A Dynamical Reduction Theory of Einstein-Podolsky-Rosen Correlations and a Possible Origin of CP Violations.

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Abstract

We show that there is essentially only one way to construct a stochastic Schrödinger equation that gives a dynamical account of the transformation of entangled into factorized states and is consistent both with quantum mechanics and required symmetries. The noisy, non-linear term is a unimodular scalar multiple of the time reversal operator that must be present whenever a Hamiltonian term in the Schrödinger equation can distinguish the factorized constituents of an entangled state. The dynamical mechanism involved in the transformation of entangled into factorized states provides an explanation for the fact that Einstein-Podolsky-Rosen correlations appear in a time determined by the response of the measuring device and independent of the distance between the particles. The dependence on the response time of the measuring device may be testable through a delay in observing the collapse of mesoscopic “Schrödinger cat” states in ion traps. It is further shown that there are situations where a two-particle interaction can induce a non-linear term by virtue of coupling to decay modes that distinguish factorized constituents of an entangled state. We show that this should happen in the neutral K-meson system where the entangled $K_L$ state is pushed slightly in the direction of a factorized constituent ($K_0$ or $\bar{K}_0$) as a consequence of the fact that these can be distinguished via the sign of the charged lepton in a semi-leptonic decay mode. The result is a CP violation that is within 20% of the experimental value.

1. Introduction

Although the most vexing conceptual problems in the foundations of quantum mechanics arise from the manner in which the theory deals with
the measurement process, our computations are normally insensitive to these problems. We collapse wave functions with a pencil and paper, confident that we are describing situations in which no future interference is possible[1]. But a small group of physicists have always had the uneasy feeling that they were missing something interesting by sweeping the collapse process under the rug. That group is now becoming larger as developments in mesoscopic physics such as the study of single-atom wave functions with trapped ions make direct experimental probes possible.

A dynamical reduction theory (DRT), is an extension of quantum mechanics that attempts to account for the collapse of the wave function. DRT has a long history dating back to the earliest days of quantum mechanics. But serious conceptual problems (such as Schrödinger’s cat) discouraged its development. Only in recent years has it become clear that we can turn the Schrödinger equation into a stochastic differential equation with a non-linear noise term that will reproduce the predictions of quantum mechanics while collapsing wave functions rapidly in the manner that we observe. The resulting enthusiasm is revealed by J.S. Bell’s remark in 1990 that the stochastic modification of quantum mechanics is the most important new idea in the foundations of quantum mechanics in his professional lifetime[2].

Although DRT’s have a non-linear term, they are not vitiated by the criticisms applicable to non-linear forms of the non-stochastic Schrödinger equation, namely Peres’[3] demonstration of violations of the second law of thermodynamics or Gisin’s[4] demonstration of superluminal telephony. The reason is that both of these arguments depend on assumptions made about linear behavior on density matrices which need not hold in DRT’s.

Following pioneering work by Bohm and Bub[5], the most successful DRT’s to date are the spontaneous localization theories of Ghirardi et al[6, 7], and Pearle[8]. They are, however, ad hoc and phenomenological, requiring the introduction of new constants of nature. Predictions are made concerning the spontaneous ejection of electrons from atoms with resulting x-ray generation, and experimental constraints have been put on the parameters[9].

In this paper we shall develop a DRT differing fundamentally from spontaneous localization theories. The differences will be found in three basic questions the theory must answer:
(a) The trigger: What initiates the collapse?

(b) The preferred basis: What determines the Hilbert space directions along which collapse occurs?

(c) The noise: What produces it, and what is its fluctuation pattern?

Our point of departure from spontaneous localization theories is the observation that the only macroscopically distinguishable states whose collapse we have to explain are those that we know how to produce, or at least can conceive of producing in a gedanken experiment. Since all such experiments involve entanglement, we need only insure that the collapse mechanism apply to entangled states. Thus rather than postulating one universal collapsing interaction (such as the “hits” of spontaneous localization theory”) that operates on all states, we shall adopt the following hypothesis:

*Any Hamiltonian that can distinguish the factorized constituents of an entangled state induces a corresponding non-linear interaction that can collapse the state into those constituents.*

The strength and preferred basis of the non-linear term will be inherited from the Hamiltonian that induces it. Since the theory will now have no new constants of nature to set scales of length and time, it will be up to the dynamics itself to insure that only macroscopically distinguishable superpositions collapse rapidly. We shall refer to this type of theory as an *induced non-linearity* (INL) theory.

By allowing ordinary Hamiltonians to induce non-linear terms we are, of course, exposing the theory to rejection if a conflict is discovered with the well-established and very accurate predictions of standard quantum mechanics. The INL theory will be partially protected by its modesty, i.e. the non-linear term has no effect on factorized states. The danger to the theory as well as its opportunities for new predictions will be found where there is competition between the linear theory controlling the dynamics of an entangled state and an induced non-linear term that may modify the dynamics. The following examples are intended to show that (1) such situations are not encountered in ordinary applications of quantum mechanics, but that (2) interesting situations nonetheless arise in which the theory can be tested.

(1) The simplest type of measurement:
Consider a Stern-Gerlach experiment in which one measures the spin of an electron in an atom by means of an inhomogeneous magnetic field. The entanglement of the spatial wave function of the atom with the spin state of an electron is effected by the inhomogeneous field. If $|\phi_o\rangle$ denotes the initial state, the transformation is:

$$|\phi_o\rangle \rightarrow \alpha |\text{up}\rangle |\uparrow\rangle + \beta |\text{down}\rangle |\downarrow\rangle.$$  \hspace{1cm} (1a)

Here $|\text{up},\text{down}\rangle$ indicate the time dependent spatial locations of the atom, and $|\uparrow,\downarrow\rangle$ indicate orthogonal spin states of the electron in the direction determined by the orientation of the magnet. After sufficient time $\tau$, which depends inversely on the strength of the field, the spatial wave functions associated with $|\text{up},\text{down}\rangle$ no longer overlap and ultimately become sufficiently well separated that one can recognize the difference between them macroscopically. It is this separating action of the field on the entangled state that recognizes the difference between its constituents, induces the non-linear term, and triggers the collapse. The preferred basis consists of the two states on the right of (1a). There is no significant competition between the collapsing mechanism and the linear dynamics because the matrix elements of the linear Hamiltonian between the spatially separated constituents are already negligible when the non-linear term is active.

(2) Mesoscopic states created with a Be$^+$ ion trap,[10]:

Although the interaction is quite different, the situation is schematically similar to example (1). An initial zero point harmonic oscillator state $|\phi_o\rangle$ of a single ion in the trap is entangled by a pair of Raman lasers. The $|\text{up},\text{down}\rangle$ of example (1) will now be classical-like coherent states, $|\alpha e^{\pm i\phi/2}\rangle$, involving gaussians separated by as much as 80nm. The $|\uparrow,\downarrow\rangle$ of example (1) will be internal atomic states. Now, however, $\phi$ is a controllable phase through which one can sweep experimentally. Using a detection beam that resonantly couples $|\downarrow\rangle$ to a an excited state producing ion fluorescence, one measures the probability $P_{\downarrow}(\phi)$ for the internal state to be $|\downarrow\rangle$, and thereby induces collapse.

