A U(1) Gauge Theory for
Antisymmetric Tensor Fields

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Abstract

We show that a U(1) gauge theory defined in the configuration space for closed $p$-branes yields the gauge theory of a massless rank $(p + 1)$ antisymmetric tensor field and the Stueckelberg formalism for a massive vector field.

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Antisymmetric tensor fields have been introduced in various theories. For instance, supergravity multiplets in many supergravity theories include antisymmetric tensor fields [1]. In p-brane theories it is known that rank-(p + 1) antisymmetric tensor fields couple naturally to p-branes [2].

In these theories, however, rank-(p + 1) [p ≥ 1] antisymmetric tensor fields have not been understood in terms of differential geometry. In order to make geometric aspects of rank-(p + 1) antisymmetric tensor fields clear, Freund and Nepomechie have introduced the space of all closed p-manifolds embedded in space-time [3]. This space is nothing other than the configuration space for closed p-branes. Hereafter we refer to the space as closed p-manifold space. Freund and Nepomechie pointed out that a rank-(p + 1) antisymmetric tensor field is geometrically characterized as a constrained connection on a U(1) bundle over the closed p-manifold space. In their paper, only the case p = 1 was discussed in detail: they derived the gauge theory of a massless rank-two antisymmetric tensor field from a U(1) gauge theory in loop space.

In a recent paper [4], the U(1) gauge theory in loop space has been reconsidered more elegantly. Thereby it became possible to derive the Stueckelberg formalism for massive tensor fields, as well as the gauge theory of a massless rank-two antisymmetric tensor field, from the U(1) gauge theory in loop space [4,5].

The purpose of the present paper is to generalize the formulation in ref.[4] by replacing the loop space by the closed p-manifold space; we indeed obtain the gauge theory of a massless rank-(p + 1) antisymmetric tensor field from a U(1) gauge theory defined in the closed p-manifold space. In addition we show that the U(1) gauge theory in closed p-manifold space yields the Stueckelberg formalism for a massive vector field.

We define a closed p-manifold space ΩpMp as the set of all closed p-manifolds embedded in D-dimensional Minkowski space M. An arbitrary closed p-manifold in M with a parametrization xμ = xμ(σ) [σ ≡ (σ1, σ2, ..., σp)] is represented as a point in ΩpMp denoted by coordinates (xμσ) with xμσ = xμ(σ) 1. The closed one-manifold space Ω1M1 is nothing other than the loop space. In what follows, we treat the pair (µσ) as a generalized index in ΩpMp. Since each closed p-manifold in M itself does not vary under reparametrizations, xμ(σ) are reparametrization scalar functions on the p-manifold. Thus the transformation rule of the coordinates (xμσ) under an infinitesimal reparametrization σ → σ + ε(σ) [ε = (ε1, ε2, ..., εp)] is determined to be

\[
x^{\mu\sigma} \rightarrow \tilde{x}^{\mu\sigma} = x^{\mu\sigma} + \varepsilon^{(\sigma)}x^{\mu}_{\alpha}(\sigma)
\]

where x^{\mu}_{\alpha}(\sigma) ≡ ∂x^{\mu}(\sigma)/∂σ^\alpha (α = 1, 2, ..., p), and ε^{(\sigma)} are infinitesimal functions of σ.  

1) In this paper, the indices κ, λ, μ, ν and ξ take the values 0, 1, 2, ..., D − 1, while each of the indices ρ, σ, χ and ω is a set of p real variables that take values necessary to parametrize a closed p-manifold.
Let us consider a U(1) gauge theory defined in $\Omega^p M^D$. The infinitesimal gauge transformation of a U(1) gauge field $A_{\mu\sigma}$ on $\Omega^p M^D$ is given by

$$\delta A_{\mu\sigma}[x] = \partial_\lambda A_{\mu\lambda}[x],$$

(2)

where $\partial_\mu \equiv \partial / \partial x^\mu$, and $\Lambda$ is an infinitesimal scalar function on $\Omega^p M^D$. Since the U(1) gauge symmetry has no relation with reparametrizations, $\Lambda$ has to be invariant under (1):

$$x'^\mu_\alpha(\sigma) \partial_\mu A_{\alpha\beta} = 0.$$  

(3)

