Cosmology in Brans-Dicke theory with a scalar potential

Caroline Santos* and Ruth Gregory
Centre for Particle Theory, Department of Mathematical Sciences
South Road, Durham, DH1 3LE

ABSTRACT

We consider the general behaviour of cosmologies in Brans-Dicke theory where the dilaton is self-interacting via a potential $V(\Phi)$. We show that the general radiation universe is a two-dimensional dynamical system whereas the dust or false vacuum universe is three-dimensional. This is in contrast to the non-interacting dilaton which has uniformly a two-dimensional phase space. We find the phase spaces in each case and the general behaviour of the cosmologies.

November, 1996

* On leave from: Departamento de Física da Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4150-Porto, Portugal
1. Introduction

Einstein's theory of General Relativity is extremely successful at describing the dynamics of our solar system, and indeed the observable universe, nonetheless, the realization that general relativity probably does not describe gravity accurately at all scales has led to various alternatives being explored, most notably the scalar-tensor family of gravity theories pioneered by Jordan, Brans and Dicke (JBD) [1], and the gravitational lagrangian inspired by low energy string theory, [2]. In fact, these two theories are not unrelated, since the low energy effective action for bosonic string theory generically takes the form

\[ S = \int d^4 x \sqrt{-g} e^{-2\phi} \left( R + 4(\nabla \phi)^2 - \frac{1}{12} H_{\mu\nu\lambda}^2 \right) \]  \hspace{1cm} (1.1)

where \( \phi \) is the dilaton, and \( H_{\mu\nu\lambda} \) is the field strength of the two form \( B_{\mu\nu} \); comparing this with the JBD action

\[ S_{JBD} = \int d^4 x \sqrt{-g} \left[ \Phi R - \omega \frac{(\nabla \Phi)^2}{\Phi} + 16\pi \mathcal{L} \right] \]  \hspace{1cm} (1.2)

shows that the scalar-tensor sectors are identical if \( \omega = -1 \).

The phenomenological importance of such extended gravity theories to modern cosmology was powerfully highlighted by La and Steinhardt [3], who suggested that by using JBD theory instead of general relativity, the “graceful exit” problem of old inflationary cosmology [4] might be ameliorated. It was rapidly realised [5] that this original scenario required a value of the JBD parameter \( \omega \) which was in conflict with the observational limits [6]. This then led to the development of various alternatives [7], however, a general study of the extended inflationary ethos [8] indicates that in order to satisfy observational constraints rather contrived models are required. Nonetheless, extended inflationary ideas still survive in some recent inflationary models (e.g. [9]).

The prominence of string theory as a theory of everything, in particular a quantum theory of gravity, means that we must examine its consequences in regimes where it departs from general relativity, in particular we expect that the early universe might display “stringy” qualities; [10] gives a selection of articles dealing with precisely this problem. Correspondingly, by field redefinitions, one can instead investigate the cosmological implications of JBD theories, which several recent authors have done in a variety of ways [11], [12], [13], [14]. Most of the recent studies of the JBD theories have focussed on the qualitative behaviour of cosmologies in pure JBD theory, or JBD plus a cosmological constant\(^\dagger\).

\(^\dagger\) It should be stressed that the ‘cosmological constant’ in this case amounts to setting \( R \rightarrow R + \Lambda \) in the JBD action.
However, since it is generally believed that the string dilaton must be massive in order to anchor it to the current observed Newton’s constant, it seems likely that more realistic JBD models will include a potential for the dilaton:

\[ S = S_{JBD} - \int d^4x \sqrt{-g}V(\Phi) \]  

(1.3)

The specific form of this potential is unknown, but it seems reasonable that at high temperatures it might look something like \( V_0 \Phi^{2n} \). In this paper therefore, we are interested in the qualitative behaviour of JBD cosmologies with just such a form for the potential, focusing in particular on \( n = 1 \): \( V(\Phi) = V_0 \Phi^2 \). (The cosmological constant models investigated in [13] have \( V(\Phi) = \Lambda \Phi \).) Clearly this is not a particularly realistic potential for the current time, since it has a minimum at \( \Phi = 0 \), which would correspond to an infinitely strong gravitational coupling, (although as we will show \( V(\Phi) \) is not necessarily to be viewed as a canonical scalar field potential) however, the model potential ought to provide some insight on the cosmological effects of a self-interacting dilaton.

Our starting point will be the recent papers of Kolitch and others[14], who showed that the field equations for cosmological models could be reduced to a two-dimensional dynamical system for any reasonable perfect fluid matter source in pure JBD theory, and also for spatially flat models with a non-zero cosmological constant. In the presence of the scalar potential, the cosmological dynamical system turns out to be generically three-dimensional, apart from radiation dominated universes in which the system once more reduces to two-dimensions. The layout of the paper is as follows. In the next section we analyse the cosmological equations of motion following from (1.3) and derive the corresponding minimal dynamical system for a spherically symmetric perfect fluid cosmology. In the third section we consider the example of empty universes as a means of testing the method and exploring the effect of the potential on the purely gravitational theory, we also consider radiation universes, since these too turn out to have a particularly simple form qualitatively similar to the vacuum spacetimes. We also consider conformally transforming to the so-called Einstein frame to motivate some of these features. In section four we present an analysis of dust and false vacuum models. Finally, we sum up our results and conclude in section five. We use a mostly plus signature for the metric.
2. The cosmological equations.