As the experiment was recently described[10], the radiative linewidth of the fluorescence that signals collapse is $\approx 20 MHz$ and hence the collapse process is so fast ($\approx 5 \times 10^{-8}$ s) that, as in example (1), there is no chance for the non-linear term to compete with the linear term, governing the internal
dynamics of the entangled state. If it becomes possible to design the ion trap to speed up the evolution of the entangled state, or if one can detect the internal state with a narrower fluorescent transition $\gamma$, the INL theory (see below) predicts an observable change in $P_1(\phi)$, namely a small displacement with respect to $\phi$ associated with a time-delay $\hbar/\gamma$ in the collapse.

(3) A typical EPR experiment:

The right side of (1a) is replaced by a state $|\Phi\rangle$ of two spin-1/2 particles, i.e., we have an entangled state already formed by some prior dynamical process. When one of the partners, say particle-1, encounters a measuring device the collapse process begins to operate immediately on the entangled state. The trigger is the Hamiltonian that recognizes the factorized constituents by an interaction with particle-1, and collapse must occur in the time $\tau$ required to distinguish the constituents. That Hamiltonian will determine a basis for particle-1 say $|\uparrow, 1\rangle, |\downarrow, 1\rangle$. The state $|\Phi\rangle$ then has a unique representation in the form:

$$|\Phi\rangle = \alpha |\uparrow, 1\rangle |a, 2\rangle + \beta |\downarrow, 1\rangle |b, 2\rangle$$

in which the two states on the right are orthonormal. (The two states $|a, 2\rangle, |b, 2\rangle$ are normalized but not necessarily orthogonal.) The collapse now goes to either of the two states on the right with probability $|\alpha|^2$ and $|\beta|^2$ respectively. Thus it is seen that the preferred basis for the collapse is once again determined by the magnet and does not depend on $|\Phi\rangle$. The collapse takes place when the two particles of the entangled state are far apart, so that matrix elements of the linear Hamiltonian between the factorized states are negligible during the collapse. Thus the linear Hamiltonian does not compete with the non-linear term.

The above examples show that under normal circumstances the collapse mechanism is either not there at all because the state is factorized, or it dominates the linear dynamics to the extent that there is no observable competition. Let us therefore try to look for situations where the linear dynamics induces a slow process through which the factorized constituents are recognized, i.e., slow enough that the linear dynamics has a chance to transform the entangled state significantly while the induced non-linear term is acting.

This may happen, for example, if the internal interaction has matrix
elements connecting the factorized constituents to distinct decay channels.
The following is a case in point that will be discussed in detail below:

(4) Decay of neutral K-mesons:

In the standard model the CP eigenstates $|K_{1,2}\rangle$ are the entangled states

$$|K_{1,2}\rangle = 2^{-1/2}(|K_0\rangle \pm |\bar{K}_0\rangle),$$

in which $|K_0\rangle = |s\rangle|d\rangle$, $|\bar{K}_0\rangle = |\bar{s}\rangle|\bar{d}\rangle$, (2)

where $s, d, \bar{s}, \bar{d}, s$ are quark and anti-quark flavors. In the absence of CP violation the states $|K_{1,2}\rangle$ would be energy eigenstates of the weak interaction Hamiltonian, the strength of which can be measured by the mass difference $\delta m$ between these two states. Now the weak interaction Hamiltonian is also able to recognize the two factorized constituents $|K_0\rangle$, $|\bar{K}_0\rangle$ through the semi-leptonic decay modes of $K_1$ or $K_2$. Here the sign of the charged lepton will determine whether the decay involved the conversion of an $s$ or $\bar{s}$ quark.

According to the INL theory this recognition of the factorized constituents of an entangled state must induce a non-linear term that will drive a collapse of the entangled state. The strength of this effect will be $\hbar/\tau$ where $\tau$ is the time it takes for the distinction to be recognizable. Since the branching ratio to semi-leptonic decay is negligible for $|K_1\rangle$ (where it is overwhelmed by the two-pion decay mode), it is essentially the lifetime of $|K_2\rangle$ (for which the branching ratio is 66% to the semi-leptonic mode) that sets the time-scale. The INL theory then obtains an induced non-linear term whose strength relative to the linear term is of order 0.001. The competition between the linear and induced non-linear term thus produces a small distortion of the $|K_2\rangle$ state so that the energy eigenstate, is not quite a CP eigenstate. In this way the INL theory produces a small CP violation which, considering the coarseness of the model, is surprisingly close to the experimental value.

One sees clearly in this example that the non-linearity is to be triggered by the linear Hamiltonian which is also to fix the strength ($\hbar/\tau$) and preferred basis ($|K_0\rangle$, $|\bar{K}_0\rangle$). Note that the linear interaction now dominates the non-linear term. However, the dynamics is such that the effect of the non-linear term shows up in a small but detectable way.

As we remarked earlier, the main criticism that has been leveled at various proposed DRT’s is that they are ad hoc. To avoid this criticism we shall, in the remainder of this introductory section, set forth the physical ideas that motivate the choices we will make below in constructing the INL theory.
The form of the noise:

Whereas a linear Hamiltonian can cycle an entangled state into a factorized state and back again, the collapse process is uni-directional. This does not mean that it is thermodynamically irreversible, however, for a pure state goes into a pure state, and there is no increase in entropy. To produce uni-directionality the dynamical transformation must be described by an operator whose domain is restricted to the entangled states, i.e. one that becomes singular on factorized states. The uni-directionality does not introduce an absolute arrow of time, but each measurement results in the deflection of a “pointer”, which assigns a direction to time in an unpredictable way. This suggests that there should be a random relative sign between the linear and non-linear terms, and so, if a convention is adopted for the linear term, we should introduce a random sign for the non-linear term. It is quite remarkable that such a simple form of noise will turn out to reproduce the statistical distribution of outcomes predicted by the recipe of conventional quantum mechanics.

A much subtler problem has to do with the intervals in which the sign fluctuation is to occur. It is important that these fluctuations not occur so frequently as to slow down the collapse process. In order to avoid introduction of a new scale-setting constant to define the interval, we must let the stochastic process itself determine when these fluctuations are to occur. In brief the idea is as follows: One can think of the noise fluctuation as deciding whether one of two gamblers will win in a fair game of chance. In spontaneous localization theories fluctuations occur in such a way that the gamblers are betting a random amount in every play until one of them loses his entire fortune. This game, known as the “gambler’s ruin”, can result in a game of long duration. However, when the noise is merely a sign fluctuation, it is possible to play the game as “double or nothing” with the stake in every play being that of the player with the smaller remaining fortune. The average length of such a game is just two plays, and so, without having to introduce a new time constant, the collapse is not significantly delayed.

Form of the non-linear term:

Because the INL theory only collapses entangled states we can deduce another important property that the theory will be required to have. Suppose we partition the Hilbert space into orthogonal “cells”, each of which is a
connected set of states that can be collapsed by the non-linear map into the same set of factorized states. If the non-linear map is to act independently on states belonging to different cells, it must be extended by linearity to superpositions of states belonging to different cells. But this requires that the map be *homogeneous*, i.e. it must retain a vestige of linearity, namely linearity with respect to multiplication by scalars.

