Is $N=4$ Yang-Mills Theory Soluble?\footnote{Talk given by PCW at The Quantum Gravity Seminar June 12-19, 1995, Moscow and at the Imperial College Workshop on "Gauge Theories, Applied Supersymmetry and Quantum Gravity", 5-10 July 1996}

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Abstract

The superconformal properties of $N = 4$ Yang-Mills theory are most naturally studied using the formalism of harmonic superspace. Superconformal invariance is shown to imply that the Green’s functions of analytic operators are invariant holomorphic sections of a line bundle on a product of certain harmonic superspaces and it is argued that the theory is soluble for a class of such operators.
Theories with extended rigid supersymmetry are the most symmetric consistent quantum field theories that we know which do not include gravity. Since field theories that involve spin $\frac{3}{2}$ particles require spin 2 particles for classical consistency, we must, in the absence of gravity, restrict our considerations to spins one and less. The celebrated no-go theorem ensures that if we want to combine internal and Poincaré group symmetries in a non-trivial way then we must include supersymmetry. Supersymmetric theories are classified by the number, $N$, of supercharges they have. In four dimensions theories with more than four supercharges contain spins greater than one and so must, for consistency, be supergravity theories. As such, the maximally supersymmetric theory with only rigid supersymmetry can have at most four supersymmetries. Such a theory has a unique particle content given by 1 spin one, 4 spin $\frac{1}{2}$ and 6 spin 0 all in the adjoint representation of a gauge group. This theory is called the $\mathcal{N}=4$ Yang-Mills theory and its action is uniquely determined once the coupling constant and gauge group are specified. Given that this theory is uniquely picked out on grounds of symmetry, one may hope that its Green’s functions can also be determined by its symmetries. Long ago, it was shown that the $N=4$ Yang-Mills theory [1] and a large class of $N=2$ rigid supersymmetric theories [2] are conformally invariant even as quantum theories. Hence the $\mathcal{N}=4$ Yang-Mills theory actually has $\mathcal{N}=4$ superconformal symmetry.

Although a systematic understanding of how to compute non-perturbative effects in quantum field theory is still lacking, in the 1980’s spectacular progress [3] was made when it was shown that one could solve a large class of two-dimensional conformal models, the so-called minimal models. Solve in this context means that one can determine explicitly the Green’s functions of these theories.

The most likely four-dimensional theory for which one could hope to achieve a similar success is the $\mathcal{N}=4$ Yang-Mills theory as it is conformally invariant and uniquely picked out on grounds of symmetry. It is well-known however, that for a generic conformal theory, conformal invariance determines only the two- and three-point Green’s functions. Given two possible Green’s functions for a given set of operators, their ratio will be a conformal invariant. Hence, the existence of conformal invariants will, in the absence of other constraints, imply the theory is not soluble. It is well-known that for four and more space-time points conformal invariants do exist. One may hope that supersymmetry by itself constrains the number of invariants. However, one finds that the basic building block of conformal invariants, the four point cross-ratio, generalises in a straightforward way to Minkowski superspace. Hence, at first sight, it would not seem likely that one could use the superconformal invariance of $N=4$ Yang-Mills theory to solve for its Green’s functions explicitly.

