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INTEGRABILITY IN NONLINEAR REALIZATION SCHEME

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It is fairly well established that $1+1$ dimensional integrable equations (e.g. KdV, Boussinesq, etc.), connected with the $W$-type algebras, play an important role in the understanding of ($W$)-string and 2D ($W$)-gravity theories [1-3]. The integrability properties of these equations stem from the presence of infinite number of involuting conserved quantities. The existence of the zero-curvature representation, second-Hamiltonian structure, Lax-pair formulation, etc., are also some of the key requirements for the integrability properties to be satisfied.

The central theme of the present note is to discuss the integrability properties of the Boussinesq equations in the framework of the universal geometric approach of nonlinear realization (NLR) method [4]. The geometrical understandings of these properties might turn out to be useful in the context of $W$-gravity theories.

The classical (centrally extended) $W_3$ algebra of Zamolodchikov

\[
[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0},
\]

\[
[L_n, W_m] = (2n - m) W_{n+m},
\]

\[
[W_n, W_m] = 16(n - m) \Lambda_{n+m} - \frac{8}{3} (n - m)(n^2 + m^2 - \frac{nm}{2} - 4) L_{n+m}
- \frac{c}{9} (n^3 - n)(n^2 - 4) \delta_{n+m,0},
\]

is a nonlinear algebra. One of the key ingredients invoked in Refs.[5,6] was to obtain an infinite dimensional linear algebra $W_3^\infty$

\[
W_3^\infty = \{ L_n, W_n, \Lambda_n, J_n^h (h \geq 5) \ldots \},
\]

by treating the spin-4 $\Lambda_n = -\frac{8}{c} \sum_m L_{n-m} L_m$ and all the other higher spin $J_n^h (h \geq 5)$ generators emerging due to commutation relations of $W$'s and/or $L$'s with the higher spin composite generators as independent generators.

We shall concentrate here on the truncated version of linear algebra (2) (i.e. "contact" $W_3^\infty$) for the application of the nonlinear realization method, where the Laurent indices of the generators with conformal spin $h$ vary from $-(h-1)$ to $\infty$. The stability subalgebra $\mathcal{H}$ that concerns our discussion here is [5]

\[
\mathcal{H} = \{ W_{-1} + 2L_{-1}, W_0, W_1, W_2, L_1, L_0, \Lambda_n (n \geq -3), J_n^h (h \geq 5, n \geq -h+1) \ldots \}. \quad (3)
\]

The left coset element $g = ("contact" W_3^\infty)/\mathcal{H}$ can be parametrized in terms of the coordinates $x, t$ and coset fields $(u,v,\psi^s,\xi^s)$ as

\[
g = e^{xW_{-2}} e^{xL_{-1}} e^{\psi L_0} \left( \Pi_{n \geq 4} e^{\psi_n L_n} e^{\xi_n W_n} \right) e^{uL_2} e^{vW_3}. \quad (4)
\]

The commutativity of $W_{-2}$ and $L_{-1}$ ensures the linear independence of $x$ and $t$ directions. Therefore, a point on the coset manifold can be parameterized by these coordinates. As a consequence, the coset fields become functions of $x$ and $t$. 

1
The basic geometrical object in NLR method is the one-differential Cartan form \( \Omega = g^{-1}dg \) in terms of which curvature, complex structure, metric, etc., of the coset manifold are determined. It can be decomposed as a sum over all the spin-\( h \) generators of the "contact" \( W^\infty_3 \)

\[
\Omega = g^{-1}dg \equiv \sum_{n=-1}^{\infty} \omega_n L_n + \sum_{n=-2}^{\infty} \theta_n W_n + \text{higher spin contributions.} \tag{5}
\]

By definition, this one-differential form satisfies the Maurer-Cartan equation

\[
d^\text{Maurer-Cartan} \Omega = \Omega \wedge \Omega = 0, \tag{6}
\]

which implies the existence of the zero-curvature representation. As higher spin composite fields form an ideal, it is essential to know only some of the lower order forms to obtain the dynamical equations if we exploit the idea of inverse Higgs-covariant reduction constraints appropriately [7,8]. These are:

\[
\begin{align*}
\omega_{-1} &= dx, \quad \omega_0 = 0, \quad \omega_1 = -3udx + 160vd t, \quad \omega_2 = du - 4\psi_3 dx + 320\xi_4 dt, \\
\omega_3 &= d\psi_3 + \left(\frac{3}{2}u^2 - 5\psi_4 \right)dx + (560\xi_5 - 240uv)dt, \\
\omega_4 &= d\psi_4 - 6\psi_5 dx + (896\xi_5 - 192v\psi_3 - 768u\xi_4)dt.
\end{align*}
\tag{7}
\]

\[
\begin{align*}
\theta_{-2} &= dt, \quad \theta_{-1} = 0, \quad \theta_0 = -6udt, \quad \theta_1 = -8\psi_3 dt, \\
\theta_2 &= -5 vdx + \left[12u^2 - 10\psi_4 \right] dt, \quad \theta_3 = dv - 6 \xi_4 dx + \left[24u\psi_3 - 12\psi_5 \right] dt.
\end{align*}
\tag{8}
\]

