}}q^\eta \Psi(k, q, \eta) \) (with \( l_P = M_P^{\frac{4z_1}{3} - 1} \)). In the particular case of the 10-dimensional model (2.9) we have \(|\delta\Psi(kq, \eta)| = l_Pk^{3/2}q^3\Psi_{k,q}(k, q, \eta) \) and from (5.1) with the use of (5.4) we obtain for \( k^2 > q^2(a/b)^2 \)

\[
|\delta\Psi|(k, q, \eta) \simeq \left( \frac{H_c}{M_P} \right)^4 \left( \frac{k\eta_c}{k\eta} \right)^{3/2} \left( \frac{\omega}{\omega_c} \right)^2 \tag{5.6}
\]

(we used that \( z_1 = \sqrt{\frac{\eta}{\eta_c}} \) and \( H_0^{(2)}(z) \sim \ln z \)). Since \( k_c \sim 1/\eta_c \) we find that \(|\delta\Psi(k, q, \eta)| < 1 \) on scales \( k^{-1} \) such that \( \eta/\eta_c > (H_c/M_P)^2(\omega/\omega_c)^{3/2}(k_c/k)^{1/4} \). From eq. (5.1) using (5.5) we obtain, for \( k^2 < q^2(a/b)^2 \)

\[
|\delta\Psi| \simeq \left( \frac{H_c}{M_P} \right) \left( \frac{\omega}{\omega_c} \right) \left( \frac{k}{k_c} \right)^{3/2} \left( \frac{\eta_c}{\eta} \right) \tag{5.7}
\]

which implies that the perturbative approach is only reliable for conformal times \(|\eta/\eta_c| > (H_c/M_P)^4(\omega/\omega_c)(k/k_c)^{3/2} \).

The presence of the internal gradients slightly changes the quantitative estimates but does not change the nature of the growing mode problem. If this is the situation the scalar fluctuations will become very soon critical and will then enter in a true non-perturbative regime. This apparent contradiction among the behaviour of the tensor inhomogeneities and the behaviour of the scalar ones might be the connected with our perturbative technique.
The fluctuations which we are discussing now are gauge-invariant only for infinitesimal coordinate transformations while some quantities which are invariant to all orders would be more suitable for the study of fluctuations which are growing outside the horizon. It can be actually shown [24] that in the \((3 + 1)\) isotropic case the linearized variables describing the scalar and tensor fluctuations in a fully covariant and gauge-invariant approach [11] grow only logarithmically outside the horizon. Unfortunately a fully covariant and gauge-invariant formalism is only formulated in the case of a homogeneous and isotropic manifold and it seems to be quite complicated to formulate it for an anisotropic manifold. Within the linearized theory discussed in this paper it would be anyway interesting to understand if it is possible to “gauge-down” the growing mode solutions arising in a anisotropic background perhaps with a suitable generalization of the “off-diagonal” gauge [24, 50] to the case of an anisotropic background.

6 Conclusions

We discussed the treatment of the scalar and tensor inhomogeneities in an anisotropic background manifold undergoing dimensional decoupling.

We showed that it is possible to study the evolution equations of the scalar modes by completely fixing the coordinate system with a suitable gauge choice which reduces, in the isotropic case, to the well known conformally newtonian gauge often employed in the analysis of the density fluctuations in the context of the inflationary models driven by a scalar field or by perfect fluid matter. The coupled system of second order differential equations describing the fluctuations of the scalar field and of a homogeneous and anisotropic manifold was diagonalized in terms of two scalar variables which become, in the absence of internal dimensions, the normal modes of oscillation obtained by perturbing scalar-tensor action to second order in the amplitude of the fluctuations. The evolution equations for the amplitude of the scalar perturbations (depending on the internal and external coordinates) were also explicitly solved in a particular 10-dimensional background motivated by String Cosmology. The scalar spectral amplitudes grow outside the horizon faster than the tensor amplitudes and the dependence on the internal coordinates does not change drastically this situation
unless the present value of the typical frequency of the internal dimensions would be much smaller (by several orders of magnitude) than the maximal amplified frequency.

The scalar and tensor modes are coupled because the tensors defined in the internal manifold are scalar eigenstates of the Laplace-Beltrami operators defined in the external manifold. In order to study the connection among the occurrence of a process of dimensional reduction and the present energy distribution of cosmic gravitons we discussed explicitly two oversimplified toy models of dimensional decoupling. Provided we keep track, in the perturbed amplitudes, of the dependence on the internal coordinates the spectral energy distribution will be a function of the curvature scale which the compactification occurs and of the typical frequency of the internal excitations. We found for both the models a two branches violet spectrum. Since the power spectra are increasing in frequency the most significant bounds come from small wave-length and constrain the curvature scale to be \( H_c < 10^{-1} M_P \) (provided \( \omega_q < \omega_c \) and with \( \omega_c \approx 10^{11} (H_c/M_P)^{1/2} Hz \)). At the same time the violet spectra are less constrained at low frequencies by the large scale observations and in particular by the \textit{COBE} bound. The estimates presented in this paper suggest that for \( \omega > \omega_q \) the slopes of the spectra (but not their amplitudes) can be reliably computed by neglecting the internal Laplacians. On the contrary the presence of the internal Laplacians affects decisively the spectral slopes (and amplitudes) for \( \omega < \omega_q \). Our considerations can be also applied to more realistic models of dimensional reduction (with the unavoidable help of numerical techniques) in order to discuss the back-reaction problems which could eventually lead to the isotropisation of the original background model. It might also be of some interest to deepen the possible phenomenological signatures of the scenarios of dimensional decoupling and their relevance for the formation of a stochastic gravity-waves background. We want finally to stress that even though the models analyzed in this paper are quite simplified the perturbative techniques which we introduced are more general. The open problem which emerges also from our discussion is of course to understand if a viable multidimensional cosmological model, free of the well known problems mentioned in the introduction, exists at all. String theory seem to be a very good candidate for this purpose and it is very tempting to speculate that the same mechanisms leading, in principle, to a graceful exit in four dimensions\cite{25} could also operate in order to stabilize the internal
dimensions producing, ultimately a completely isotropic universe. It could also be possible that classical field configurations (like Dirac monopole configurations polarized along the internal dimensions) can offer suitable mechanisms for the stabilization of the internal space [43] and in this direction the work is still in progress [51].

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