Almost all supersymmetric theories of interest admit a superfield description based on superfields that are constrained. For example, the Wess-Zumino multiplet and the superspace field strengths of the $\mathcal{N}=1$ and 2 Yang-Mills theories are described by chiral superfields. One encouraging sign for $\mathcal{N}=4$ is that, in superconformal theories, one can determine [4] the anomalous dimension of any chiral operator if one knows its weight under $R$-transformations and one can also determine the Green’s functions which depend only on superfields of a given chirality [5]. This is not the case for operators and correlators in generic conformal field theories.
A chiral superfield, \( \varphi \), can be defined on Minkowski superspace which has coordinates
\[
x^{\alpha \dot{\alpha}}, \theta^{\alpha i}, \bar{\theta}^{\dot{\alpha} i}; \quad \alpha, \dot{\alpha} = 1, 2, \ldots, N.
\] (1)
and is subject to the constraint \( \bar{D}_{\dot{a}} \varphi = 0 \) where \( \bar{D}_{\dot{a}} \) is the superspace dotted spinor derivative. The meaning of this constraint becomes clear if we make a coordinate change to \( (y^\mu = x^\mu - i \theta^{\alpha i} (\sigma^i)_{\alpha \dot{\alpha}} \bar{\theta}^{\dot{\alpha} i}, \theta^{\alpha i}, \bar{\theta}^{\dot{\alpha} i}) \) whereupon \( \bar{D}_{\dot{a}} \varphi = 0 \) becomes simply \( \frac{\partial}{\partial y^\mu} \varphi = 0 \) so that \( \varphi \) depends only on \( y^\mu \) and \( \theta^{\alpha i} \), the coordinates of chiral superspace. The ability to solve a given chiral sector of the theory is a consequence of the fact that there exist no superconformal invariants which are Grassmann even, non-nilpotent, and which depend only on \( y^\mu, \theta^{\alpha i} \) where \( \alpha \) labels different points of chiral superspace corresponding to the different operators in a Green’s function [5].

The above discussion can be given a more mathematically sophisticated formulation. The complex conformal group is \( SL(4, \mathbb{C}) \), while the complex superconformal group is \( SL(4|N, \mathbb{C}) \). We can regard complex space-time and complex Minkowski superspace as coset spaces of these groups divided by suitable subgroups, \( H \) and \( H_s \) respectively. Chiral superspace is the coset space \( H_{sc} \backslash SL(4|N, \mathbb{C}) \) where \( H_s \) is a subgroup of \( H_{sc} \).

The fields of interest to us transform under induced representations. We recall that given a coset space \( H \backslash G \), a field transforming under an induced representation is a field defined on the group which satisfies the condition \( \phi(hg) = D(h) \phi(g) \forall \ g \in G, \ h \in H \) and \( D(h) \) is the matrix representation carried by \( \phi \). The action of the group is then defined to be \( U(g_1) \phi(g) = \phi(g) = \phi(gg_1) \). Although we can work with fields defined on the group, it is often more convenient to choose coset representatives \( s(x) \) and consider \( \phi \) to depend only on \( x \) by writing every \( g \) in the form \( g = hs(x) \). The action of the group then becomes
\[
U(g_1) \phi(s(x)) = D(h_1) \phi(s(x')).
\] (2)
where \( h_1 \) is an appropriate element of the isotropy group depending on \( g_1 \) and \( s(x) \) which is given by \( s(x)g_1 = h_1 s(x') \). Carrying out this procedure for a chiral superspace we find that \( \varphi \) depends on only the coordinates \( y^\mu, \theta^{\alpha i} \). We can recover the more usual formulation of chiral superfields defined on Minkowski superspace by using only \( h \in H_s \) instead of \( H_{sc} \) to gauge away parts of any \( g \in G \). Then \( \varphi \) depends on \( y^\mu, \theta^{\alpha i}, \bar{\theta}^{\dot{\alpha} i} \), but is subject to \( \varphi(hg) = D(h) \varphi(g) = 0 \) for \( h \in H_{sc} \) but \( h \notin H_s \). This latter constraint is none other than the chiral constraint \( \bar{D}_{\dot{a}} \varphi = 0 \). Hence, using the coset representatives for chiral superspace automatically solves the chiral constraint.