In this section we present a brief derivation of the dynamical system representing a spherically symmetric cosmology in JBD theory with the scalar potential. Varying the action (1.3) with respect to the metric and scalar field yields the following equations of motion:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{\omega}{\Phi^2} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{\Phi} \nabla_\mu \nabla_\nu \Phi + g_{\mu\nu} \left( \frac{\Box \Phi}{\Phi} + \frac{\omega}{2\Phi^2} (\nabla \Phi)^2 + \frac{V(\Phi)}{2\Phi} \right) = \frac{8\pi}{\Phi} T_{\mu\nu}
\]

(2.1)

\[
R - \frac{\omega}{\Phi^2} (\nabla \Phi)^2 + \frac{2\omega}{\Phi} \Box \Phi - \frac{dV(\Phi)}{d\Phi} = 0
\]

(2.2)

Contracting (2.1) and substituting in (2.2) gives

\[
\Box \Phi = \frac{8\pi}{3 + 2\omega} T + \frac{1}{3 + 2\omega} \left( \Phi \frac{dV(\Phi)}{d\Phi} - 2V(\Phi) \right)
\]

(2.3)

as a reduced equation of motion for \( \Phi \).

Following the cosmological principle, we will assume a standard Friedman-Robertson-Walker (FRW) form for the metric:

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

(2.4)

Substituting into (2.1) and (2.3) gives the following equations

\[
\frac{\dot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{2k}{a^2} + \frac{\dot{\Phi}}{a\Phi} = \frac{8\pi}{(3 + 2\omega)\Phi} \left[ \omega p + (\omega + 1) \rho \right] + \frac{V(\Phi)}{2\Phi}
\]

(2.5)

\[
\left( \frac{\dot{a}}{a} + \frac{\dot{\Phi}}{2\Phi} \right)^2 + \frac{k}{a^2} = \frac{2\omega + 3}{12} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 + \frac{8\pi \rho}{3\Phi} + \frac{V(\Phi)}{6\Phi}
\]

\[
-\frac{1}{a^3} \frac{d}{dt} (\dot{a} a^3) = \frac{8\pi}{3 + 2\omega} \left( T + \frac{\Phi}{8\pi} \frac{dV(\Phi)}{d\Phi} - \frac{V(\Phi)}{4\pi} \right)
\]

for the universe. Note that we have assumed the stress energy takes the form of a perfect fluid, \( T_{ab} = \text{diag}(\rho, p, p, p) \) in an orthonormal frame. This satisfies the conservation equation

\[
\dot{\rho} = -3 \frac{\dot{a}}{a} (\rho + p)
\]

(2.6)

Following [14] we transform to conformal time

\[
\eta = \int \frac{dt}{a(t)}
\]

(2.7)
and we take an equation of state

\[ p = (\gamma - 1) \rho \]  

(2.8)

for the fluid. Note that this implies

\[ \rho \propto a^{-3\gamma} \]  

(2.9)

from (2.6).

Denoting \( \frac{d}{dt} \) by a prime, and defining

\[
X = \left[ \frac{2\omega + 3}{12} \right]^{1/2} \frac{\Phi'}{\Phi} = A \frac{\Phi'}{\Phi}  
\]

(2.10)

\[
Y = \frac{a'}{a} + \frac{\rho'}{2\Phi}  
\]

we see that for the potential \( V(\Phi) = V_0 \Phi^2 \), the latter two equations of (2.5) become:

\[
X' = -2XY + (1 - 3\gamma/4) \frac{8\pi \rho a^2}{3A\Phi} 
\]

(2.11)

\[
Y^2 + k = X^2 + \frac{8\pi \rho a^2}{3\Phi} + \frac{V_0 \Phi a^2}{6} 
\]

(2.12)

we may then, [14], differentiate this constraint equation for \( Y \), using the \( X \)-equation and the equation of motion for \( \rho \) to finally obtain

\[
X' = -2XY + \frac{(1 - 3\gamma/4)}{A} (Y^2 + k - X^2 - \frac{Z}{6}) 
\]

\[
Y' = (1 - 3\gamma/2)(Y^2 + k - X^2) - 2X^2 + \frac{\gamma Z}{4} 
\]

(2.13)

\[
Z' = 2ZY 
\]

where \( Z = V_0 \Phi a^2 \), as the three-dimensional dynamical system representing the general perfect fluid cosmological models for a JBD theory with the scalar potential \( V_0 \Phi^2 \). (The method can be generalised for \( V(\Phi) = V_0 \Phi^n \) in which case \( Z = V_0 \Phi^{n-1} a^2 \), and \( X' \) acquires additional \( Z \)-terms.)

Note that (2.13) represents the most general case scenario. If \( V_0 = 0 \), the \( Z \)-equation disappears and we recover the Kolitch and Eardley scenario [14]. Also, if \( \rho = 0 \) (as we will consider in the next section) (2.12) can be used to eliminate \( Z \), again leading to a two-dimensional dynamical system, and in fact, we can also reduce (2.13) to a two-dimensional system in a radiation universe, since \( \rho \propto a^{-4} \) which allows us to eliminate \( X \) using (2.12).
Before proceeding to an analysis of (2.13) for the various cosmologies, we conclude this section with a few general remarks. First, note that in order for the Ricci term in the action (1.2) to have the correct sign, we require $\Phi > 0$, i.e. $Z > 0$. From (2.13) we note that since $Z'|_{Z=0} = 0$, the dynamical system trajectories will not cross $Z = 0$. We will also take $\omega > -3/2$, as is conventional, which implies that $A > 0$; for the case of string theory, $A = \frac{1}{2\sqrt{3}}$. For a physical cosmology, we will require $\rho > 0$, i.e.,

$$Y^2 + k - X^2 - \frac{Z}{6} \geq 0 \quad (2.14)$$

from the constraint equation (2.12). Using the dynamical system equations, (2.13), verifies that $[Y^2 + k - X^2 - \frac{Z}{6}]|_{\rho=0} = 0$ and hence trajectories do not cross $\rho = 0$. In other words, a positive energy trajectory will remain a positive energy trajectory.