We shall, of course, have to examine carefully what the map does on approaching the boundary of a cell. In order to join the entangled state before collapse to a factorized state after collapse that satisfies the ordinary Schrödinger equation, the singularity on the cell boundary will have to be weak enough that both a wave function and its time derivative change continuously. Since quantum mechanical probabilities are computed from a quadratic form in the wave function, it follows that probability must be conserved across the boundary. This suggests that the non-linear theory should also retain another property of the linear theory, namely that probability be conserved without the necessity of having to renormalize wave functions.

As is suggested by example (3) above, there is an intimate connection between the INL theory and the problem of EPR correlations. The INL theory is required to explain the rapidity of collapse of the wave function associated with an entangled state when a measurement is made. The existence of EPR correlations reveals the most baffling aspect of the collapse process: If the measurement is made on one of the particles of an entangled pair, the time required for the collapse is independent of the distance from the point of measurement to the other particle and depends only on the reaction time of the measuring device. While this means, in particular, that the collapse happens before there is time for a light signal to get from the point of measurement to the other particle, the speed of light plays no intrinsic role in the EPR problem. The problem of quantum non-locality, the violation of outcome independence [2], is not going to be resolved by producing a Lorentz covariant form of the theory although that may be desirable for other reasons.

There is, however, a type of symmetry that will “explain” what we observe and which we shall be able to implement in the INL theory: Suppose that in the absence of two-particle interactions in the Hamiltonian, the INL theory can be constructed in such a way that its form is invariant under arbitrary
unitary transformations of the form:

\[ \Psi \rightarrow \Psi' = (U^{(1)} \otimes U^{(2)}) \Psi, \] (3)

i.e. any unitary transformation that acts separately on the two particles. We shall say that such a theory enjoys \textit{one-particle unitary invariance}. Since space-time symmetries in a quantum theory are implemented by such transformations, we see that one-particle unitary invariance guarantees that \textit{the INL theory will preserve any space time symmetry enjoyed by the linear theory} whether it be Lorentzian or Galilean. It also means that \textit{the non-linear term will be unchanged in form if one applies a translation operator to just one of the two particles, and hence the non-linear term will produce an effect that is independent of the distance between the two particles.}

By displaying a collapse mechanism that is completely insensitive to the space-time symmetries of the linear theory, the INL theory essentially decouples it from the notion of “event” in space-time. Thus, in a sense, the INL theory dissolves the non-locality problem by asserting that the notion of locality is meaningful only to the extent that the linear theory is a good approximation to dynamics.

We are now ready to embark upon the construction of the INL theory. The above discussion has supplied very strong constraints. We list them here for convenient reference. As we implement each of them below the form of the theory will emerge:

\textbf{(A)} The non-linear interaction is uniquely determined by a Hamiltonian that distinguishes the factorized constituents of an entangled state.

\textbf{(B)} The non-linear map must be weakly singular on factorized states.

\textbf{(C)} The non-linear map must enjoy one-particle unitary invariance.

\textbf{(D)} The non-linear map must be homogeneous.

\textbf{(E)} The Schrödinger equation with the non-linear term must conserve probability.

\textbf{(F)} The noise consists of a randomly fluctuating sign of the non-linear term.

\textbf{(G)} The occurrence of a fluctuation must be determined by the stochastic process itself.

Because these constraints suffice to essentially fix the form of the theory,
and since we do not have to introduce any new constants, the INL theory will neither be ad hoc nor phenomenological.

2. The modified Schrödinger equation.

Since entanglement plays a special role in the INL theory, we begin by introducing a simple formalism in which the distinction between entangled and factorized states is clearly manifest. To keep the notation as simple as possible we will develop the essential features of the theory for the simplest entangled system consisting of a pair of spin-1/2 particles in which only the spin degree of freedom is explicitly indicated. After developing the theory in this context the generalization to more complex systems will be described.

The state of two spin-1/2 particles can be represented by:

$$|\Omega\rangle = C_{00}|0\rangle|0\rangle + C_{01}|0\rangle|1\rangle + C_{10}|1\rangle|0\rangle + C_{11}|1\rangle|1\rangle$$  \hspace{1cm} (4a)

with the $C_{ij}$ assuming complex values. Dependence on space coordinates is not indicated. We represent the state $|\Omega\rangle$ by the matrix of coefficients i.e.

$$|\Omega\rangle \rightarrow C \equiv \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix},$$  \hspace{1cm} (4b)

so that scalar products become

$$\langle \Omega' | \Omega \rangle = Tr(C'^\dagger C),$$  \hspace{1cm} (5)

and the normalization condition is:

$$\langle \Omega | \Omega \rangle = Tr(C^\dagger C) = 1.$$  \hspace{1cm} (6)

It follows from this (see Appendix A) that

$$0 \leq |\det C| \leq 1/2,$$  \hspace{1cm} (7)

and that $\det C = 0$ if and only if the state is a factorized state. At the other extreme $|\det C| = 1/2$ corresponds to completely entangled states.

Operators that act on the spin of particle-1 act on $C$ from the left, and those that act on particle-2 act on $C$ from the right. Thus the most general unitary transformations that act on the particles separately are of the form

$$C \rightarrow ACB,$$  \hspace{1cm} (8)
in which \( A, B \) are two-by-two unitary matrices. Under Lorentz transformation, for example, \( C \) is transformed into \( ACB \) in which \( A \) and \( B \) are the Wigner rotations associated with the Lorentz transformation. These may be different for the two particles and are momentum dependent. Since \( |\det(ACB)| = |\det C| \) for unitary \( A, B \) it follows that the properties of being a factorized or maximally entangled state are unitary invariant properties.

In this language the ordinary Schrödinger equation of the the system \( S \) is:

\[
\frac{dC}{dt} = -iH_1 C - iCH_2 + R_{12}(C)
\]  

(9)

where \( H_1 \) and \( H_2 \) are the one-particle Hamiltonians, and \( R \) is a linear, two particle interaction, e.g. a spin-spin interaction (see eq. (35)). Thus the \( H_j \)'s contain the kinetic terms and any interactions which the particle spins may experience with an external field. In the typical EPR experiment there will be no \( R \) term and one of the \( H_j \)'s contains the interaction with a Stern-Gerlach magnet which measures the spin of the particle.

To construct a dynamical reduction theory we add a term, writing

\[
\frac{dC}{dt} = \mathcal{M}(C) - iH_1 C - iCH_2 + R_{12}(C).
\]  

(10)

We refer to the added term as the non-linear term and the remaining terms as the Hamiltonian or linear terms. Our task is to find an appropriate form for the non-linear term.

Observe first that \( \mathcal{M}(C) \) must be singular on factorized states in order to avoid recycling a factorized into an entangled state. Now we note that \( C \to C^{-1} \) has the right sort of singularity, since the factorized states are characterized by \( \det C = 0 \). However when spins are transformed by unitary transformations we have \( C \to ACB \) but \( C^{-1} \to B^{-1}C^{-1}A^{-1} \). We can, however, arrange to have the requisite singularity and still have the linear and non-linear terms transform in the same way if we use the map \( C \to C^{\dagger -1} \) where \( \dagger \) is the hermitian conjugate. In fact if \( \nu \) is any real number we can take the map to be: \( \mathcal{M}(C) = |\det C|^\nu C^{\dagger -1} \), and we will then have \( \mathcal{M}(ACB) = A\mathcal{M}(C)B \). But we saw above that the map must be homogeneous, and it is easy to check that for \( n \times n \) matrices this requires \( \nu = 2/n \). Thus for the present situation with \( n = 2 \) we are led to the map:

\[
C \to \hat{C} \equiv |\det C|C^{\dagger -1}.
\]  

(11)
This map has remarkable properties.

