Domain Walls from M-branes

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ABSTRACT

We discuss the vertical dimensional reduction of M-branes to domain walls in $D = 7$ and $D = 4$, by dimensional reduction on Ricci-flat 4-manifolds and 7-manifolds. In order to interpret the vertically-reduced 5-brane as a domain wall solution of a dimensionally-reduced theory in $D = 7$, it is necessary to generalise the usual Kaluza-Klein ansatz, so that the 3-form potential in $D = 11$ has an additional term that can generate the necessary cosmological term in $D = 7$. We show how this can be done for general 4-manifolds, extending previous results for toroidal compactifications. By contrast, no generalisation of the Kaluza-Klein ansatz is necessary for the compactification of M-theory to a $D = 4$ theory that admits the domain wall solution coming from the membrane in $D = 11$.

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It has recently been observed that some unusual and interesting features emerge when one pushes the process of vertical dimensional reduction of \( p \)-brane solutions to the stage where the transverse space orthogonal to the \( d = p + 1 \) dimensional world-volume becomes only one dimensional. More precisely, we may begin with extremal \( p \)-brane solutions in \( D \)-dimensional supergravity, which are described in terms of an harmonic function \( H \) on the \((D - p - 1)\)-dimensional transverse space \([1]\):

\[
\begin{align*}
 ds^2_{\tilde{p}} &= H^{-\frac{d_\tilde{p}}{2}} dx^\mu dx^\nu \eta_{\mu\nu} + H^\frac{d_\tilde{p}}{2} dy^m dy^m , \\
 e^\phi &= H^{\frac{2a}{\Delta}} ,
\end{align*}
\]

where \( \partial_m \partial_n H \equiv 0, \epsilon = \pm 1 \) according to whether the solution is elementary or solitonic, \( d_\tilde{p} = D - d - 2 \), and the constants \( a \) and \( \Delta \), related by \( a^2 = \Delta - 2d_\tilde{p}(D - 2) \), characterise the dilaton coupling in the Lagrangian

\[
e^{-1} \mathcal{L} = R - \frac{1}{2}(\partial \phi)^2 - \frac{1}{2^n!} e^{a \phi} F_n^2 .
\]

The \( n \)-rank field strength \( F_n \) has \( n = p + 2 \) in the elementary case, or \( n = (D - p - 2) \) in the solitonic case. In the \( p \)-brane solution, it is given by

\[
\begin{align*}
 \text{elementary} : & \quad F_{m\mu_1\ldots\mu_n} = \epsilon_{\mu_1\ldots\mu_n} \partial_m H \\
 \text{solitonic} : & \quad F_{m_1\ldots m_n} = \epsilon_{m_1\ldots m_n} \partial_l H .
\end{align*}
\]

By stacking up an infinite periodic array of single \( p \)-brane solutions along an axis in the transverse space, one can construct a solution which, in the continuum limit, becomes independent of the associated coordinate. The solution may then be dimensionally reduced along this coordinate, to give a \( p \)-brane in \((D - 1)\) dimensions. It is described in terms of a function that is harmonic on the dimensionally-reduce transverse space. The process may be continued until eventually the \( p \)-brane is living in a spacetime whose total dimension is only \((p + 2)\). The solution describes a domain wall, and has the form given by (1) and (3), with \( H = 1 + m|y| \).

The above discussion of the dimensional reduction of the \( p \)-brane solutions has a parallel at the level of the theory itself, namely that all the dimensionally-reduced \( p \)-branes are solutions of the dimensionally-reduced Lagrangian. This is straightforward to understand until the vertical descent reaches the domain wall discussed above. At this point, a significant difference between the elementary and solitonic solutions emerges. In fact the elementary case continues to follow the standard pattern, but in the solitonic case, the domain-wall solution requires the presence of a cosmological constant term in the \((p + 2)\)-dimensional
Lagrangian, of the form $m^2 e^{i\phi}$. However, such a cosmological term is not generated in the standard Kaluza-Klein reduction procedure. The resolution of this puzzle is that a slight generalisation of the standard Kaluza-Klein reduction is needed here, and in fact, the form of the domain-wall solution already indicates the nature of this generalisation. We can see from the expression (3) for the solitonic 1-form field strength in the $(p + 3)$-dimensional solution that we have $F = mz$, where $z$ is the additional transverse-space coordinate that will be compactified in the final reduction step that gives the domain wall. It follows that the 0-form potential (or axion) for this field strength must be of the form $A_0 = mz$. Thus the axion depends on the coordinate $z$ that is to be compactified.