The \( N = 4 \) Yang-Mills theory has a superspace formulation [7] based on harmonic superspace [6], the latter being an extension of Minkowski superspace to include a Grassmann even internal space which is the coset space of two Lie groups. It is usual to take the superfields to be defined initially on the product of Minkowski superspace and the larger of the internal groups associated with the internal coset and subject them to constraints. For our purposes, it will be more useful to follow the discussion of the chiral superfield given above where we in effect solved these constraints. The complex superconformal group for \( N \)-extended supersymmetry is \( SL(4|N; \mathbb{C}) \) and, if we divide by an appropriate subgroup \( H_N \), we can construct the harmonic superspace analogue of chiral superspace. This superspace is not harmonic superspace itself, but a new superspace which is usually
called analytic superspace. We choose coset representatives which can, by appropriate swapping of rows and columns, be parameterised by the elements of $GL(\frac{N}{2}|2, \mathbb{C})$ matrix, $X$ [5],

$$X = \begin{pmatrix} x & \lambda \\ \pi & y \end{pmatrix}$$

(3)

The matrix $x$ contains the coordinates of space-time, $\lambda$ and $\pi$ are spinors and $y$ are the coordinates of the internal coset space. The transformation [5][9]of $X$ under an arbitrary group element $g_1 \in \text{SL}(4|N; \mathbb{C})$, which can be written $g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D \in GL(\frac{N}{2}|2, \mathbb{C})$ is

$$X \rightarrow X' = [A + CX]^{-1} [B + XD]$$

(4)

The superfields of interest to us transform under induced representations of $H_N \setminus \text{SL}(4|N; \mathbb{C})$, but they carry a representation of $H_N$ which is non-trivial only for a $U(1)$ subgroup (strictly $\mathbb{C}^*$ in the complex case). In fact, they transform as [9]

$$U(g_1) \phi(X) = (\text{sdet} \ [A + XC])^{-q} \phi(Xg_1)$$

(5)

where $q$ is the $U(1)$ weight of $\phi$, that is to say, $\phi$ is a section of the holomorphic line bundle $L^q$ over analytic superspace, where $L$ itself is associated with charge 1.

A generic field which depends on the Grassmann even coordinates $y$ and those of space-time would, when Taylor expanded in $y$, contain an infinite number of space-time dependent fields. This is avoided by taking $\phi$, to be a holomorphic section of $L^q$. Since the internal space is compact, there are only an finite number of such sections and so only a finite number of space-time dependent fields contained in $\phi$. For example, if $N = 2$ the internal space is $U(1) \setminus \text{SU}(2) = CP^1$ and $\phi$ corresponds to a holomorphic section of $O(q)$, the $q$th power of the standard line bundle on $CP^1$. It is well known that there are only $q + 1$ such sections and $\phi$ may be expanded around the south pole as

$$\phi = \sum_{i=0}^{a} \phi_i(x, \lambda, \pi)y^q$$

(6)

yielding a finite number of ordinary space-time dependent fields once we also Taylor expand about $\lambda = \pi = 0$.

Like chiral superfields, analytic superfields have their dilation weights linearly related to their weights under certain internal $U(1)$ transformations. The $N = 4$ Yang-Mills superfield strength transforms under an induced representation of $\text{SL}(4|4; \mathbb{C})$ associated with the subgroup $H_4$ and has $U(1)$ weight one. We are interested in Green’s functions of observables. We take the observables to be the gauge invariant quantities $\text{Tr}W^n$. The Green’s function

$$G = \langle \text{Tr}[W(X_1)]^{n_1} \ldots \text{Tr}[W(X_p)]^{n_p} \rangle$$

(7)

will then obey the superconformal Ward identities. These imply that [5] [9]

$$G(X_1, \ldots, X_p) = \prod_{j=1}^{p} (\text{sdet} \ [A + X_j C])^{-n_j} G(X_1', \ldots, X_p').$$

(8)
The question we must answer is: does the above equation determine \( G \) up to constants, given that it can only depend analytically on the \( y \)'s?