(i) On its domain, which consists of non-singular matrices, \( C \rightarrow \tilde{C} \) is homogeneous, maps unitary matrices into themselves, and is an automorphism of the group of non-singular matrices, i.e.

\[
C_1 C_2 = \tilde{C}_1 \tilde{C}_2, \text{ if } \det C_1 \neq 0, \det C_2 \neq 0. \quad (12a)
\]

so that in particular

\[
A \tilde{C} B = A \tilde{C} B, \text{ for unitary } A, B \quad (12b)
\]

This is the indispensible property needed to insure the one-particle unitary invariance of the theory to which we referred in the introduction.

(ii) Consider any set \( S \) of entangled states that have the same so-called “Schmidt normal form”, i.e. states of the form \( \gamma_1 |0\rangle |0\rangle + \gamma_2 |1\rangle |1\rangle \) with \( \gamma_1 \neq 0 \) and \( \gamma_2 \neq 0 \) in which \( |0\rangle, |1\rangle \) and \( |0\rangle', |1\rangle' \) are arbitrary orthonormal bases for the two particles. Then \( C \rightarrow \tilde{C} \) leaves the set \( S \) invariant. To see this observe that the states of \( S \) are of the form \( C = U^\Gamma V \) where \( U \) and \( V \) are fixed unitary matrices (determined by the bases) and \( \Gamma \) ranges over all diagonal matrices such that neither of its diagonal elements \( \gamma_1, \gamma_2 \) vanish. The asserted invariance property follows from (12b), i.e. \( \tilde{U}^\Gamma \tilde{V} = U^\Gamma V \), and a direct calculation showing that \( \tilde{\Gamma} \) is diagonal and has no vanishing diagonal elements if this is true for \( \Gamma \). The significance of this property will become clear below. It assures that the collapse takes place within sets of the form \( S \) which we shall call “cells”. The homogeneity of the map will then permit the extension of the non-linear map to linear combinations of states belonging to different cells. For example the state on the right side of (1b) can be written as such a linear combination by expressing \( |a, 2\rangle, |b, 2\rangle \) in terms of any orthogonal basis and the predictions of the theory will be independent of the choice of that basis.

(iii) \( C \rightarrow \tilde{C} \) is a duality, i.e.

\[
\tilde{C} = C. \quad (13)
\]

and is related in a simple way to time-reversal. To see this recall that the time-reversal operator \( T \) is an anti-unitary map that can be defined on spin states by:

\[
T(\lambda |0\rangle + \mu |1\rangle) = \lambda^* |1\rangle - \mu^* |0\rangle \quad (14)
\]
from which we deduce
\[ \tilde{C} = e^{-i\text{arg} (\det C)} T(C). \]  
(15)

Since \( T \) is non-singular we see that the singularity of the map arises from the phase factor and is merely an \textit{ambiguity} in the phase when the determinant goes to zero. This is a very important conclusion because, as we will see, such a mild singularity causes no problems when we try to match up solutions across the domain boundaries of the map.

We must now put in the dependence on the Hamiltonian that induces the non-linear term. Its task is to define a basis along which the collapse takes place and provide the energy distinguishing the different collapsed states. Let us try to do this in the simplest possible way and guess the form of our modified Schrödinger equation to be:
\[ \frac{d\tilde{C}}{dt} = \Lambda_1 \tilde{C} + \tilde{C} \Lambda_2 - iH_1 C - iCH_2 + R_{12}(C), \]  
in which the matrices \( \Lambda_j \)'s are going to be determined by the inducing Hamiltonian.

Conservation of probability now imposes a strong constraint. Thus from (6) we must have:
\[ \frac{d}{dt} \text{Tr}(C^\dagger C) = 0. \]  
(17)
This will be satisfied if the \( \Lambda_j \)'s are hermitian and
\[ \text{Tr}(\Lambda_1 + \Lambda_2) = 0 \]  
(18).

Next consider the typical EPR experiment with a Stern-Gerlach magnet in which the linear dynamics recognizes the factorized constituents by acting on particle-1. Thus we set \( \Lambda_2 = 0 \). Since \( \Lambda_1 \) is hermitian and has zero trace by (18) we can choose a basis for particle-1 in which \( \Lambda_1 \) is diagonal and of the form:
\[ \Lambda_1 = \eta/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  
(19)
It is clear that the parameter \( \eta \) which has the dimensions of energy, should be \( \hbar/\tau \) where \( \tau \) is a suitable measure of the time it takes the magnet to resolve the spins. A dimensionless factor of order unity can, of course, be introduced that would have to be determined by experiment.
To test this hypothesis and to obtain guidance on introducing the noise we turn in the next section to an examination of the solution of (16) with this form of \( \Lambda_1 \).

4. Solution of the equation with a non-linear term only.

Consider the case where \( R_{12}(C) = 0 \). It follows from the properties (12) that we may, as in the linear Schrödinger equation, transform away the one-particle linear terms in (16) by going to the interaction picture, i.e.

\[
C \rightarrow e^{-iH_1 t} C e^{-iH_2 t}.
\]

Let us further assume that we only make a measurement on particle-1 which we represent by (19). Equation (16) now has the form

\[
dC/dt = \Lambda_1 \tilde{C},
\]

subject to the requirement \( \det C \neq 0 \) defining the domain of the map.

Now note that the right side transforms states of the form \( \gamma_1 |0\rangle|0\rangle + \gamma_2 |1\rangle|1\rangle \) into one another (diagonal \( C \)) and also transforms states of the form \( \gamma_1 |0\rangle|1\rangle + \gamma_2 |1\rangle|0\rangle \) into one another (anti-diagonal \( C \)). An arbitrary initial state can be decomposed into a linear combination of these two types. From the homogeneity of the map we can then solve (21) for the two cases separately and combine the results by linearity. Since the equation has the same form for both we now examine (21) when the initial \( C \) is diagonal. Let

\[
y_0 = |C_{00}|^2, \quad y_1 = |C_{11}|^2,
\]

which are the probabilities at any time of finding the system in the state \( |0\rangle|0\rangle \) and \( |1\rangle|1\rangle \), respectively. We take the initial values to be

\[
y_0(0) = \alpha, \quad y_1(0) = 1 - \alpha.
\]

We then obtain from (21) with (11) and (19):

\[
\frac{dy_0}{dt} = \eta \sqrt{y_0 y_1}, \quad \frac{dy_1}{dt} = -\eta \sqrt{y_0 y_1}.
\]