Normally in the Kaluza-Klein procedure, one performs a truncation in which all the fields are taken to be independent of the compactification coordinate. By making this requirement, one ensures that the truncation is consistent, i.e., that all the solutions of the lower-dimensional theory are also solutions of the higher-dimensional one. However, this $z$-independent truncation is slightly more restrictive than is actually necessary. For consistency, one need require only that the higher-dimensional equations of motion, after substituting the Kaluza-Klein ansatz, should be independent of $z$. In particular, this means that a potential such as $A_0$ above can be allowed to have a linear dependence on $z$, provided that it always appears via its exterior derivative $dA_0$. This generalisation of the ansatz gives rise to a cosmological term in the lower-dimensional Lagrangian, since it implies that the reduction of the 1-form field strength now yields a “0-form field strength” as well as a 1-form. It also gives rise to mass terms for certain of the previously-massless gauge fields, and thus we may describe the resulting theory as a massive supergravity. This type of generalised Kaluza-Klein dimensional reduction was first considered in [2] where the axion in type IIB theory in $D = 10$ was taken to have an additional term that is linearly dependent on the compactifying circular coordinate. This gives rise to a maximally supersymmetric massive supergravity in $D = 9$. This procedure was generalised in [3] to compactify M-theory to various massive supergravities in lower dimensions. (It was observed earlier, in the context of compactifying the heterotic string to $D = 4$, that wrapping the 5-brane on a 3-torus to give rise to a membrane in $D = 4$ would require some ansatz that went beyond the usual Kaluza-Klein dimensional reduction [4, 5].) The occurrence of the cosmological term explains how the domain-wall, which comes from a solution in the higher dimension, can continue to be a solution in this lower dimensional massive theory. It should remarked, however, that just as the domain wall is a solution of the massive theory but not the massless one, so $p$-branes that are solutions of the massless theory will not be solutions of
the massive one.

The generalised Kaluza-Klein reduction described above entails making the ansatz

\[ A_0(x, z) = m z + A_0(x) \]  

for the axion \( A_0 \), while making the standard \( z \)-independent ansatz for all the other fields. We may view \( z \) as a 0-form defined locally on the \( S^1 \) compactification manifold, whose exterior derivative gives the volume form \( dz \) on \( S^1 \). This formulation lends itself to an immediate generalisation, in which we consider a Kaluza-Klein compactification on an \( n \)-dimensional manifold \( M_n \), with a generalised ansatz for an \( (n-1) \)-form potential \( A_{n-1} \), of the form

\[ A_{n-1}(x, z) = m \omega_{n-1}(z) + A_{n-1}(x) + A_{n-2}^Q(x) \wedge \Omega_1^Q (z) + \cdots, \]  

where \( x \) denotes the lower-dimensional coordinates, \( z \) denotes the coordinates on the internal manifold \( M_n \), \( \Omega_0^Q \) denotes the set of harmonic \( q \)-forms on \( M_n \), and \( \omega_{n-1} \) is an \( (n-1) \)-form defined locally on \( M_n \), whose exterior derivative gives the globally-defined volume form: \( d\omega_{n-1} = \Omega_n \). Provided that \( A_{n-1} \) appears only via its exterior derivative in the higher-dimensional equations of motion, the inclusion of the extra term \( m\omega_{n-1} \) in its ansatz will not upset the consistency of the Kaluza-Klein truncation.\(^1\) As in the case of the 0-form example that we discussed previously, the dimensional reduction of kinetic term for the field strength \( F_n = dA_{n-1} + \cdots \) will now generate a cosmological term in the lower-dimensional theory, which consequently will admit domain-wall solutions.