Finally, it will be of interest to note whether the cosmological models corresponding to the dynamical system trajectories are expanding or contracting. From (2.10) we note that for expansion $Y - X/2A > 0$, therefore we will be interested in where this surface is relative to the physical regions and the possible trajectories. It will mostly be the case that trajectories will lie on one side or the other of this surface, and hence will represent cosmologies that are either eternally expanding or contracting, however, particularly for small $A$, it may well be the case that some trajectories cross this line, in which case they will correspond to cosmological solutions which ‘bounce’ in the Brans-Dicke frame, that is, they start off contracting, reach a minimum size and re-expand (or vice-versa). However, it should be pointed out that universes which bounce in the JBD frame, may well not be true bounce universes when viewed from the ‘Einstein’ frame, a conformally related metric in which the gravitational part of the action appears in Einstein form. For a universe to bounce in the Einstein frame requires $Y$ to change sign, something which is only possible in $k = 1$ cosmologies.

3. Vacuum and radiation cosmologies.

We consider vacuum cosmologies first, since these should display the main features of the effect of the potential, however, we also include radiation universes since these too are effectively two-dimensional and are exactly soluble. We first derive the two-dimensional dynamical system governing the radiation/vacuum cosmologies, determining the form of their solution, and plotting the curves on the two-dimensional phase plane. Using also
the original $X, Y, Z$-variables, we identify the various qualitative behaviours possible, re-interpreting key solutions in terms of the original cosmological parameters $\Phi(t), a(t)$.

Setting $\gamma = 4/3$ in (2.13) gives

$$
X' = -2XY \\
Y' = -(Y^2 + k + X^2) + \frac{Z}{3} \\
Z' = 2ZY
$$

(3.1)

The key feature that enables radiation (and hence also vacuum) universes to be simplified is that for $\gamma = 4/3$, (2.9) implies $\rho \propto a^{-4}$, which means that the constraint equation (2.12) reads

$$
Y^2 + k = \frac{8\pi \rho_0 V_0}{3Z} + X^2 + \frac{Z}{6}
$$

(3.2)

This in turn allows us to decouple $X$ from (3.1) leading to

$$
Y' = -2(Y^2 + k) + \frac{8\pi \rho_0 V_0}{3Z} + \frac{Z}{2}
$$

(3.3)

In order to reduce this to a standard form, we write

$$
W = YZ \quad (= Z'/2)
$$

(3.4)

and obtain

$$
Z' = 2W \\
W' = \frac{Z^2}{2} - 2kZ + \frac{B}{2}
$$

(3.5)

where we have written $B = 16\pi \rho_0 V_0/3$ for convenience.

It is not difficult to see that

$$
W^2 = \frac{Z^3}{6} - kZ^2 + \frac{BZ}{2} + C
$$

(3.6)

satisfies (3.5), where $C$ is a non-negative constant from (3.2). Since $W = Z'/2$, (3.6) is in fact an elliptic equation, and hence $Z(\eta)$ can be written in closed form:

$$
Z = Z_1 + \frac{Z_3 - Z_1}{\text{sn}^2 \left[ -\sqrt{\frac{Z_3-Z_1}{6}} \eta, \sqrt{\frac{Z_3-Z_1}{Z_3-Z_1}} \right]}
$$

(3.7)

where the $Z_i$ are the roots of the cubic $W^2(Z)$. However, since this leads to rather involved general expressions for $\Phi$ and $a$ in terms of integrals of elliptic functions, it proves to be more illuminating to proceed with the qualitative picture.
First we identify the critical points of the \((Z, W)\)-system, i.e. points at which \(W' = Z' = 0\) which represent an equilibrium solution of this system of equations. From (3.5), (3.6), these are
\[
\begin{align*}
    P_+ &= (2 + \sqrt{4 - B}, 0); \quad \text{with } C = C_0(B), \quad k = 1 \\
    P_{0,-} &= (0, 0); \quad B = C = 0, \quad k = 0, -1
\end{align*}
\]  
(3.8)

We will return to the classification of these critical points later.

Next, note that if \(W^2(Z)\) is always positive then \(W\) is always defined. This is clearly true for all \(B\) and \(C\) when \(k = 0, -1\), and for \(B \geq 3\) if \(k = 1\). If, however, \(W^2(Z) < 0\) at any point, then \(W\) cannot be defined and the corresponding trajectory will consist of disjoint segments. This means that there are three main forms that the \((Z, W)\) phase plane can take, which are illustrated in figure 1, according to whether the cubic \(W^2(Z)\) has three, two or one real root(s) for the \(C = 0\) trajectory. (Figure 1d is the special case \(B = 0\) of \(k = 1\).) Although these figures are shown for \(k = 1\), the diagrams are qualitatively the same for \(k = 0, -1\) with these cases generically taking the qualitative form of figure 1c, although for \(B = 0\) (vacuum) the figure includes a critical point at the origin. Not surprisingly the three different pictures translate into three (slightly) different qualitative behaviours for the cosmologies.

Before turning to this however, we first investigate a little further the parameter ranges for each option. Note that \(W^2(Z)\) always has one non-positive root, \(Z_1 \leq 0\). Whether or not it has additional real roots depends on whether the discriminant
\[
D = B^3 - 3B^2k^2 + 18kB + 9C^2 - 48k^3C
\]  
(3.9)
is negative or zero.