Due to inverse Higgs-covariant reduction (IH-CR) constraints, all the forms associated with the coset generators are set equal to zero as they transform homogeneously under the left action of \( W^\infty_3 \) symmetry. Mathematically these are expressed as

\[
\omega_n = 0 \quad \forall \quad n \geq 2, \quad \theta_n = 0 \quad \forall \quad n \geq 3. \tag{9}
\]

It can be seen that \( \omega_2 = 0, \omega_3 = 0, \omega_4 = 0, \theta_3 = 0 \) lead to the following kinematical relations due to inverse Higgs effect [7]

\[
\begin{align*}
\psi_3 &= \frac{u'}{4}, \quad \psi_4 = \frac{u''}{20} + \frac{3}{10}u^2, \\
\xi_4 &= \frac{v'}{6}, \quad \psi_5 = \frac{1}{30} \left( \frac{u'''}{4} + 3uu' \right), \tag{10}
\end{align*}
\]

and the dynamical equations due to covariant reduction [8]

\[
\begin{align*}
\dot{u} &= -\frac{160}{3} v', \quad \dot{v} = \frac{u'''}{10} - \frac{24}{5} u u', \tag{11}
\end{align*}
\]
where primes and dot stand for the derivatives w.r.t. coordinates $x$ and $t$ respectively. Equation (11) is nothing but the Boussinesq equations realized on spin-2 and spin-3 fields $u$ and $v$. Due to IH-CR constraints, all the higher spin coset fields can be expressed in terms of the two basic fields $u, v$ and the space derivatives on them. Geometrically, this amounts to singling out a 2D geodesic surface $(x, t, u(x, t), v(x, t))$ from the starting infinite dimensional coset manifold and the Boussinesq equations (11) correspond to the embedding conditions on this surface. Furthermore, the constraints (9) lead to the reduction of the one-differential Cartan forms to $sl(3, R)$ valued $\Omega^{red}$ given by

$$\Omega^{red} = (L - 5 r W_2 - 3 u L_1) dx + [W_2 - 6 u W_0 - 2 u' W_1 + 160 v L_1 + (9 u^2 - \frac{1}{2} u'') W_2] dt.$$  

(12)

modulo higher spin composite generators which form an ideal. It is straightforward to check that the Maurer-Cartan equation $d^x \Omega^{red} + \Omega^{red} \wedge \Omega^{red} = 0$ lead to the rederivation of the Boussinesq equations (11). Identifying $\Omega^{red} = A_\mu dx^\mu$, it is clear that the curvature two-form $(F_{\mu\nu})$ emerging from the Maurer-Cartan equation is zero for the Boussinesq equations, i.e.,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0.$$  

(13)

This explains the zero-curvature representation. The Drinfel'd-Sokolov type of Lax-pair can also be obtained from (13) which can be re-expressed as

$$F_{tr} = \left[ \frac{\partial}{\partial t} + A_t, \frac{\partial}{\partial x} + A_x \right] = 0.$$  

(14)

Here the $sl(3, R)$ valued connections $A_t$ and $A_x$ coincide with the coefficients of $dt$ and $dx$ in the reduced one-differential Cartan form (12).

It is well known that the second Hamiltonian structure

$$\{u(x, t), u(y, t)\} = \frac{2}{c} \left[ \frac{1}{6} \frac{\partial^3}{\partial y^3} - 2 u(y) \frac{\partial}{\partial y} - \frac{\partial u}{\partial y} \right] \delta(x - y),$$

$$\{u(x, t), v(y, t)\} = -\frac{2}{c} \left[ 3 v(y) \frac{\partial}{\partial y} + \frac{\partial v}{\partial y} \right] \delta(x - y),$$

$$\{v(x, t), v(y, t)\} = \frac{5}{100c} \left[ \frac{1}{48} \frac{\partial^3}{\partial y^3} + \frac{5}{4} u(y) \frac{\partial^3}{\partial y^3} + \frac{15}{8} \frac{\partial u}{\partial y} \frac{\partial^2}{\partial y^2} + \frac{9}{8} \frac{\partial^2}{\partial y^2} - 12 u^2 \right] \frac{\partial}{\partial y} + \left[ \frac{1}{4} \frac{\partial^3 u}{\partial y^3} - 12 u \frac{\partial u}{\partial y} \right] \delta(x - y).$$  