In this letter, we shall concentrate on an example of particular interest, namely compactifications of \( D = 11 \) supergravity, or M-theory. Since this has just a 4-form field strength, it implies that we can consider either generalised Kaluza-Klein reductions of the above kind on 4-manifolds, or else standard Kaluza-Klein reductions on 7-manifolds, depending

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\(^1\) The consistency of the truncation even without the inclusion of the extra term is far from obvious in generic compactifications. The \( S^1 \) compactification is rather special in that the truncation to the \( z \)-independent harmonics on \( S^1 \) is guaranteed to be consistent, since one is setting all the non-singlets under the \( U(1) \) isometry group to zero, and so the retained singlet modes cannot, via the non-linear terms in the field equations, generate sources for the non-singlet modes in the higher-dimensional equations of motion. By contrast, in a compactification on a manifold \( M_n \) such as K3, Calabi-Yau, or a Joyce manifold, the non-linear terms involving products of harmonic forms on \( M_n \) would in general be expected to generate higher harmonics that would act as sources for fields that have been set to zero in the truncation. Indeed this would be the case for such a compactification of a generic theory, and it seems that it is supersymmetry that forbids the appearance of these source terms, and thus ensures the consistency of the truncation [6, 7], in supergravity theories.
on whether we make a solitonic ansatz or an elementary ansatz. The corresponding M-branes, \( i.e. \) a 5-brane or a membrane, vertically reduce to domain-wall solutions in 7 or 4 dimensions respectively.

First we shall look at the solitonic ansatz for the 4-form field strength. This will illustrate the new features arising from the need for the generalisation of the Kaluza-Klein reduction procedure. The simplest choice for the internal 4-manifold \( M_4 \) is the 4-torus. This case has in fact already been discussed in [3], from a slightly different viewpoint in which the \( D = 11 \) theory is first compactified on a 3-torus, giving rise to a set of field strengths in \( D = 8 \) that includes a 1-form coming from the 4-form of \( D = 11 \). Then, its associated axionic potential \( A_0 \) is subjected to the generalised Kaluza-Klein reduction of the form (4), generating the massive theory in \( D = 7 \). In our present discussion, the same result is achieved in one step, by making the generalised ansatz

\[
A_3(x,z) = mz_1 dz_1 \wedge dz_2 \wedge dz_3 + A_3(x) \\
+ A_2^i(x) \wedge dz_i + \frac{2}{3} A_1^{ij}(x) \wedge dz_i \wedge dz_j + \frac{1}{6} A_0^{ijk}(x) \wedge dz_i \wedge dz_j \wedge dz_k
\]

in \( D = 11 \), rather than \( A_0(x,z_4) = mz_4 + A_0(x) \) in \( D = 8 \). As has been shown in [3], the resulting theory in \( D = 7 \) has a topological mass term for \( A_3 \), ordinary mass terms for the four Kaluza-Klein vectors, and a cosmological term. The remaining six 1-form potentials, and four 2-form potentials, which come from \( A_3 \) in \( D = 11 \), are massless, as are the six 0-form potentials coming from the metric. The four 0-form potentials coming from \( A_3 \) in \( D = 11 \) are eaten when the four Kaluza-Klein vectors become massive.