Combination of (2) and (3) leads to the condition $\delta(x'^\mu_\alpha(\sigma) A_{\mu\sigma}) = 0$, which shows that $x'^\mu_\alpha(\sigma) A_{\mu\sigma}$ is gauge invariant. Hence $x'^\mu_\alpha(\sigma) A_{\mu\sigma}$ will be written in terms of transverse components of $A_{\mu\sigma}$. However, such an expression is incompatible with the fact that $x'^\mu_\alpha(\sigma) A_{\mu\sigma}$ is Lorentz scalar, since the transverse components are dependent on a choice of coordinate system. Consequently, we conclude that

$$x'^\mu_\alpha(\sigma) A_{\mu\sigma}[x] = 0.$$  

(4)

The reparametrization-invariant condition for $A_{\mu\sigma}$ is found to be

$$x'^\mu_\alpha(\sigma) \partial_\mu A_{\alpha\beta} + \delta_\mu(\sigma - \bar{\rho}) A_{\alpha\bar{\rho}} = 0,$$

(5)

where $\delta(\sigma) \equiv \prod_\alpha \delta(\sigma^\alpha)$ and $\delta_\mu(\sigma) \equiv \partial \delta(\sigma) / \partial \sigma^\mu$.

We now assume the existence of a reparametrization-invariant measure $[dx] = d^D x \{dy\}$ of $\Omega^p M^D$. Here $d^D x \equiv \prod_\mu dx^\mu$ is defined from $x^\mu \equiv \int \frac{d^D x}{\sim x^\mu(\bar{\sigma})}$ ($\sim \equiv \int \sim x^\sigma$), which denote a point in $M^D$, and $\{dy\}$ is a measure defined from $y^\alpha(\sigma) \equiv x^\mu(\sigma) - x^\mu$. In a similar fashion to the U(1) gauge theory in loop space [3,4], we insert the damping factor $\exp(-L/v^2)$ with $L \equiv \int \frac{d^D x}{\sim x^\mu} \det(-\eta_{\mu\nu} x'^\mu_\alpha(\sigma) x'^\nu_\beta(\sigma))$ into an action for $A_{\mu\sigma}$ so that it becomes well-defined. Here $v (> 0)$ is a constant with dimensions of [length]$^p$, and $\eta_{\mu\nu}$, diag$\eta_{\mu\nu} = (1, -1, -1, \ldots, -1)$, is the metric tensor on $M^D$. The action for $A_{\mu\sigma}$ with the damping factor is given by

$$S_R = k \int \frac{[dx]}{V_R} \mathcal{L} \exp \left( - \frac{L}{v^2} \right)$$

(6)

with the Maxwell-type lagrangian

$$\mathcal{L} = -\frac{1}{4} \epsilon_{\rho\mu\lambda\sigma} \mathcal{G}^{\rho\mu\lambda\sigma} \mathcal{F}_{\nu\bar{\rho}\mu\bar{\lambda}} \mathcal{F}_{\lambda\bar{\sigma},\nu\bar{\sigma}},$$

(7)

where $k$ is a constant, $V_R \equiv \int \{dy\} \exp(-L/v^2)$, and

$$\mathcal{F}_{\mu\bar{\rho},\nu\bar{\sigma}} \equiv \partial_{\mu\bar{\rho}} A_{\nu\bar{\sigma}} - \partial_{\nu\bar{\sigma}} A_{\mu\bar{\rho}}.$$  

(8)

We employ Einstein's convention for indices in $\Omega^p M^D$; for example, $V^\mu\bar{\rho} W_{\nu,\bar{\sigma}} = \sum_\mu \int \frac{d^D x}{\sim x^\mu} V^\mu\bar{\rho} W_{\nu,\bar{\sigma}}$. 

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The (inverse) metric tensor $G^\mu_\nu,\nu\sigma$ can be constructed so as to incorporate reparametrization invariance into the lagrangian $L$. (Such a metric was actually defined in the U(1) gauge theory in loop space [4].) In the present paper, however, we use the metric $G^\mu_\nu,\nu\sigma = \eta^\mu_\nu \delta(\rho - \sigma')$, disregarding the reparametrization invariance of $L$. We understand that the integrations with respect to $\eta^0(\sigma')$ are carried out after applying the Wick rotation $\eta^0(\sigma') \rightarrow -i\eta^0(\sigma')$.