(15)

for the Boussinesq equations is nothing other than the $W_3$ algebra. In the framework of NLR method, this connection can be understood by the choice of the Hamiltonian $(H)$ in terms of the coset fields because the following Hamilton equations

$$\frac{\partial u}{\partial t} = \{u, H\}, \quad \frac{\partial v}{\partial t} = \{v, H\}.$$  

$$H = -\frac{10c}{3} \int dx v(x, t).$$  

(16)
lead to the derivation of Boussinesq equations (11) if we use the brackets (15). On the coset manifold this Hamiltonian corresponds to the translation generator \( W_{-2} \). This can be understood as follows. If the 2D theory is considered on a complex plane, the spin-3 fields \( v \) can be decomposed in Laurent modes \( v(z) = \sum W_n z^{-n-3} \) and the contour integration in \( H \) will lead to \( W_{-2} \).

To establish the complete integrability of the Boussinesq equations, it is essential to obtain involving conserved quantities under NLR scheme. For this purpose additional Cartan forms have to be computed. Some of these are

\[
\begin{align*}
\omega_5 &= dv_5 + u dv_3 + \left( \frac{1}{2} u^3 - 5uv_4 + 2v_3^2 - 40v^2 - 7v_6 \right) dx \\
&+ (192u^2v - 336uv_3 - 704v_3\xi_4 - 160uv_4 + 1344\xi_7) dt,
\end{align*}
\]

\[
\begin{align*}
\omega_6 &= dv_6 + 2udv_4 + (8v_3v_4 - 12uv_5 - 8\psi_7) dx \\
&+ (192u\xi_8 + 768uv_6 + 768u^2\xi_4 - 640v_4\xi_4 - 1664v_3\xi_5) dt.
\end{align*}
\]

(17)

\[
\begin{align*}
\theta_4 &= d\xi_4 + (3uv - 7\xi_3) dx + \left[ 20v_3^2 + 20uv_4 - 14v_6 - 8u^3 - 80v^2 \right] dt,
\end{align*}
\]

\[
\begin{align*}
\theta_5 &= d\xi_5 + u dv_3 + (6u\xi_4 - 4uv_3 - 8\xi_6) dx \\
&+ \left[ 56v_3v_4 + 320uv_4 + 12uv_5 - 12u^2\psi_3 - 16\psi_7 \right] dt,
\end{align*}
\]

(18)

The application of IH-CR constraints leads to the derivation of the conserved quantities. For instance, the Boussinesq equations (11) can be regarded as the conservation laws emerging due to IH-CR constraints applied on \( \theta_3 \) and \( \omega_2 \). This leads to the determination of the first two conserved quantities (i.e. \( H_1 = \frac{3}{2} \int dx u(x, t) \), \( H_2 = -\frac{48u}{3} \int dx v(x, t) \)) which correspond to the translation generators \( L_{-1} \) and \( W_{-2} \) on the coset manifold. The next nontrivial conservation law emerges from the IH-CR constraints applied on \( \theta_5 \). In fact, when its \( dt \) projection is set equal to zero due to covariant reduction procedure, we obtain

\[
\frac{\partial \xi_5}{\partial t} = 16\psi_7 + 12u^2\psi_3 - 12uv_5 - 56v_3\psi_4.
\]

(19)

Besides (10), using the additional inverse Higgs relations

\[
\begin{align*}
\xi_5 &= \frac{\xi_4}{7} + \frac{3}{7} u v, \\
\xi_6 &= \frac{\xi_5}{4} + \frac{3}{4} u \xi_4, \\
\xi_7 &= \frac{\xi_6}{9},
\end{align*}
\]

(20)

\[
\begin{align*}
\psi_6 &= \frac{\psi_4}{7} + \frac{2}{7} u^2 - \frac{7}{7}, \\
\psi_7 &= \frac{\psi_6}{8} + \psi_3\psi_4 = \frac{d}{dx} \left[ \frac{\psi_6}{8} + \frac{\psi_3^2}{10} + \frac{u^3}{40} \right],
\end{align*}
\]

(21)
that emerge when $dx$ projections of various forms are set equal to zero, it can be seen that

$$\frac{\partial \xi_5}{\partial t} = \frac{\partial}{\partial x} \left[ 2u\psi_6 + \frac{1}{5} u^3 - 2u\psi_4 - \frac{16}{3} \psi_3^2 \right].$$