- If \(D < 0\) for \(C = 0\), then \(D < 0\) for some range of \(C < C_0\), say, where \(C_0\) is the critical value of \(C\) for which \(D(B, C_0) = 0\):
\[
C_0 = \frac{8}{3}k^3 - kB + \frac{1}{3} \left(4k^2 - B\right)^{3/2}
\]  
(3.10)

Therefore, for \(C < C_0\), \(W^2(Z)\) has two additional positive roots for some range of \(C\) and we have diagram 1a. This will be the case for \(k = 1, B < 3, C < C_0\). A graph of \(C_0(B)\) is shown in figure 2.
Fig. 1: The (Z,W) phase plane for the radiation cosmologies. Critical points are indicated by a disc, the disallowed $C < 0$ and $Z < 0$ regions are shaded out. The plots are shown for $k = 1$ cosmologies, but the general form of $k = 0, -1$ cosmologies is qualitatively the same as figure 1c.
Fig. 2: A plot of the parameter region in $(B, C)$ space in which the discriminant, $D$, is negative. When $D$ can be negative the $(Z, W)$ phase plane takes the form of figure 1a and we have bounce cosmologies. This can only happen for $k = 1$.

- If $D = 0$ for $C = 0$ then $W^2(Z)$ has one (repeated) root for $C = 0$. If $k = 1$, $B = 0$ then
$D$ is negative on $(0, C_0)$ and we have diagram 1d, otherwise all other trajectories have no roots and are continuous. Examining (3.9) shows that $B = 3k^2$, 0 for $D = C = 0$, and the repeated root is therefore

$$Z_+ = 3k \ ; \ Z_0 = 0$$

(3.11)

according to whether $B = 3k^2$ or $B = 0$. If $k = 1$, $B = 3$, $Z_+ = 3$, and we have diagram 1b. If $k = 0, -1$, we can only have $B = 0$, and we get diagram 1c (with the critical point at the origin as already mentioned).

- If $D > 0$ for $C = 0$ then $W^2(Z)$ has no zeros and we have diagram 1c. This will be the case for all positive $B$ for $k = 0, -1$, and for $B > 3$ if $k = 1$.

In terms of interpreting cosmological solutions it is easier to examine the original $X, Y$ variables. Using (3.2), (3.4), and (3.6) it is easy to see that

$$X^2 = \frac{C}{Z^2}$$

$$Y^2 = \frac{Z}{6} - k + \frac{B}{2Z} + \frac{C}{Z^2}$$

(3.12)

which can be used to produce parametric plots of the $(X, Y)$ plane. Although these are broadly similar for radiation and vacuum spacetimes, the vacuum plot is a true two-dimensional dynamical system whereas once the energy density of radiation is nonzero, the plot becomes a projection and contains apparent crossings of trajectories.

In translating from the $(Z, W)$ plane to the $(X, Y)$ plane there will be, broadly speaking, three qualitatively different pictures.

- Figure 1a contains trajectories along which $D < 0$, and hence $Y$ has a zero. If we order the roots of $W^2(Z)$ as $Z_1 < Z_2 \leq Z_3$, then these trajectories have two branches: the $Z \in [0, Z_2]$ branch and $Z > Z_3$ branch. For $Z > Z_3$ the trajectory starts at $(0, -\infty)$ with $Z$ infinite, and crosses the axis at $X = \pm \sqrt{C\Lambda}/Z_3$ where $Z$ is at a minimum, approaching $(0, \infty)$, $Z \to \infty$. Since $X$ always has the same sign, $\Phi$ is either monotonically increasing or decreasing along such trajectories. This means that it must be $a(\eta)$ that is causing $Z$ to be infinite, reducing to a minimum, and increasing again, i.e. this is a bounce cosmology both in the Einstein and the JBD frame. For the other branch, $Z$ starts at zero, is maximized at $Z_2$, returning to zero again. The corresponding trajectories start at $(\pm \infty, \infty)$ and tend to $(\pm \infty, -\infty)$. In this case, $X$ once again has a fixed sign, however, it is no longer small, hence $\Phi$ either increases from 0 to $\infty$ or vice versa. These solutions will also be bounce universes in either frame. For the critical $D = 0$ trajectory there are again two branches, $Z > Z_0$
and $Z < Z_0$, where $Z_0 = Z_2 = Z_3$ is the repeated root. Each of these branches corresponds to four trajectories, each of which interpolates between the images of the critical point $P_+$ in the $(X, Y)$ plane: $(\pm \sqrt{CA}/Z_0, 0)$, and $(0, \pm \infty)$ for $Z > Z_0$, or $(\pm \infty, \pm \infty)$ for $Z < Z_0$. For $D > 0$ there are no zeros of $Y$ and trajectories interpolate between $\{(\pm \infty, \pm \infty), Z = 0\}$, and $\{(0, \pm \infty), Z = \infty\}$.

- Figure 1b has one $D = 0$ trajectory which terminates at $(0,0)$; all others are of the $D > 0$ form.
- Figure 1c has $D > 0$ for all trajectories hence no trajectories cross or terminate on the $X$-axis and the phase plane consists of two disjoint parts with a ‘barrier’ across the plane separating positive and negative $Y$.

Such are the broad features of the phase planes, however, since vacuum cosmologies are a special case, we analyse them first before proceeding to the general case.

### 3.1. The vacuum cosmologies.

Although we have solutions for $X$ and $Y$, we first show how they form a two-dimensional dynamical system. For $\rho = 0$, (2.12) can be used to eliminate $Z$ from (2.13) leaving

\[
X' = -2XY \\
Y' = Y^2 + k - 3X^2
\]

independent of the value of $A$. This dynamical system has four possible critical points:

\[
S_{0,1} = (0, \pm \sqrt{\frac{-k}{3}}, 0) \leftrightarrow P_-, P_0;
\]

\[
S_{2,3} = (\pm \sqrt{\frac{k}{3}}, 0) \leftrightarrow P_+, P_0
\]

depending on $k$. These points are classified in table 1, and the phase planes are shown in figure 3.