Let us now try to derive the local gauge theory of a massless rank-$(p + 1)$ antisymmetric tensor field on $M^D$. The simplest solution of (3) is

$$\Lambda^{(0)}[x] \equiv \int d^D \sigma \frac{q_0}{p!} \Sigma^{\mu_1,\mu_2,...,\mu_p}(x(\sigma)) \lambda_{\mu_1,\mu_2,...,\mu_p}(x(\sigma)),$$  \hspace{1cm} (9)

where $\Sigma^{\mu_1,\mu_2,...,\mu_p}(\sigma) \equiv x^{\mu_1}(\sigma)x^{\mu_2}_2(\sigma)\cdots x^{\mu_p}_p(\sigma)$, $q_0$ is a constant with dimensions of $[\text{length}]^{-p}$, and $\lambda_{\mu_1,\mu_2,...,\mu_p}$ is an infinitesimal antisymmetric tensor function on $M^D$. The solution of (4) associated with $\Lambda^{(0)}$ is given by

$$A^{(0)}_{\mu \rho,\rho \nu}(x) \equiv \frac{q_0}{p!} \Sigma^{\nu_1,\nu_2,...,\nu_p}(\sigma) B_{\mu_1,\nu_1,\nu_2,...,\nu_p}(x(\sigma)),$$  \hspace{1cm} (10)

where $B_{\mu_1,\nu_1,\nu_2,...,\nu_p}$ is a rank-$(p + 1)$ antisymmetric tensor field on $M^D$. It is verified that $A^{(0)}_{\mu \rho,\rho \nu}$ satisfies the condition (5). Substitution of (9) and (10) into (2) yields the gauge transformation

$$\delta B_{\mu_1,\nu_1,\nu_2,...,\nu_p}(x) = \frac{1}{p!} \partial_{\mu_1} \lambda_{\mu_1,\nu_1,\nu_2,...,\nu_p}(x).$$  \hspace{1cm} (11)

Substituting (10) into (8), we obtain the field strength of $A^{(0)}_{\mu \rho,\rho \nu}$:

$$F^{(0)}_{\mu \rho,\rho \nu}(x) = \frac{q_0}{p!} \delta(\rho - \sigma) \Sigma^{\lambda_1,\lambda_2,...,\lambda_p}(\sigma) F_{\mu \lambda_1,\lambda_2,...,\lambda_p}(x(\sigma))$$  \hspace{1cm} (12)

with

$$F_{\mu \lambda_1,\lambda_2,...,\lambda_p}(x) \equiv \frac{1}{(p + 1)!} \partial_{\mu} B_{\lambda_1,\lambda_2,...,\lambda_p}(x),$$  \hspace{1cm} (13)

which is invariant under the transformation (11). The field strength of $A^{(0)}_{\mu \rho,\rho \nu}$ is thus written in terms of that of $B_{\mu_1,\nu_1,\nu_2,...,\nu_p}$. This result makes easy to derive an action for $B_{\mu_1,\nu_1,\nu_2,...,\nu_p}$ from (6).

The lagrangian (7) with (12) becomes

$$L^{(0)} = - \frac{q_0^2}{4(p!)} \delta(\rho) \int \frac{d^D \sigma}{c} \Sigma^{\lambda_1,\lambda_2,...,\lambda_p}(\sigma) \Sigma^{\nu_1,\nu_2,...,\nu_p}(\sigma) \times F_{\lambda_1,\lambda_2,...,\lambda_p}(x(\sigma)) F_{\nu_1,\nu_2,...,\nu_p}(x(\sigma)).$$  \hspace{1cm} (14)
Recalling \( x^\mu(\vec{\sigma}) = x^\mu + y^\mu(\vec{\sigma}) \), we carry out the Taylor expansion of the \( F(x(\vec{\sigma}))^2 \) term around \( (x^\mu) \):

\[
F_{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_p}(x(\vec{\sigma})) F_{\kappa_1 \cdots \kappa_p \mu_1 \cdots \mu_p}(x(\vec{\sigma})) = F_{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_p}(x) F_{\kappa_1 \cdots \kappa_p \mu_1 \cdots \mu_p}(x) + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \prod_{j=1}^{D-1} \sum_{\mu_j=0}^{\kappa_j} g^{\mu_j}(\vec{\sigma}) \partial_{\mu_j} \right) F_{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_p}(x) F_{\kappa_1 \cdots \kappa_p \mu_1 \cdots \mu_p}(x). \tag{15}
\]