Let us extract the solution by a method that will readily generalize when we discuss higher spin. Define a dimensionless variable \( \tau \):

\[
d\tau = \eta \sqrt{y_0 y_1} dt.
\]
The solution now has a different form for \( \eta > 0 \) and \( \eta < 0 \). For \( \eta > 0 \) we have

\[
y_0 = \alpha + \tau, \quad y_1 = 1 - \alpha - \tau,
\]

whence for \( 0 \leq \tau < 1 - \alpha \) we have

\[
|\eta|t = \int_0^\tau d\tau[(\alpha + \tau)(1 - \alpha - \tau)]^{-1/2} = \arcsin(1 - 2\alpha) - \arcsin(1 - 2\alpha - 2\tau).
\]

from which:

\[
\tau = \frac{1}{2}\{1 - 2\alpha + \sin(|\eta|t - \arcsin(1 - 2\alpha))\},
\]

For \( \tau = 1 - \alpha \), i.e. for

\[
t = t_0(\alpha) = |\eta|^{-1}(\pi/2 + \arcsin(1 - 2\alpha)),
\]

one sees from (26) that the right sides of (24) vanish, and the process terminates with \( y_0 \) having the value 1 and \( y_1 \) having the value 0. One may check that \( y_0 \) and \( dy_0/dt \) are continuous at \( t = t_0 \), but that \( d^2y_0/dt^2 \) changes discontinuously from \((\eta/2)^2\) to zero at this point. This is a manifestation of the singularity of the mapping upon arrival at the factorized states. But, as we have anticipated, the singularity is so mild that we have the required continuity in the function and its first derivative that enables us to extrapolate across the temporal boundary, joining the factorized state to a solution of the linear Schrödinger equation.

Next observe that if the sign of \( \eta \) is reversed one obtains the same solution as in (26) but with the sign of \( \tau \) reversed. Thus the process will now terminate when \( \tau = \alpha \) and the factorized state will have \( y_0 = 0 \) and \( y_1 = 1 \). If the initial state is completely entangled, i.e. if \( \alpha = 1/2 \), the factorization in both cases requires time

\[
t_0 = \frac{\pi}{2\eta}.
\]

Thus the time scale for completion of the process is determined by the parameter \( \eta \) of the measuring device and has nothing to do with the time required for a signal to get from one particle to the other.

5. Form of the noise.
We are now ready to put in the noise and turn (16) into a stochastic
differential equation. The solution obtained above immediately suggests a
simple and natural way to do this: We observe that a fluctuating sign of the
parameter $\eta$ will alter the outcome. Moreover this has an intuitive physical
justification: From (15) the map $C \rightarrow \hat{C}$ is a scalar multiple of the time-
reversal operator. Measuring devices break time reversal symmetry by having
a pointer deflect in one direction rather than another. Thus a randomly
fluctuating sign of $\eta$ expresses the ambiguity in the direction of time until a
measurement is registered.

But there are many possible hypotheses we can make about the *intervals*
in which the random sign fluctuations occur: are they finite or infinitesimal,
sporadic or regular? To decide among these we can use two basic tests that
must be passed by a satisfactory model: (i) The predictions of quantum
mechanics must be recovered. (ii) The delay in producing the factorization
because of oscillations in the sign must not be such as to destroy the rapidity
of achieving factorization, i.e. the time scale should remain of order $1/\eta$.

It is convenient to consider the noise as a function of the parameter $\tau$ in
(25) (which is invariant under Lorentz boosts). The simplest choice would
be to let the sign fluctuate randomly in any interval $\delta\tau$. Let us see that this
will pass test (i) but will not pass test (ii):

In this model, according to (26), $y_0$ increases and $y_1$ decreases by the
same amount $\delta\tau$ or vice versa according to the sign choice. If this fluctuates
randomly we will generate a random-walk with boundary problem (sometimes
called the “gamblers ruin”), [11] i.e. one in which the walk terminates
when $y_0$ reaches either 0 or 1. In our situation the probability of a left move
and a right move in the $\tau$ variable are equal. Suppose that $y_0$ is an integer
multiple of $\delta\tau$. Then if $p(y_0)$ is the probability of ending with $y_0 = 1$ and
$y_1 = 0$ starting from some given value $y_0$ one sees that:

$$p(y_0) = \frac{1}{2}p(y_0 + \delta\tau) + \frac{1}{2}p(y_0 - \delta\tau)$$

(31)

which implies that $p$ is linear in its argument. Since we must have $p(0) = 0$, $p(1) = 1$ we obtain $p(y_0) = y_0$.

Thus for randomly fluctuating sign we deduce that if $y_0$ starts off with
value $\alpha$, the probability that the process will terminate with $y_0 = 1$ is $\alpha$. *But*
this is just the square of the amplitude of the final state in the initial state and so is just the prediction made by conventional quantum mechanics.

But let us now see that a difficulty with this model is revealed when we ask for the duration of the process. The fluctuating sign means that the factorization is delayed. If we divide up the τ interval [0, 1] into segments of size δτ, then, starting from y₀ = α, one can show[11] that the expected number of moves required to end the process will be α(1 − α)/(δτ)^2. But from (25) the time to traverse an interval δτ is linear in dτ and hence an arbitrarily small subdivision will sooner or later delay the factorization to the point that (30) is no longer a reliable estimate.

To overcome this difficulty we must have a fluctuation scheme that does not become arbitrarily rapid. However there is no natural choice of sub-interval in τ. Indeed, since we had to assume that y₀ was an integer multiple of δτ we would have to let δτ become arbitrarily small. A way out is to allow a fluctuation at those times that are singled out by the process itself. Thus suppose we think of y₀ and y₁ as the fortunes of two gamblers who play “double-or-nothing” with the bet in each play being the current fortune of the player who has less. Let a sign fluctuation of η take place whenever there is a play and let y₀ win if η is positive and lose if it is negative. In this model of the noise the fluctuations are finite but not sporadic. Let us see that both tests are now passed:

First observe that for z ≤ 1/2 the probability p(z) of obtaining y₀ = 1 will satisfy

\[ p(z) = \frac{1}{2} p(2z) \]  

(32a)

which has the solution \( p(z) = z \) as before. Thus test (i) is passed. But now the average number of moves required to end the game is

\[ 1/2 + 2(1/4) + 3(1/8) + \cdots = 2, \]

(32b)

so the expected time for termination is merely doubled from (30). Thus the mean collapse time is now given by

\[ t₀ = \pi/η \]

(33),

and we see that test (ii) is also passed.
We now have a theory of EPR correlations in systems of two spin-1/2 particles that reproduces the quantum mechanical predictions and happens in a time characteristic of the measuring device.