This massive theory in \( D = 7 \) is maximally supersymmetric; namely it has the same number \( N = 2 \) of supersymmetries as the usual massless \( D = 7 \) maximal supergravity. However, unlike a normal Poincaré or de Sitter supergravity, it does not admit any solution, such as Minkowski or anti-de Sitter spacetime, that preserves all the supersymmetry. In fact, its analogous “natural” ground state is the domain wall solution that we mentioned previously. This can be seen at the level of the dimensional reduction of solutions by noting that in the above generalised reduction, the 5-brane solution in \( D = 11 \) is vertically reduced to a domain wall in \( D = 7 \). The metric in \( D = 11 \) is given by (1), with \( H \) chosen to be harmonic in just a 1-dimensional subspace of the 5-dimensional transverse space, \( i.e. \)

\[
 ds_{11}^2 = (1 + m |y|)^{-1/3} dx'' dx' \eta_{\mu \nu} + (1 + m |y|)^{2/3} (dy'^2 + d\bar{s}_4^2) ,
\]

where \( d\bar{s}_4^2 \) is the metric on the compactifying 4-torus. The 4-form field strength is given by \( F_4 = m \Omega_4 = m dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \), which is consistent with the ansatz (5) for the
generalised dimensional reduction of the 3-form potential. The domain wall preserves one
half of the supersymmetry of the $D = 7$ massive supergravity.

We can now easily extend the above discussion to the case where $d^2z'^2$ is the metric
on any Ricci-flat compactifying 4-manifold. The metric (7) remains a solution for the 11-
dimensional supergravity [8]. We shall consider the example of a K3 compactification in
some detail. Clearly the solution (7) again describes a domain wall after reduction down to
$D = 7$. Our principal interest is to investigate the structure of the supergravity theory in
$D = 7$ that we obtain by making the generalised dimensional reduction on K3. As in the
previous toroidal compactification, so also in this case the generalised dimensional reduction
on K3 will give a theory with the same supersymmetry as that given in a standard Kaluza-
Klein reduction on K3. Thus in this case, we will obtain a massive $D = 7$ theory with
$N = 1$ supersymmetry. The analogue of the generalised ansatz (6) in this case will be

$$A_3(x, z) = m\omega_3 + A_3(x) + A_1^a(x) \wedge \Omega_2^a(z), \quad (8)$$

where $d\omega_3 = \Omega_1$ is the volume form of K3, and the summation in the final term is over
the 22 harmonic 2-forms $\Omega_2^a$ on K3. Note that in the K3 compactification, there are no
1-form potentials coming from the metric, since there are no harmonic 1-forms on K3, unlike
on the 4-torus. There are, however, a total of 58 massless scalars coming from the metric,
corresponding to the 57 varieties of volume-preserving Ricci-flat deformations, together with
the overall scale deformation. Upon substitution of the ansätze into the $D = 11$ Lagrangian,
we find that the kinetic term for $F_4$ reduces to terms including a cosmological term, while
the $F_4 \wedge F_4 \wedge A_3$ term in $D = 11$ gives rise to a topological mass term for $A_3$ in $D = 7$. The
$D = 7$ Lagrangian has the form

$$\mathcal{L} = eR - \frac{1}{2}e (\partial \phi)^2 - \frac{1}{2}m^2 e^{-4\sqrt{10}\phi} - \frac{8}{185} e^{-4\sqrt{10}\phi} F_4^2$$
$$\quad + \frac{1}{288} m \epsilon^{MNPRST} F_{MNPO} A_{RST} + \mathcal{L}_{\text{matter}}, \quad (9)$$

where $\phi$ is the scalar field associated with the scaling mode of K3, and $\mathcal{L}_{\text{matter}}$ represents the
Lagrangian for the 57 scalars and 22 vectors of the matter multiplets, and their couplings
to the supergravity multiplet. If we truncate out the matter multiplets, the resulting $N = 1$
pure massive supergravity presumably coincides with the topologically massive theory
obtained in [9].