All the differential coefficients at \( (x^\mu) \) in this Taylor series vanish after integration with respect to \( x^\mu \), since \( |x^\mu| < \infty \). Thus the action (6) with the lagrangian (14) is written as

\[
S_{\text{R}}^{(0)} = -\frac{kq_0^2}{4(p^!)^2} \delta(\vec{0}) \int d^D x F_{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_p}(x) F_{\kappa_1 \cdots \kappa_p \mu_1 \cdots \mu_p}(x) \frac{1}{V_{D}} \int \left\{ d\sigma \right\} \frac{d^p \sigma}{\sigma} \sum_{\lambda_1 \cdots \lambda_p} \Sigma_{\kappa_1 \cdots \kappa_p} \exp \left( -\frac{L}{\sigma} \right) \]

\[
= \frac{(-1)^p(D-p)!(D-1)}{4D!} \frac{\partial \log V_{D}}{\partial(1/\sigma^2)} \sum_{P} \text{sgn}(P) \eta^{\lambda_1 \sigma_{P(1)}} \cdots \eta^{\lambda_p \sigma_{P(p)}} \frac{1}{V_{D}} \delta(\vec{0}). \tag{16}
\]

by using the formula

\[
\frac{1}{V_{D}} \int \left\{ d\sigma \right\} \frac{d^p \sigma}{\sigma} \sum_{\lambda_1 \cdots \lambda_p} \Sigma_{\kappa_1 \cdots \kappa_p} \exp \left( -\frac{L}{\sigma} \right) \]

\[
= \frac{(-1)^{p+1}p!(D-p)!}{D!} \frac{\partial \log V_{D}}{\partial(1/\sigma^2)} \sum_{P} \text{sgn}(P) \eta^{\lambda_1 \sigma_{P(1)}} \cdots \eta^{\lambda_p \sigma_{P(p)}}. \tag{17}
\]

Here \( P \) denotes a permutation of the numbers \( 1, 2, \ldots, p \), and \( \text{sgn}(P) \) takes \( 1(-1) \) for an even(odd) permutation. The summation \( \sum_{P} \) extends over all possible permutations. Defining the constants \( k \) and \( q_0 \) so as to satisfy the normalization condition

\[
-\frac{(p+2)!(D-p)!}{2D!} \frac{\partial \log V_{D}}{\partial(1/\sigma^2)} k^2 \delta(\vec{0}) = 1, \tag{18}
\]

we arrive at the well-known action \([1,2,6]\)

\[
S_{\text{R}}^{(0)} = \frac{(-1)^{p+1}}{2(p + 2)!} \int d^D x F_{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_p}(x) F_{\kappa_1 \cdots \kappa_p \mu_1 \cdots \mu_p}(x). \tag{19}
\]

The condition (18) is understood with suitable regularizations of the divergent quantities. As a consequence, the \( U(1) \) gauge theory in \( \Omega \) \( MP \) leads to the gauge theory defined by (19).

Since \( A_{\mu \vec{\sigma}} \) is a functional field on \( MP \), it appears that \( A_{\mu \vec{\sigma}} \) contains an infinite number of local component fields. In fact, in addition to \( A_{\mu \vec{\sigma}}^{(0)} \), equation
(4) has an infinite number of fundamental solutions written in terms of local fields on $M^D$. Among them, we next examine a fundamental solution consisting of vector and scalar fields on $M^D$.

Consider the following solution of (3):

$$
\Lambda^{(1)}[x] \equiv \int \frac{d^\sigma}{\sqrt{\sigma}} q_1 \sqrt{-\Sigma^2(\sigma)} \lambda(x(\sigma)) ,
$$

(20)

where $\Sigma^2(\sigma) \equiv \Sigma_{\mu_1 \mu_2 \cdots \mu_p}(\sigma)\Sigma_{\mu_1 \mu_2 \cdots \mu_p}(\sigma)$, $q_1$ is a constant with dimensions of [length]$^{-p}$, and $\lambda$ is an infinitesimal scalar function on $M^D$. The solution of (4) associated with $\Lambda^{(1)}$ consists of a vector field $A_\mu$ and a scalar field $\phi$ on $M^D$:

$$
A^{(1)}_{\mu \sigma}[x] \equiv q_1 \sqrt{-\Sigma^2(\sigma)} \left( \delta_{\mu}^\nu x^\nu_{p} Q_{\lambda_1 \lambda_2 \cdots \lambda_{p-1}}(\sigma) A_\mu(x(\sigma)) + \epsilon q_1 h_1 (\sigma) \cdots h_{p-1}(\sigma) Q_{\lambda_1 \lambda_2 \cdots \lambda_{p-1}}(\sigma) \phi(x(\sigma)) \right) ,
$$