<table>
<thead>
<tr>
<th>TABLE 1: The critical points of a vacuum cosmology.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = -1$</td>
</tr>
<tr>
<td>$S_{0,1}$</td>
</tr>
<tr>
<td>$S_{2,3}$</td>
</tr>
</tbody>
</table>
Fig. 3: The $(X,Y)$ phase plane diagrams for the vacuum cosmologies. The shaded areas represent the disallowed regions where $Z < 0$. Critical points are indicated with a dot. Figure 3d shows a closeup of the critical points in figure 3b, the grey line represents $a' = 0$ for $A = 0.75$. 
In order to interpret the phase diagrams, we must translate from the $X, Y$ variables to a cosmological solution $\Phi(t), a(t)$. We begin with the critical points.

(i) Critical point solutions.

- $k = 0$: Critical point $(0, 0)$: For $X = Y = 0$, $a$ is constant and we observe from (2.12) that $\Phi = 0$. Hence this “cosmology” is simply Minkowski spacetime.

- $k = 1$: Critical points $(\pm \frac{1}{\sqrt{3}}, 0)$: Here the constraint equation (2.12) gives $Z \equiv 4$, a constant. Solving for $a$ then gives

\[
\begin{align*}
a(t) &= \frac{\mp t}{2\sqrt{3}A} \\
\Phi(t) &= \frac{48A}{V_0 t^2}
\end{align*}
\]  

This now has a non-trivial JBD scalar field. The upper (lower) branch corresponds to the $+(-)$ critical point, has $t \in (-\infty, 0)$ ($t \in (0, \infty)$) and is collapsing to (expanding from) a singularity.

- $k = -1$: Critical points $(0, \pm 1)$: Here (2.12) gives $\Phi \equiv 0$ again, however, $a'/a = \pm 1$ is non-zero. Integration gives $a \propto e^{\mp \eta}$ hence $a = |t|$. The $+(-)$ root corresponds to $t > (<) 0$ and corresponds to an expanding (contracting) universe, however it should be pointed out that this is a coordinate transform of Minkowski spacetime, and is known as the Milne universe.

In addition to the critical point solutions, there are also some simple solutions to (3.13).

(ii) Constant scalar exact solutions.

An examination of (3.13) readily reveals that the $Y$-axis (or segments thereof) represents a solution. Inputting $X = 0$ gives

\[
Y(\eta) = \begin{cases} 
\tan \eta & k = 1 \\
-\coth \eta & k = -1
\end{cases}
\]

- $k = 0$: The $Y$-axis consists of two solution segments, $Y > 0$ and $Y < 0$, for which

\[
a(t) = e^{\pm Ht} \quad ; \quad \Phi = \frac{6H^2}{V_0}
\]

- $k = 1$: The whole of the $Y$-axis is a trajectory, which corresponds to the cosmological solution

\[
a(t) = H^{-1} \cosh Ht \quad ; \quad \Phi = \frac{6H^2}{V_0}
\]
\( k = -1 \): There are again two solution segments, since (2.12) requires \( Y^2 \geq 1 \). The cosmological solutions are given by

\[
a(t) = H^{-1} \sinh H|t| \quad ; \quad \Phi = \frac{6H^2}{V_0}
\]

Note that these are the de-Sitter family of solutions.

(iii) Attractor solutions.

Finally, it is apparent from figure 1 that there are three main attractors:

\[
\mathcal{A}_0 = \{ X = 0, Y \to \infty \}
\]

\[
\mathcal{A}_\pm = \{-Y \propto \pm X \to \infty \}
\]

We have already discussed the attractor \( \mathcal{A}_0 \) which corresponds to the de-Sitter family of universes, therefore it only remains to discuss \( \mathcal{A}_\pm \). Clearly, from (3.12) \( Z \to 0 \) along each of these trajectories, and to leading order we find:

\[
Y \propto \frac{1}{2\eta}, \quad Z \propto -\eta, \quad \text{with} \quad \eta \to 0^-
\]

\[
a(t) \propto |t|^{\frac{2A+1}{2A+1}}, \quad \Phi \propto |t|^{\frac{1}{2A+1}}, \quad \text{with} \quad \begin{cases} 
  t \to 0^- & \text{for } 6A \pm 1 > 0, \\
  t \to \infty & \text{for } 6A - 1 < 0.
\end{cases}
\]

\[
a(t) \propto e^{Ht}, \quad \Phi \propto e^{-3Ht}, \quad \text{with } t \to \infty \text{ for } 6A = 1
\]

Thus the attractor \( \mathcal{A}_+ \) always corresponds to a (power law) contracting universe with a diverging JBD scalar. The \( \mathcal{A}_- \) attractor has a vanishing JBD scalar and is contracting for \( A > 1/2 \), expanding for \( A < 1/2 \). This behaviour can be seen clearly from considering the \( A' = 0 \) line \( Y = X/2A \). For \( A > 1/2 \), this line lies in the physically inaccessible \( \rho < 0 \) regions for \( k = 0, -1 \), only briefly crossing the \( \rho > 0 \) region for \( k = 1 \) giving rise to a small number of bounce solutions in that case. For \( A < 1/2 \), the \( A' = 0 \) line lies (almost) entirely in the allowed \( \rho > 0 \) region, with a small exception now for \( k = -1 \) where it crosses the \( \rho < 0 \) barrier. This means that there are many bounce solutions for \( A < 1/2 \). In particular, the vacuum attractor \( \mathcal{A}_- \) now becomes a late time expanding solution.