It is of interest to see what happens if we take the above topologically massive $D = 7$
$N = 1$ supergravity plus matter, and compactify it by one further dimension, using the
standard Kaluza-Klein ansatz on a circle. The resulting theory in $D = 6$ has a cosmological
term, and an off-diagonal mass term involving a 2-form and a 3-form field strength. This theory is dual to the one obtained by performing a K3 compactification of the type IIA theory in $D = 10$, where the 3-form potential takes the generalise form (8). Put another way, this means that an ordinary $S^1$ reduction of M-theory followed by a generalised K3 reduction gives a theory in $D = 6$ that is equal to the one obtained by first making a generalised K3 reduction of M-theory, followed by an ordinary $S^1$ reduction. Although this would obviously be true if the reductions were all of the standard Kaluza-Klein kind, this is not as trivial as it sounds in the present case, since if instead of K3 we used $T^4$ for the generalised reduction, then the order in which the ordinary and the generalised reduction steps are performed matters. In fact in general, reductions of the ordinary and the generalised forms do not commute. This was observed in the case of $S^1$ reductions in [3].

Now we shall consider the situation where the 4-form field strength carries an electric charge instead, giving a membrane solution and an 8-dimensional transverse space. We wish to perform a vertical dimensional reduction to a domain wall in $D = 4$, by taking the harmonic function governing the solution in $D = 11$ to depend on only one of the eight transverse coordinates, giving

$$ds_{11}^2 = (1 + m|y|)^{\frac{2}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + (1 + m|y|)^{\frac{1}{3}} (dy^2 + d\tilde{z}_7^2).$$

In this electric case, the potential $A_3$ for the membrane solution in $D = 11$ is given by $A_3 = H e_3$, where $e_3 = dx^0 \wedge dx^1 \wedge dx^2$ is the volume form on the membrane world volume, and $H = 1 + m|y|$. Thus unlike the vertical reduction of the 5-brane that we discussed previously, here the 3-form potential is independent of the compactifying coordinates of the 7-metric $d\tilde{z}_7^2$. Correspondingly, it is not obligatory in this case to consider a generalised Kaluza-Klein ansatz for the dimensional reduction of the $D = 11$ theory to $D = 4$ in order to obtain a domain wall solution. The reason why the standard Kaluza-Klein reduction is able to give a theory that admits domain wall solutions is that now we will end up in $D = 4$ with a theory that includes a 4-form field strength. The relevant terms in the $D = 4$ Lagrangian are of the form

$$\mathcal{L} = eR - \frac{1}{2} e (\partial \phi)^2 - \frac{1}{48} e e^{\phi} F_4^2.$$  

This admits an electric membrane solution (i.e., a domain wall) of the standard $p$-brane type. Note that one could dualise $F_4$ to a “0-form field strength,” or cosmological term,

\footnote{Generalised reductions of M-theory to $D = 4$ will be discussed in [13].}
giving the Lagrangian
\[ \mathcal{L} = eR - \frac{1}{2} e (\partial \phi)^2 - \frac{1}{2} e m^2 e^{-a \phi}, \]  
where \( m \) is the constant of integration coming from solving the equation of motion for \( F_4 \).

We can recognise this dualised form of the Lagrangian as being of the same kind, with a cosmological term, as the \( D = 7 \) Lagrangian that we obtained earlier by generalised Kaluza-Klein reduction. Indeed, it was observed in [2] that an alternative way of understanding the generalised reduction procedure was by first dualising the 1-form field strength in \( (D + 1) \) dimensions to a \( D \)-form field strength, then performing a standard Kaluza-Klein reduction to \( D \) dimensions, and finally dualising the resulting \( D \)-form field strength to a 0-form field strength, giving the cosmological term. It should be emphasised, however, that the Lagrangian (12) is not completely equivalent to (11): in the former, since \( m \) is a given fixed parameter in the Lagrangian, it does not admit a Minkowski ground state if \( m \) is non-zero, and conversely, it does not admit the domain-wall ground state if \( m \) is zero. On the other hand, the Lagrangian (11) admits both the Minkowski and the domain wall ground states.