6. Competition between linear and non-linear terms

A major goal of the theory is universality, i.e. that the non-linear terms be associated with linear terms that induce them. This suggests the idea that we look for situations in which both are present and compete. The usual EPR experiment is not of this type, for the Hamiltonian term associated with the magnet merely displaces an electron. Suppose, however, that a non-linear term is induced by a spin-spin term in the Hamiltonian. This cycles the spin states against the action of the non-linear term that is driving towards the factorized state. Let us see how the competition plays out:

We write (16) in the form
\[ \frac{dC}{dt} = \Lambda \tilde{C} + R_{12}(C) \] (34),
with
\[ (R(C))_{jk} = \sum_{m,n=0}^1 R_{jkmn}C_{mn}. \] (35)

To model the spin-spin interaction we only need a non-vanishing value denoted \( \gamma \) for the two \( R_{jkmn} \) that exchange \( C_{00} \leftrightarrow C_{11} \), and we take \( \Lambda \) to be of the form (20). We then obtain a pair of coupled differential equations for \( C_{00} \) and \( C_{11} \) from (16):
\[ \frac{dC_{00}}{dt} = \frac{1}{2} \left( \eta \frac{|C_{00}C_{11}|}{C_{00}^*} + i\gamma C_{11} \right), \]
\[ \frac{dC_{11}}{dt} = \frac{1}{2} \left( -\eta \frac{|C_{00}C_{11}|}{C_{11}^*} - i\gamma C_{00} \right). \] (36)

We shall solve (36) with \( \eta > 0 \) in the region where the non-linear term is defined, i.e. where \( \text{det} C \neq 0 \), so that neither \( C_{00} \) nor \( C_{11} \) vanish.
It is straightforward to obtain equations for $d[C_{00}]/dt$ and for $d[C_{11}]/dt$ from (36) and thereby to obtain an equation for $d[C_{00}C_{11}^*]/dt$. One can also obtain an equation for $d(C_{00}C_{11}^*)/dt$ and thence for the relative phase of $C_{00}$ and $C_{11}$. The result is the following. Let

$$C_{00} = \cos(\theta/2), \quad C_{11} = \sin(\theta/2)e^{i\phi}, \quad (37)$$

so that $\{C_{00}, C_{11}\}$ is represented by a point on a sphere with polar latitude $\theta$ and azimuth $\phi$. We choose units in which $\gamma = 1$ so that $|\eta| > 1$ is the regime of macroscopic measurement. After some algebra we obtain from (36):

$$d\theta/dt = -\eta + \sin \phi, \quad d\phi/dt = \cos \phi \cot \theta. \quad (38)$$

We then deduce that there is an integral of the motion involving $\theta$ and $\phi$:

$$\sin \theta \cos \phi (\tan(\pi/4 + \phi/2))^\eta = \text{constant}. \quad (39)$$

Let us examine this solution in various regions: $|\eta| > 1$ means domination by the non-linear term and $|\eta| << 1$ means domination by the linear term. First consider all cases in which $|\eta| > 1$. One then sees from (38) that $d\theta/dt$ is always has the same sign so that $\theta$ changes monotonically. Moreover the slope has absolute value exceeding $|\eta - 1|$, whence for large $\eta$ the factorization at $\sin \theta \to 0$ is achieved in times of order $1/\eta$ as we found for a macroscopic measuring device. In all cases for $|\eta| > 1$ factorization will ultimately come about. If $|\eta|$ is not too much bigger than unity, the effect will be a slow collapse.

A particularly interesting case of this kind may be realizable, as we noted in the introduction, as a result of recently developed techniques for studying mesoscopic entangled states, the so-called “Schrödinger cat states” of trapped Be$^+$ ions[10]. The delay $\hbar/\eta$ predicted by the INL theory is just the inverse of the width $\gamma$ of the fluorescent transition by means of which the factorized constituent containing $|\downarrow\rangle$, is recognized. If $\phi(t)$ is the time-evolving coherent state phase, the INL theory in first approximation implies that the probability $P_\downarrow(\phi)$ should shift to $P_\downarrow(\phi')$ where $\phi'(t) = \phi(t + \delta t)$ with $\delta t$ inversely proportional to the width of the fluorescent transition.

It is interesting to observe from (38) that in all situations with $|\eta| > 1$ the phase $\phi$ will tend to the same value $\text{sign}(\eta)\pi/2$ as one approaches the
factorization at $\sin \theta = 0$. This limit is achieved smoothly, so that we can join the unfactored and factored solutions together smoothly in time across the operator domain boundary. For macroscopic $|\eta|$ the factor $(\tan(\pi/4 + \phi/2))^n$ in (39) forces $\phi$ to remain very nearly constant until the state is almost factorized. It then changes rapidly from whatever its initial value was to $\text{sign}(\eta)\pi/2$.

Now we consider the cases in which $|\eta| \leq 1$. Suppose first that $\eta = 0$. Then the stationary states will be the states

$$s_\pm = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle \pm |1\rangle|1\rangle),$$

(40)

This corresponds to $\theta = \pi/2$ and $\phi = 0$ or $\phi = \pi$. Now with non-zero $\eta$ we see from (38) that there is a stationary solution with $\theta = \pi/2$ and $\sin \phi = \eta$. *Because the state is stationary it can never reach a factorization boundary and so, in accordance with the INL theory, no noise fluctuation ever occurs.*

Thus we find that the stochastic equation actually has a stationary solution in which $s_\pm$ are modified to:

$$s'_\pm = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle \pm e^{i \text{arcsin}(\eta)}|1\rangle|1\rangle).$$

(41)

The effect of the phase in the second term is to admix $s_-$ and $s_+$.

We shall refer to the phase $\text{arcsin} \eta$ as the “non-linearly induced phase”.

7. Application of the theory to the neutral K meson system.

Let us now apply the model of the last section to the neutral K system. Equations in the following are to be understood as valid up to normalization constants that are irrelevant to the conclusions and are left out. Higher orders in small quantities are also left out. In the absence of CP violation the eigenstates of the weak interaction Hamiltonian are $K_{1,2} = K_0 \pm \overline{K}_0$. Because of the CP violation the actual eigenstates are are $K_S, K_L$ which are not quite the same. One finds that:

$$K_L = K_2 + \epsilon K_1 = K_0 - (1 - 2\epsilon) \overline{K}_0 = K_0 - e^{i\delta} \overline{K}_0,$$

(42)

in which the experimental value is

$$\epsilon = 1.6(1 + i) \times 10^{-3},$$

(43)
so that
\[ \delta_{\text{experimental}} = 2i(1 + i)1.6 \times 10^{-3}. \] (44)

Now let us apply the theory above. The spin-spin interaction plays the role of the weak interaction mass matrix that splits the \( K_1, K_2 \) masses\([12]\). In our model that splitting is by \( \gamma \) so we take it to be the experimental \( K_1, K_2 \) mass difference which has imaginary part because the masses are unstable. Thus we set:

\[ \gamma = (1 + i)3.5 \times 10^{-6} \text{eV}. \] (45)

The parameter \( \eta \) according to the INL theory is determined using (33) from the time required to distinguish the factorized states. In this case \( K_0, \overline{K_0} \) are distinguished by means of the sign of the charged lepton in a semi-leptonic decay which reveals whether an \( s \) or an \( \bar{s} \) was converted. The semi-leptonic branching ratio is 66\% of \( K_L \) and negligible in \( K_S \) decays so that we compute the collapse time \( t_0 \) required to recognize a factorized state from the \( K_L \) lifetime divided by 0.66. Thus, we can compute \( \eta \) (up to sign) from (33) and obtain

\[ \eta = \pm 2.6 \times 10^{-8}. \] (46)

Thus the non-linearly induced phase is found to be:

\[ \delta_{\text{theory}} = \arcsin(\eta/\gamma) = \pm 2i(1 + i)1.9 \times 10^{-3}. \] (47)

Considering the coarseness of the model it is quite surprising that we have gotten the experimental result to within 20\%.