We are now in a position to state the general behaviour of a vacuum cosmology. Apart from a few special \( k = 1 \) trajectories which cross the \( X \)-axis, the late time behaviour of a vacuum universe is determined by which quadrant of the \((X, Y)\) plane it lies in. Trajectories with \( Y > 0 \) attract to \( \mathcal{A}_0 \) - the de-Sitter universes, and those with \( Y < 0 \) attract to \( \mathcal{A}_\pm \) according to whether \( X = \pm |X| \). The exceptions to this rule are a small set of \( k = 1 \) cosmologies which correspond to trajectories crossing the \( X \)-axis. These correspond to \( D < 0 \) trajectories and either have \( Z \geq Z_3 \) or \( 0 < Z \leq Z_2 \). The former trajectories lie
very close to the $Y$-axis and appear to be perturbations of de-Sitter universes with a very nearly constant dilaton. The latter trajectories are perturbations of the $\mathcal{A}_\pm$ attractors either having a very small dilaton ($\mathcal{A}_-$) or a very large dilaton ($\mathcal{A}_+$). We should also point out that in addition to these bouncing cosmologies, for $k = 1$ there are also a handful of vacillating and coasting cosmologies as indicated in the closeup figure 1d. A vacillating cosmology has two zeros of $a'$ and corresponds to an expanding universe which slows down, recontracts for a short period, then re-expands (or vice versa). The coasting cosmology is tangent to the $a' = 0$ line, and represents a universe which expands, slows down and halts expansion, then re-expands again (and vice versa).

3.2. The radiation cosmologies.

The classification of the radiation cosmologies is extremely similar to the vacuum cosmologies, as can be seen from the $(X, Y)$ phase plots shown in figure 4, however, there are a few subtle differences that should be highlighted.

As we have already mentioned, the radiation $(X, Y)$ plot is a projection, and therefore contains several features that are a consequence of this. Most obvious is the apparent crossing of trajectories. Recall that different trajectories correspond to different choices of $C$ in (3.12), therefore, when $Z_1 = \sqrt{\frac{C_1}{C_2}} Z_2$ the value of $X$ at $Z_1$ and $Z_2$ is the same. If in addition $Z_1 = 6B/Z_2$, then the $Y$ values are the same. Since $Z$ ranges from zero to infinity on those trajectories which have a minimum non-zero value of $|Y|$, this means that any one such trajectory will intersect any other such trajectory once. One implication of this is that the boundary of the physically allowed region is no longer a trajectory (or projective trajectory) as it was in the vacuum case. The physically allowed region is given from (3.2):

$$Y^2 + k - X^2 = \frac{Z}{6} + \frac{B}{2Z}$$

(3.21)

In vacuum, $Z \geq 0$ gave $Y^2 + k \geq X^2$, hence the shaded regions in figure 3. If, however, $B \neq 0$, the right hand side of (3.21) now has a minimum value $\sqrt{\frac{B}{3}}$ for $Z = \sqrt{3B}$, i.e.

$$Y^2 + k - X^2 \geq \sqrt{\frac{B}{3}}$$

(3.22)

as shown in figure 4. Therefore, unless we are at the critical point $P_0$ for $B = 3$, $Z$ cannot remain at $\sqrt{3B}$ along a trajectory. This means that each trajectory reaches the minimum value at most once, and for $Z \in (0, \infty)$ trajectories, at least once. Thus the curve given by equality in (3.22) forms an envelope for the physical trajectories.
Fig. 4: The ($X,Y$) phase plane diagrams for the radiation cosmologies. The shaded areas represent the disallowed regions where $Z < 0$. These figures are for $k = 1$, the general figure for $k = 0, -1$ taking the appearance of figure 4c. Figure 4d shows the trajectories for different spatial curvatures with $B$ chosen to give the same disallowed region; the solid trajectories are those with $k = 1$, the dashed, $k = 0$, and the dotted, $k = -1$. 
Another straightforward difference is that for \( k = 0,-1 \), there can be no critical points, since \((3.2)\) implies \(|Y|, Z > 0\) strictly, and hence \(|Z'\) is strictly positive. Thus the intersection of the forbidden region with the \( Y\)-axis at \((0, \left( \frac{B}{3} \right)^{\frac{1}{3}})\) does not represent a critical point, merely a stationary point on the trajectory \( Y(Z) = \sqrt{\frac{Z}{6} + \frac{B}{2Z}} \). For \( k = 1 \), we must have \( B \leq 3 \) for a critical point, in which case the points correspond to \( P_+ \) as defined in \((3.8)\) for the \((Z, W)\)-plane, and occur at \( X = \sqrt{C_0/Z_+} \), where \( Z_+ \) is as defined in \((3.11)\). The critical points are classified in table 2.

<table>
<thead>
<tr>
<th>( k = -1 )</th>
<th>( k = 0 )</th>
<th>( k = +1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_+ )</td>
<td>—</td>
<td>((\frac{\sqrt{C_0}}{2+\sqrt{4-B}}, 0), B &lt; 3, \text{SADDLE})</td>
</tr>
<tr>
<td>( P_{0,-} )</td>
<td>VACUUM</td>
<td>ONLY</td>
</tr>
</tbody>
</table>

Finally, note that the ‘attractors’ \( \mathcal{A}_\pm \) are subtly different for the radiation case. Although to leading order \( \mathcal{A}_\pm \) and the solutions are the same, to sub-leading order,

\[
|Y| \sim |X| + \frac{B}{4\sqrt{C}}
\]

Therefore, qualitatively the overall behaviour of the radiation cosmologies is the same as the vacuum cosmologies. Universes starting at \( Y > 0 \) generically attract to de-Sitter universes at late times, and those with \( Y < 0 \) to the \( \mathcal{A}_\pm \) solutions. For \( k = 1 \) and sufficiently low radiation density \((B < 3)\) there will be a small number of trajectories crossing the axis, and hence universes which are a perturbation of the full de-Sitter/ \( \mathcal{A}_\pm \) universes, however, as we consider initial values \(|Y| \to \infty\), these form a smaller and smaller set, tending to measure zero. In addition, there will also be a small number of coasting and vacillating cosmologies.