(22)

The use of inverse Higgs kinematical relation $\xi_5 = \frac{\xi_4}{\bar{t}} + \frac{3}{\gamma} u v$, leads to

$$\frac{\partial (u v)}{\partial t} = \frac{\partial}{\partial x} \left[ 2u\psi_4 - \frac{11}{5} u^3 - \frac{4}{5} \psi_3^2 - \frac{80}{3} v^2 \right],$$

(23)

as the next nontrivial conservation law. This result can also be derived from the Maurer-Cartan equations with the appropriate choice of IH-CR constraints (see, e.g. Ref. [9] for details). Similar arguments for the higher order Cartan forms lead to the following set of commuting quantities besides $H_1$ and $H_2$

$$H_4 = c \int dx (uv)(x,t), \quad H_5 = -c \int dx \left[ \frac{(u')^2}{20} + \frac{4 u^3}{5} + \frac{80 v^2}{3} \right],$$

$$H_7 = c \int dx \left[ \frac{u' u''}{3200} + \frac{9 u (u')^2}{400} + \frac{(u')^2}{6} + \frac{3 u^4}{50} + 4 u v^2 \right],$$

$$H_8 = -c \int dx \left[ \frac{u'' u'''}{30} + \frac{800 v^3}{9} - \frac{3 v (u')^2}{2} - 2 u u'' v + 8 u^3 v \right].$$

(24)

Here the subscripts of these conserved quantities stand for the naive conformal dimensions. Written in terms of the generators, the above quantities

$$\{ L_{-1}, \ W_{-2}, \ \Phi_{-4}, \ \bar{S}_{-5} \}$$

form a Cartan subalgebra in the "contact" $W_3^\infty$, where $\Phi = \frac{48}{c} (TW), S = \frac{1}{c} (W^2 - \frac{128}{3c} T^3 + \frac{4}{3} (\partial T)^2), \ldots$ etc. These set of generators correspond to the translation generators on the coset manifold. The composite generators of this set are connected to the evolution parameters on the coset manifold for the derivation of the higher order Boussinesq hierarchy equations [9].

To summarize, the IH-CR constraints lead to (i) the covariant kinematical relationships between higher spin coset fields and the essential fields, (ii) the dynamical equations for the essential fields, and (iii) the commuting conserved quantities for the dynamical equations. It has been demonstrated that the Boussinesq equations correspond to the embedding conditions on a 2D geodesic surface $(x, t, u(x,t), v(x,t))$ into the starting infinite dimensional coset manifold. The zero-curvature representation for these equations turns out in the form of the Maurer-Cartan equation which is satisfied by the one-differential Cartan form. The Drinfel’d-Sokolov type of Lax formulations emerge when the vanishing curvature two-form is expressed in terms of the commutators of the covariant derivatives with $sl(3, R)$ valued gauge connections [5]. These gauge connections turn out to be the coefficients of the $dx$ and $dt$
projections of the reduced Cartan form under NLR. The Miura maps for the derivation of the modified Boussinesq equations correspond to the covariant relationships among the essential coset fields when one covariantly goes from one coset manifold to another one [5]. Finally, the commuting conserved quantities correspond to the translation generators on the coset manifold and they form a Cartan subalgebra in the "contact" $W_{\infty}^\infty$. The composite generators of this set correspond to the evolution generators for the higher order Boussinesq hierarchy equations [9]. The involuting properties correspond to the linear independence of the coordinate directions on the coset manifold. It is an interesting venture to provide a geometrical basis for the integrability properties of the super Boussinesq equations [10] and other well known $1+1$ dimensional integrable systems.

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References


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Малик Р.П.
Интегрируемость в нелинейной схеме реализации

В рамках универсального геометрического подхода нелинейного метода реализации обсуждаются некоторые ключевые особенности свойств интегрируемости уравнений Буссинеска, связанных с $W_3$ алгеброй Замолодчикова. В работе также кратко дискутируются геометрический смысл этих уравнений, их вторичная гамильтонова структура, формулировка лакс-пары, представления нулевой кривизны и коммутирующие величины.

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Malik R.P.
Integrability in Nonlinear Realization Scheme

In the framework of the universal geometric approach of nonlinear realization method, some of the key features of the integrability properties of the Boussinesq equations, connected with the $W_3$ algebra of Zamolodchikov, are discussed. The geometrical origins for these equations, its second Hamiltonian structure, Lax-pair formulation, zero-curvature representation, involuting conserved quantities, etc., have also been concisely dealt with under the nonlinear realization scheme.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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