(21)

where $Q_{\mu_1 \mu_2 \cdots \mu_p}(\sigma) \equiv \Sigma_{\mu_1 \mu_2 \cdots \mu_p}(\sigma)/\sqrt{-\Sigma^2(\sigma)}$, $\partial_\sigma \equiv \partial/\partial \sigma^\nu$, and $\epsilon$ is a constant with dimensions of [length]$^{-p+1}$. Note that $A^{(1)}_{\mu \sigma}$ satisfies the condition (5). From (2) with (20) and (21), we have the gauge transformation

$$
\delta A_\mu(x) = \partial_\mu \lambda(x) , \quad \delta \phi(x) = \frac{q_1}{\epsilon} \lambda(x) .
$$

(22)

By direct calculation, the field strength of $A^{(1)}_{\mu \sigma}$ is obtained as follows:

$$
F^{(1)}_{\mu \nu \sigma \tau}[x] = q_1 \delta(\sigma - \sigma') \left\{ \sqrt{-\Sigma^2(\sigma)} F_{\mu \nu}(x(\sigma)) - \epsilon q_1 h_1 (\sigma) \cdots h_{p-1}(\sigma) Q_{\lambda_1 \lambda_2 \cdots \lambda_{p-1}}(\sigma) \phi(x(\sigma)) \right\} ,
$$

(23)

where

$$
\Pi_{\mu_1 \mu_2 \cdots \mu_p}^{\nu_1 \nu_2 \cdots \nu_p}(\sigma) \equiv \frac{1}{(p-1)!} \delta_{\mu_1}^{[\nu_1} \delta_{\mu_2}^{\nu_2} \cdots \delta_{\mu_p}^{\nu_p]} + \epsilon q_1 h_1 (\sigma) \cdots h_{p-1}(\sigma) Q_{\lambda_1 \lambda_2 \cdots \lambda_{p-1}}(\sigma) \phi(x(\sigma)) ,
$$

(24)

$$
F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad \bar{A}_\mu \equiv A_\mu - (\epsilon/q_1) \partial_\mu \phi .
$$

Obviously the right-hand side of (23) is gauge invariant.
Substituting (23) into (6) and following the procedure for deriving (19), we finally arrive at the action in the Stueckelberg formalism:

\[ S^{(1)}_R = \int d^Dx \left[ -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2} m^2 A_\mu(x) \tilde{A}^\mu(x) \right] . \] (25)

In deriving this action, we have imposed a certain normalization condition on the coefficient of the \( F(x)^2 \)-term. Then we find that the mass \( m \) is proportional to \( v^{-1/p} \).

In conclusion, the gauge theory of a massless rank-(\( p + 1 \)) antisymmetric tensor field and the Stueckelberg formalism for a massive vector field have been derived from the U(1) gauge theory in \( p \)-manifold space. In terms of differential geometry, \( A_{\mu \nu} \) is a connection on a U(1) bundle over \( \Omega^p M^D \); as seen from (10) and (21), \( B_{\mu \nu, \rho \cdot \cdot \cdot \rho} \) and the pair of \( A_\mu \) and \( \phi \) are regarded as constrained connections on this U(1) bundle.

Recently, couplings of strings and massive local fields have been discussed from the point of view of the Lorentz force in loop space [7]. In a similar manner, we can consider couplings of \( p \)-branes and massive local fields such as \( \tilde{A}_\mu \).

A Yang–Mills theory in loop space has also been studied. An essential idea in this theory is to take a loop group or an affine Lie group as the gauge group. It was shown that the Yang–Mills theory with the loop gauge group leads to a non-abelian Stueckelberg formalism for massive second-rank tensor fields [8], and that the Yang–Mills theory with the affine gauge group yields the Chapline–Manton coupling [9]. As a next study, it will be interesting to generalize the Yang–Mills theory in loop space by replacing the loop space by the \( p \)-manifold space. For the case \( p = 3 \), a candidate for the gauge group will be the Mickelsson–Faddeev group [10]. This subject will be discussed in the future.

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References