3.3. The Einstein frame.

Finally, we would like to remark on the naturalness of the de-Sitter solutions within this family by making a conformal transformation to the Einstein frame. To do this, we set

\[
g_{ab} = \Phi^{-1} \hat{g}_{ab}
\]

\[\text{(3.24)}\]
so that

\[ \sqrt{-g} \Phi R = \sqrt{-\hat{g}} \hat{R} + 3\delta \ln \Phi - \frac{3}{2} (\nabla \ln \Phi)^2 \]  

(3.25)

and the gravitational part of the action appears in Einstein form.

Under such a transformation, the radiation energy-momentum tensor is conformally invariant, and the potential transforms as

\[ \sqrt{-g} V(\Phi) = \sqrt{-\hat{g}} V_0 \]  

(3.26)

i.e. the potential appears as a cosmological constant in the Einstein frame. Thus, for \( \Phi \) constant, we would expect de-Sitter solutions.

4. Dust and false vacuum cosmologies.

In this section we analyse the dynamical system (2.13) for the cases of dust (\( \gamma = 1 \)) and false vacuum (\( \gamma = 0 \)). As before, we identify the critical points of the dynamical system, classifying them, and identify the physical regions of parameter space finding sample trajectories by numerical integration. Finally, we re-interpret these solutions as cosmologies, in terms of the original cosmological parameters \( \Phi(t) \) and \( a(t) \).

4.1. The dust cosmologies.

For dust, \( \gamma = 1 \), and the dynamical system is

\[ X' = -2XY + \frac{1}{4A}(Y^2 + k - X^2 - \frac{Z}{6}) \]

\[ Y' = -\frac{1}{2}(Y^2 + k) - \frac{3X^2}{2} + \frac{Z}{4} \]  

(4.1)

\[ Z' = 2ZY \]
Fig. 5: The three-dimensional phase diagram for a $k = 0$ dust cosmology with $A = 1$. The $A_0$ attractor is clearly visible.
This apparently has six possible critical points:

\[
Q_{0,1} = (0, \pm \sqrt{-k}, 0) \leftrightarrow S_{0,1} \\
Q_{2,3} = (\pm \sqrt{\frac{k}{3}}, 0, 4k) \leftrightarrow S_{2,3} \\
Q_{4,5} = (\pm 2A \sqrt{\frac{-k}{1 + 12A^2}}, \pm \sqrt{\frac{-k}{1 + 12A^2}}, 0)
\]

however, substitution of \(Q_{4,5}\) into (2.12) indicates that it is a non-physical critical point, as it corresponds to \(\rho < 0\). For \(k = 0\), these are all coincident at the origin; for \(k = -1\), only \(Q_{0,1}\) are relevant, and for \(k = 1\), only \(Q_{2,3}\). Table 3 summarizes the information concerning the critical points for the dust filled universe.

Since the only physical critical points correspond to the vacuum critical points \(S_{0-3}\), we can refer to section 3.1 for the solutions to which these correspond, namely, Minkowski spacetime or the Milne universe for \(S_{0,1}\) and (3.15) for \(S_{2,3}\).

Clearly, since the dynamical system is now inherently three-dimensional, a two-dimensional plot will not convey all the information, however, since the sign of \(Z'\) is determined by the sign of \(Y\), most of the useful information can be gleaned from an \((X, Y)\) plot. For the purposes of visualisation, we include fig. 5 which is a three-dimensional plot of the \(A = 1, k = 0\) dust system. Note how the trajectories strongly attract to \(Z \to \infty, X \to 0\) in the \(Y > 0\) region. This corresponds to the de-Sitter attractor \(A_0\) of the vacuum system. It is not difficult to understand this, since for small \(X, \Phi\) is very nearly constant, and \(a \to \infty\) since \(Z \to \infty\). In other words, \(\rho \propto a^{-3} \to 0\) and we rapidly approach a vacuum situation.

For \(Y < 0, Z \to 0\) rapidly along trajectories, and therefore we do not necessarily asymptote a vacuum situation. Indeed, since \(Z \simeq 0\), we expect, and observe, that the cosmologies will take the form of the Kolitch and Eardley dust cosmologies, whose two-dimensional dynamical system is obtained from ours by setting \(Z = 0\).
Fig. 6: The $(X, Y)$-projections of the dust cosmologies. The $A_0$ attractor is evident in all of these. Figure 6a corresponds to the projection of figure 5. The grey line corresponds to $a' = 0$ in each case.

Figure 6 shows a selection of two-dimensional projections of the phase space onto the
(X, Y) plane. Note that \( A_+ \) is weakly attractive where the divergence of \( \Phi \) mitigates the divergence of \( \rho \) in (2.12) to allow the cosmology to more closely mimic the vacuum scenario.

To summarize, dust cosmologies which have \( Y > 0 \) generally attract to de-Sitter cosmologies. If \( Y < 0 \) then \( A_+ \) appears to weakly attract local trajectories, but otherwise there is no apparent simple classification of the late time behaviour as in the radiation-vacuum cases. For \( k = 1 \), as with the radiation/vacuum cosmologies, there exist trajectories crossing the X-axis. These correspond to bouncing cosmologies and take one of two forms. If \( Z \geq Z_{\text{min}} \) then the trajectory stays close to the Y-axis and is a perturbation of the de-Sitter universe. If \( 0 \leq Z \leq Z_{\text{max}} \) then the trajectory start out from \((-\infty, \infty, 0)\) and either asymptotes \( A_+ \) or \( A_- \), the bulk of such trajectories taking the former course.

### 4.2. The false vacuum cosmologies.

For false vacuum, \( \gamma = 0 \), and the dynamical system is

\[
X' = -2XY + \frac{1}{A}(Y^2 + k - X^2 - \frac{Z}{6}) \\
Y' = Y^2 + k - 3X^2 \\
Z' = 2ZY
\]  

This has possible critical points

\[
R_{0,1} = (0, \pm \sqrt{-k}, 0) \leftrightarrow S_{0,1} \\
R_{2,3} = (\pm \sqrt{\frac{k}{3}}, 0, 4k) \leftrightarrow S_{2,3} \\
R_{4,5} = (\pm A\sqrt{\frac{-k}{1 - 3A^2}}, \pm \sqrt{\frac{-k}{1 - 3A^2}}, 0)
\]  

These are classified in table 4.