From the usual arguments, it is straightforward to determine the two- and three-point functions using conformal invariance. In particular, one finds for the Abelian theory that

\[
G_{12} = \langle W(X_1)W(X_2) \rangle = (s\text{det} \ X_{12})^{-1} = \frac{y_{12}^2}{x_{12}^2} + O(\lambda_{12})
\]

(9)

where \( X_{12} = X_1 - X_2 \). Any Green's function can be written in terms of the product of two-point Green's functions times a superconformal invariant, for example

\[
\langle \text{Tr}[W(X_1)]^2 \text{Tr}[W(X_2)]^2 \text{Tr}[W(X_3)]^3 \text{Tr}[W(X_4)]^2 \rangle = G_{12}^2 G_{34}^2 \times I
\]

(10)

where \( I \) is an invariant.

However, unlike the case of chiral superspace, there are superconformal invariants in analytic superspace. It can be shown [9] that they are either of the form

\[
\frac{s\text{det} \ X_{ij} \text{det} \ X_{kl}}{s\text{det} \ X_{ik} \text{det} \ X_{jl}}
\]

(11)

or of the form of supertraces of the \( X_{ij} \)'s such as

\[
\text{Str}\{X_{ij}^{-1}X_{ik}X_{kl}^{-1}X_{li}\}
\]

(12)

We refer to these as type I and type II invariants respectively. The above superconformal invariant \( I \) can then be taken to be the most general function of these invariants subject to the constraint that the Green's function be an analytic function of \( y_{ij} \). Any \( N \) point function is an invariant holomorphic section of a line bundle on \( N \) copies of analytic superspace, the particular line bundle being determined by the \( U(1) \) weight of the operators involved. The Green's function will be determined up to constants if there exist only a finite number of such sections of the appropriate bundle. We are unaware of any general theorems that classify the number of such sections and have had instead to rely on an explicit examination of the poles in \( y_{ij} \) of the function \( I \) to find the restrictions analyticity places upon it.

The requirement of analyticity places very strong constraints on the form of the function \( I \). For example, if the function \( I \) is composed of only type I invariants, i.e. super cross-ratios, then it is easy to convince oneself that analyticity requires that any Green's function is given by a sum of terms each of which is a product of two point functions multiplied by arbitrary constants. Hence once we make this restriction the Green's functions are determined up to constants and take a form reminiscent of free field theory. However, when we allow for the possibility of type II invariants the situation becomes much more complicated. The calculations one has to do to check analyticity are long and complicated, but have been carried out for the four point Green's functions of \( N=2 \) analytic operators [10]. The result is that such Green's functions can be completely determined for operators of charge two and three, but arbitrary functions can occur in the Green's functions of higher charge operators. It is likely that this calculation can be extended to higher point
Green’s functions in $N = 2$ and $N = 4$ theories with similar conclusions. One would also expect to find that the Green’s functions in $N = 4$ Yang-Mills theory are determined up to constants by superconformal invariance alone for a class of sufficiently low dimension analytic operators.

In some senses it is to be expected that one cannot determine the Green’s functions for all operators since this would implicitly require a definition of the theory with an action involving operators of arbitrarily high dimension. However, the Green’s functions could be further restricted by requiring that they satisfy physical properties such as crossing and unitarity. One may also be able to use the bootstrap programme to determine higher point Green’s functions in terms of lower point Green’s functions.

One can also apply the superconformal techniques of this article to study the operator product expansions of analytic operators in superconformal field theories [8]. In particular, one finds that the $N = 4$ energy momentum tensor supermultiplet has a very simple operator product expansion which has striking similarities with the operator product expansion for the energy momentum tensor in two dimensional spacetime. A heuristic argument for solvability of $N = 4$ Yang-Mills based on the OPE was given in reference [8].

It would be of interest to extend the analysis outlined here to spontaneously broken superconformal symmetry. If one could calculate some Green’s functions in this phase one might hope to be able to verify the predictions of duality directly. Finally, it is also possible to study anomalous superconformal Ward Identities, for example in $N = 2$ theories; this was done in [11] where it was used to derive the ‘Matone Identity’ [12] for the Seiberg-Witten prepotential [13].

References