The seven-dimensional internal metric \( d\bar{s}_7^2 \) may be taken to be any Ricci-flat metric. The simplest choice is the 7-torus, which was discussed in [3], but we may instead consider other possibilities, such as \( K3 \times T^3 \) (which was discussed in the context of the Minkowski vacuum state in [10]), \( Y \times S^1 \), where \( Y \) is any 6-dimensional Calabi-Yau space, and \( J \), where \( J \) is any 7-dimensional Joyce manifold. These last cases are compact Ricci-flat manifolds with \( G_2 \) holonomy [11]. The \( K3 \times T^3 \), \( Y \times S^1 \) and \( J \) compactifications will give respectively \( N = 4 \), \( N = 2 \) and \( N = 1 \) supergravity in \( D = 4 \). By including the 4-form field strength in \( D = 4 \), all these supergravity theories admit domain wall ground states as well as Minkowski ground states. The domain wall ground states are the vertical dimensional reductions of the membrane in \( D = 11 \).

The counting of fields in \( D = 4 \), determined from the Betti numbers of the compactifying manifolds, is as follows. For the \( K3 \times T^3 \) compactification, we obtain 28 2-form field strengths and 134 scalars [10], together with the 4-form field strength that can support the domain wall solution. For a \( Y \times S^1 \) compactification, we get \((b_2 + 1)\) 2-form field strengths, \((2b_3 + b_2 + 1)\) scalars and the 4-form, where \( b_2 \) and \( b_3 \) are the Betti numbers of the Calabi-Yau manifold \( Y \). For the \( J \) compactifications, we get \( b_2 \) 2-form field strengths, \( 2b_3 \) scalars, and the 4-form, where \( b_2 \) and \( b_3 \) are the Betti numbers of the Joyce manifold \( J \). In counting the scalars in this last example, we have made use of results in [12] for the counting of Lichnerowicz zero modes on manifolds of exceptional holonomy.

In summary, in this letter we have studied the vertical dimensional reduction of \( p \)-brane
solutions to domain walls in $D = p + 2$. In particular, we focussed on the domain wall solutions in 7 and 4 dimensions which are the vertical reductions of the solitonic 5-brane and elementary membrane in M-theory. These reductions can be achieved by taking the harmonic function governing the M-brane in $D = 11$ to be independent of 4 or 7 transverse-space directions respectively, and compactifying these directions on a Ricci-flat 4-manifold or 7-manifold. We also studied how M-theory is dimensionally reduced on these manifolds. In order for the dimensionally-reduced theory in $D = 7$ to be able to admit, as it must, the solitonic domain wall solution coming from the 5-brane, it is necessary to generalise the usual Kaluza-Klein ansatz to allow the 3-form potential to have an additional term whose exterior derivative is a constant multiple of the volume form of the compactifying 4-manifold. This does not spoil the consistency of the Kaluza-Klein truncation, since the 3-form potential always enters in the equations of motion via its exterior derivative. This generalised dimensional reduction gives rise to a supergravity theory in $D = 7$ with a topological mass term and a cosmological term, together possibly with other mass terms depending on the choice of compactifying 4-manifold. The supersymmetry of the massive theory also depends on the choice for the 4-manifold; for example we get $N = 2$ for the 4-torus, and $N = 1$ for K3.

The situation for the 4-dimensional theory obtained by compactification on a Ricci-flat 7-manifold is different. In this case, the standard Kaluza-Klein ansatz for the dimensional reduction is adequate. The reason why there is nevertheless a domain wall solution is that there is a 4-form field strength in $D = 4$, which is dual to a cosmological term. The major difference in this case is that the dimensionally-reduced theory admits both domain wall solutions and the usual $p$-brane solutions, as well as a Minkowski spacetime vacuum solution. By contrast, in the generalised Kaluza-Klein reduction to $D = 7$, the resulting massive theory is different from the usual massless theory obtained by the standard reduction; the former admits only the domain wall solution, while the latter admits only the usual $p$-brane solutions and the Minkowski vacuum. Finally, we remark that yet more general compactifications utilising the cohomology classes of the internal manifold are possible. These will be the subject of a forthcoming publication [13].

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References