8. The EPR problem.

Leaving out two-particle interactions one sees that in virtue of (13b) the INL theory predicated on equation (16) has the same form if for any unitary transformations \( A, B \) one makes the substitutions:

\[ \Lambda_1 \rightarrow AA_1A^\dagger, \quad H_1 \rightarrow AH_1A^\dagger, \]
\[ \Lambda_2 \rightarrow B^\dagger\Lambda_2B, \quad H_2 \rightarrow B^\daggerH_2B, \quad \text{and} \quad C \rightarrow ACB. \] (48)

This means that the non-linear term will share whatever space-time symmetries are enjoyed by the linear theory. It also means that if one only measures particle-1, the equation is unaltered by any unitary transformation applied
to particle-2, in particular to any translation of particle-2. *This is just the observed EPR effect in which the correlation appears in a time that is independent of the separation between the particles and depends only on the response time of the measuring device.*

In examining the puzzling behavior of EPR correlations, Mermin [15] suggested that we might do well to ask: “What is it about the way we think about the world that makes us so puzzled?” Since we have now seen that the “spooky” behavior of EPR correlations is obtained in the INL theory from the insensitivity of the non-linear term to the particular form of space-time symmetry assumed, and since this in turn emerges from the peculiar structure of the non-linear term, we might rephrase Mermin’s question as follows: Can we find a different way to think about the transformations of wave functions that will eliminate our puzzlement?

The way we usually think about the transformation of a wave function $\psi$ is through a differential equation, i.e. we write:

$$\psi \rightarrow \psi + dt(\frac{d\psi}{dt}),$$

(49)

in which $d\psi/dt = f(\psi, t)$ and the form of $f$ determines the dynamics. Thus even if $f$ is non-linear we are thinking of changes in $\psi$ to the wave function as *additive* changes.

But an interesting thing about our $C$ matrices representing two particle systems above is that they have a *multiplicative* structure as well as an additive structure. In general that multiplicative structure may be useless because, unlike the additive structure, inverses don’t always exist. But within the sets of states that are driven to factorization by the stochastic process, inverses both exist and by their existence distinguish the elements of a set from its boundary.

Let us see then what dynamics looks like if we reformulate it via the multiplicative property. Define a “cell” as a connected set of states which can be collapsed into any one of a given set of orthogonal, factorized states. Let $C_o$ be a state inside a cell and $C$ a state on the boundary. With the trace norm (6) defining a metric we can, for any $\epsilon > 0$, make a chain of states $C_o, C_1, \ldots, C_{n-1}, C$ such that all but the last are in the cell and such that the distance between each state and its successor is $\epsilon$. Now consider the
operator

\[ Z_j = C_{j+1}C_j^{-1} \]  

(50)

which is well defined for \( j = 0, \cdots, n - 1 \) and which maps each \( C_j \) of the sequence to its successor. By letting \( \epsilon \) become as small as we like we can define a continuous set of transformations taking us from \( C_0 \) to the boundary. Note that these are not unitary, in general, but they nonetheless map each \( C_j \) to a matrix of the same norm. Moreover each \( Z_j \) acts only on particle-1 since it acts from the left. Indeed if we examine how the \( Z_j \)'s transform under one-particle unitary transformations \( A, B \) applied to the two particles we see that:

\[ Z_j = C_{j+1}C_j^{-1} \rightarrow (AC_{j+1}B)(AC_jB)^{-1} = AZ_jA^\dagger, \]

(51)

which makes it plain that its properties are independent of how one describes particle-2. Equivalently, by operating from the right with operators of the form \( C_j^{-1}C_{j+1} \), we could also accomplish the same thing by acting only on particle-2.

Once we recognize that \( C_0 \) can be dynamically mapped to \( C \) by a sequence of transformations \( Z \) that act only on one of the particles, we see that an EPR correlation is induced without “doing or sending” anything to the other one.

To describe this process let us use (16) to obtain a more natural way of characterizing the dynamics of collapse than (16) itself. We have:

\[ C_{j+1}C_j^{-1} = I + (C_{j+1} - C_j)C_j^{-1} \rightarrow I + dt(dC/dt)C^{-1} \]

(52)

so that from (16) with just the non-linear term we obtain:

\[ Z = 1 + A V(C)dt, \quad V(C) \equiv |\det C|(CC^\dagger)^{-1} \]

(53)

which essentially replaces the differential equation with a prescription for computing evolution directly as a sequence of operations on particle-1.

Since the additive properties of wave functions are the useful ones for Hamiltonian processes, and the multiplicative properties are the useful ones for collapse processes, one suspects that a more natural description of quantum mechanics will be obtained if we embed Hilbert space in a normed ring.
the topology supplied by the trace norm metric, $\sqrt{Tr(CC^\dagger)}$. We conjecture that within such a framework the problem of quantum non-locality will appear a great deal less mysterious.

9. Generalizing the theory

a. Hilbert spaces of higher dimension.

Up to this point we have confined our analysis to two spin-$1/2$ particles. In order to create a general theory of we must now turn to an investigation of Hilbert spaces of arbitrary dimension $n$, and consider what happens when $n$ becomes very large. This is necessary e.g. in order to investigate the EPR effect in the context that it was originally proposed by Einstein, Podolsky, and Rosen[13] which involves coordinate and momentum rather than spin wave functions. The Hilbert spaces will now be infinite dimensional, but can always be approximated by spaces of very large but finite dimension[14].

Referring to the homogeneity requirement in the discussion leading to (12) we note that for Hilbert spaces of dimension $n$ we must re-define the map with the power $\nu = 2/n$ of the determinant:

$$\tilde{C} = |\det C|^{2/n}C^{\dagger-1}. \quad (54)$$

It is then straightforward (See Appendix B) to generalize the arguments used for $n = 1/2$. Since we are primarily interested in what happens for $n$ large, let us take advantage of a simplification that occurs if the dimension is a power of two. We can then think of the measurement as a sequence of $\log_2(n)$ measurements analogous to the spin-$1/2$ measurement in each of which the state is collapsed into one that lies in a space with half the dimension of its predecessor. In the first step we write (20) as an $n \times n$ matrix in which the units appearing in (20) are replaced by projectors onto a pair of conjugate subspaces of dimension $n/2$. One will then find that the time required for the reduction of a completely entangled state in dimension $n$ to a completely entangled state in dimension $n/2$ is again given by (33). Repeating the process we find that the total reduction time to a simple factorized state is:

$$t = \frac{\pi}{\eta} \log_2(n). \quad (55)$$

Suppose then that we wish to apply the method above to an EPR experiment involving the coordinate and momentum wave functions instead of spin. The
number of phase space cells of volume $\hbar^3$ for a macroscopic measurement would be of order $10^{100}$. Since one coherent state per phase space cell suffices to span the Hilbert space, it suffices to consider a Hilbert space of dimension $n = 10^{100}$ for which the logarithmic factor is only 300. Therefore 300 repetitions of the process above will reduce the wave function so that even for energy differences as small as $1\,\text{eV}$ the reduction time would be smaller than one picosecond.

While we have not investigated the limit $n \to \infty$, the determinantal factor in (48) that might cause problems is formally unity in that limit. We thus have grounds to be optimistic that the infinite limit will be sensible.

b. Multi-particle systems.