As before, we can refer to the vacuum critical point solutions for \( R_{0-3} \), what is less clear is that \( R_{4,5} \) are also essentially the same cosmological solutions. To see this note that \( Z = 0 \) requires either \( \Phi = 0 \) or \( a = 0 \), (2.12) then implies \( \Lambda = 0 \) or \( \infty \) respectively – a somewhat artificial limit. Setting \( \Phi = 0 \) in order to get a meaningful cosmology then gives \( a \propto |t| \). This is, in each case, a coordinate transform of Minkowski spacetime and is equivalent to the \( P_0 \) critical point of the \((Z, W)\)-plane.

As with the dust cosmologies, the false vacuum dynamical system is also three-dimensional, and as before, we plot the two-dimensional projection onto the \((X, Y)\)-plane in figure 7. The false vacuum case however, seems to be the opposite of dust, since for
Y > 0 there is no obvious attractor, while for Y < 0 the $A_\pm$ attractors show up clearly. We have checked this behaviour by compactifying to the Poincaré sphere and analysing the critical points at infinity which confirms that there is no attractor for Y > 0. Again, this is not difficult to understand, since the only difference between the false vacuum dynamical system (4.3) and the vacuum one (3.13) is the presence of the extra term in the $X'$ equation, which is proportional to $\lambda a^2/\Phi$. But ($\ln a^2/\Phi)' \propto (Y - X/A)$ hence if $Y < X/A$, as it is for both $A_\pm$ for $A \geq 1$ or for $A_+$ for all $A$, then this term is rapidly damped and we approach the vacuum situation. For $Y > 0$ however, this extra term in $X'$ grows and we depart increasingly from vacuum at late times. Thus, in distinction to all the other cosmologies we have considered, the expanding false vacuum cosmologies do not attract to de-Sitter universes at late times.

5. Conclusions

In this paper we have presented a dynamical systems analysis of the FRW cosmologies in Brans-Dicke theory with a potential for the scalar JBD field. Our analysis was based on that of Kolitch and others, [14], who found that cosmology in pure JBD theory could be expressed as a two-dimensional dynamical system. In the presence of a scalar potential, we found that unless the matter source was radiation, the dynamical system was inherently three-dimensional. In spite of this we were able to see clearly that the asymptotic late time behaviour of the expanding cosmologies was generically that of a de-Sitter universe, the specifics depending on the topology of the spatial sections of the cosmology, and the
Fig. 7: The (X, Y)-projections for the false vacuum cosmologies. Note the change in the attractors for Y < 0 for the different values of A. The grey line corresponds to a' = 0 in each case.

only exception being that of a false-vacuum source. Since the potential chosen is actually
equiv alen t to an Einstein cosmological constant, these results are in general agreement with
the statement that general relativity is an attractor for Brans-Dicke theory[12], although
this statement comes with certain caveats, as the false vacuum case illustrates.

In order to get an idea of when general relativity is a late time attractor, consider the
intermediate equation for $X'$, (2.11), substituting the behaviour of $\rho$ from (2.9):

$$X' = -2XY + (1 - 3\gamma/4) \frac{8\pi \rho_0}{3A} a^{2-3\gamma}\Phi^{-1}$$

Now,

$$(\ln[a^{2-3\gamma}\Phi^{-1}])' \propto (2 - 3\gamma)Y - 3(1 - \gamma)X/A$$

therefore, if $\gamma > 2/3$, $[a^{2-3\gamma}\Phi^{-1}]$ will decrease along a positive $Y$ trajectory, and $X$ will
be driven to zero. $\Phi$ will therefore become constant and we will asymptote a de-Sitter
universe. For $\gamma < 2/3$ however, we do not expect a late time de-Sitter solution. Since
$\rho + 3p = (3\gamma - 2)\rho$, $\gamma < 2/3$ corresponds to a matter source which does not satisfy the
strong energy condition, therefore it appears that general relativity is an attractor for
matter obeying the SEC.

Clearly this model is somewhat artificial since it neither anchors $\Phi$ at a specific value,
nor behaves as we might expect a scalar potential to behave, however, it does illustrate
differences that crop up between pure JBD theory and those with a scalar potential. In
particular the appearance of a strong attractor for expanding universes. However the
model also displays many similarities with the pure JBD case, for instance, the presence
of vacillating or coasting cosmologies for $k = 1$.

Perhaps the most important conclusion to draw from this work is that for “ordinary
matter” (i.e. matter obeying the strong energy condition) general relativity is an attractor,
therefore, independent of the specifics of a dilaton potential, i.e. how and at what energy
scale the dilaton acquires a mass and gravity becomes Einstein in nature, the universe
should exit the stringy era of gravity in a reasonably familiar Einstein form. On the other
hand, at higher energies, when an inflationary phase may have occurred, there is no strong
attractor(s) – very much in contrast to pure JBD [14] – hence (as was hinted in [15]) the
behaviour of inflationary universes in this case could be strongly affected by the behaviour
of the dilaton potential.

Acknowledgements

We would like to thank Bob Johnson and Filipe Bonjour for conversations and assis-
tance with numerical aspects of this problem. C.S. would like to thank the Theoretical
Physics Group of the University of Oporto, in particular Prof.E.S.Lage and Dr.J.C.Mourão (IST-Lisbon) for help in the early stages of this project. This work was supported by a JNICT fellowship BD/5814/95 (C.S.), and a Royal Society University Research Fellowship (R.G.).

References

    J.Barrow and P.Parsons, *The behaviour of cosmological models with varying G*, gr-qc/9607072