To generalize the INL theory to $n$-particles, the $C_{ij}$ will be replaced by $C_{i_1, i_2, \ldots, i_n}$. If all but two of the indices are held fixed we obtain a two-particle matrix associated with some pair of particles. The concepts of inverse, adjoint, determinant, and factorization are all applicable to this matrix, as is the question of whether a linear term in the Hamiltonian induces a non-linear term with respect to this pair of variables. One expects that like all systems of more than two particles the dynamics will be difficult.

c. Lorentz invariance.

Because, as we have noted, the non-linear term has one-particle unitary invariance, the need to make the theory explicitly Lorentz covariant is more a luxury than a necessity. We expect that it will be possible for the following reasons: We can replace the Schrödinger equation by a Tonomaga-Schwinger equation, letting the energy parameters on the right side be replaced by energy densities. The non-linear term transforms properly when the spins are transformed by Wigner rotation. The domain of the non-linear operator is defined covariantly since if in one frame a set of states can be mapped by a device into the same factorized state, that will be true in every frame. Moreover our prescription for the noise has been defined covariantly. Thus, for given initial data, we have a prescription for computing the wave function on any space like surface, and for every process there is a process of the same probability in any frame.
11. Comments

The most pressing need is to find definitive experimental tests. These are necessarily exotic because we must have competition between the linear dynamics that entangles the state and the and non-linear dynamics that recognizes the factorized constituents. When we have spatially well-separated wave functions it is relatively easy to recognize factorized constituents but difficult to have the linear dynamics alter the form of the entangled state during the collapse process. When we have closely spaced spatial wave functions, the linear dynamics influences the entangled state, but it is difficult to provide a mechanism that recognizes the factorized constituents. The ion trap technology provides a mesoscopic regime in which lasers can be used to have both terms compete, and the K-meson decay exploits a fortuitous aspect of the weak-interaction dynamics whereby the semi-leptonic decay mode acts to recognize the constituents.

To conclude let us test the INL theory against the desiderata listed by Shimony[2] for a non-linear modification of quantum mechanics:

(a) It is not restricted to situations of measurement.

(b) When applied to macroscopically distinguishable states it produces rapid collapse. There is no gestation of Schrödinger cats for unacceptable time durations.

(c) It reproduces quantum mechanical predictions where it ought to.

(d) Collapse happens in a finite time. There are no stochastic “tails”.

(e) One cannot send superluminal messages.

(f) The dynamics accounts for the occurrence of definite outcomes of measurements performed with actual apparatus.

Professor Shimony lists two other desiderata of which the INL theory fulfils the important one: It is not explicitly covariant but it has appeared in his lifetime.

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Appendix A

Any completely entangled state of two spin-1/2 particles can be written:

$$|\Omega\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} |i\rangle A|i\rangle,$$

in which $A$ is an anti-unitary operator[16]. Thus for a completely entangled state $|\det C| = |\det A|/2 = 1/2$. Since $|\det C| = |\det(UCV)|$ for any pair of unitary or anti-unitary transformations $U,V$ we can choose a basis in which $C_{01} = 0$. Then $\det C$ vanishes if and only if one of the two coefficients $C_{00}$ or $C_{11}$ vanishes. But if either vanishes along with $C_{01}$ the state factorizes. Conversely if the state factorizes the determinant vanishes by direct calculation. Now note that $|\det C|$ is majorized by $|C_{00}| |C_{11}| + |C_{01}| |C_{10}|$. Each of the two terms is bounded by one half the sum of the squares of the factors, the upper bound being achieved only when the two are equal. Since the sum of the four squared moduli is unity it follows that $|\det C|$ is bounded by $1/2$ and achieves the upper bound only if $|C_{00}| = |C_{11}|, |C_{01}| = |C_{10}|$, and $|C_{00}|^2 + |C_{01}|^2 = 1/2$. Choosing a one particle basis so that either $C_{00}$ or $C_{10}$ vanishes one obtains one of the familiar entangled states.

Appendix B

For Hilbert space dimension $n$ we saw that to obtain the homogeneity property we must define the map $C \rightarrow \hat{C}$ by

$$\hat{C} = |\det C|^{2/n} C^{1-1}. \quad (B1)$$

Let $\pi_1$ and $\pi_2$ be projection operators into complementary subspaces of dimension $m$ and $n-m$. A measurement that determines in which of these two subspaces a given state lies generalizes the notion of a Stern-Gerlach filter for spin-1/2. The matrix $\Lambda$ given by (20) will now be replaced by

$$\Lambda = \eta/2 \begin{pmatrix} \frac{1}{m}\pi_1 & 0 \\ 0 & \frac{1}{n-m}\pi_2 \end{pmatrix}, \quad (B2)$$

which has zero trace as required to conserve probability.
For an initial state with $C$ diagonal one puts:

$$y_j = |C_{jj}|^2, \ j = 0, \ldots, n-1,$$

and obtain from (16):

$$\frac{dy_j}{dt} = \eta(y_0 \cdots y_{n-1})^{1/n} \left\{ \begin{array}{ll}
m^{-1}, & j = 0, \ldots, m-1 \\
-(n-m)^{-1}, & j = m, \ldots, n-1.
\end{array} \right.$$

(B2)

For $\eta > 0$ put

$$d\tau = \eta(y_0 \cdots y_{n-1})^{1/n}dt,$$

whence with initial values

$$y_j(0) = \alpha_j, \quad j = 0, \ldots, n-1, \quad \sum_j \alpha_j = 1,$$

one obtains

$$y_j = \left\{ \begin{array}{ll}
\alpha_j + n\tau/m, & j = 0, \ldots, m-1 \\
\alpha_j - n\tau/(n-m), & j = m, \ldots, n-1.
\end{array} \right.$$

(B5)

so that

$$\eta t = \int_0^\tau d\tau \left\{ \prod_{j=0}^{m-1} (\alpha_j + \frac{n\tau}{m}) \prod_{j=m}^{n-1} (\alpha_j - \frac{n\tau}{n-m}) \right\}^{-1/n}.$$  

(B6)

which must be inverted to obtain $\tau$ in terms of $t$. One does not have a closed form analogous to (28) in general. The process terminates at

$$\tau = (1 - m/n) \min_{j \geq m}(\alpha_j).$$

(B7)

Let us now focus on the special case of a completely entangled initial state, so that we choose

$$\alpha_j = 1/n, \quad \forall n,$$

for which we have

$$\eta t = 2 \int_0^\tau d\tau \left( \frac{1}{n} + \frac{n\tau}{m} \right)^{-m/n} \left( \frac{1}{n} - \frac{n\tau}{n-m} \right)^{-1+m/n}.$$

The termination time $t_o$ is obtained setting $\tau = 1 - m/n$ which after some algebra gives an expression for $t_o$ in terms of a hypergeometric function namely:

$$\eta t_o = 2(1 - \frac{m}{n})F(1, 1 + \frac{m}{n}, 1 - \frac{m}{n})$$

(B9)
Thus in particular for $m = 1$ we have:

$$\eta t_o = 2\left(1 - \frac{1}{n}\right) F\left(1, 1, 1 + \frac{1}{n}, 1 - \frac{1}{n}\right) \rightarrow n \rightarrow \infty. \quad (B10)$$

whereas for $m = n/2$ we have

$$\eta t_o = \frac{\pi}{2}. \quad (B11)$$

References


