Super-Yangian \( Y(gl(1|1)) \) and Its Oscillator Realization

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ABSTRACT

On the basis of graded RTT formalism, the defining relation of the super-Yangian \( Y(gl(1|1)) \) is derived and its oscillator realization is constructed.

1. Introduction

Yangian \( Y(g) \) of a simple Lie algebra \( g \), first introduced by V. G. Drinfeld, is a deformation of the universal enveloping algebra \( U(g[t]) \) of a current algebra \( g[t] \). It is a kind of Hopf algebra and the tensor products of its finite-dimensional representations produce rational solutions of the quantum Yang-Baxter equation (QYBE).

In the last decade, Yangians associated with simple Lie algebra have been systematically studied both in mathematics and physics, and have many applications in such theoretical physics as quantum field theory and statistical mechanics. Yangian structure is the underlying symmetry of many types of integrable models. For example, 1-D Hubbard model on the infinite chain, the Haldane-Shastry model and the Polychronakos-Frahm model have Yangian symmetry; in the massive 2-D quantum field theory, a infinite-dimensional symmetry generated by nonlocal conserved currents is connected to the Yangian.

As generalizations of Yangians of simple Lie algebras, the Yangians associated with the simple Lie superalgebra, which we will call super-Yangian in this letter, also need to be studied. Actually, some structural features of super-Yangian have been investigated by M. Nazarov and R. B. Zhang. In ref. 6, the quantum determinant of the super-Yangian \( Y(gl(m|n)) \) is described, while in refs. 7 and 8 the super-Yangian \( Y_q(gl(m|n)) \) is associated
with the Perk-Schultz $R$ matrix is constructed, its structural properties and the relationship between its central elements and the Casimir operators of quantum supergroup $U_q(gl(m|n))$ is discussed, in particular, the classification of the finite-dimensional irreducible representations of the super-Yangians $Y(gl(1|1))$ and $Y(gl(m|n))$ is given.

In this letter, on the basis of the graded RTT formalism, we derived the defining relations of the super-Yangian for the Lie superalgebra $gl(1|1)$ and give its oscillator realization. In section 2, we briefly review the graded RTT formalism and the corresponding graded Yang-Baxter equation (GYBE). In section 3, we give the algebraic relation that super-Yangian $Y(gl(1|1))$ satisfies and construct its oscillator realization. Finally, we make some remarks and discussions.

2. Graded RTT Relation and GYBE

In the supersymmetric case, space is graded and the tensor product has the following property

$$(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)}AC \otimes BD$$

where $p(A)$ denotes the degree of $A$. Now the graded RTT relation with the spectral parameters takes the form

$$R_{12}(u-v)T_1(u)\eta_{12}T_2(v)\eta_{12} = \eta_{12}T_1(u)\eta_{12}T_2(v)\eta_{12}R_{12}(u-v)$$

(2.2a)

where $T_1(u) = T(u) \otimes 1$ and $T_2(u) = 1 \otimes T(u)$ and $(\eta_{12})_{ab,cd} = (-1)^{p(a)p(b)}\delta_{ac}\delta_{bd}$, and GYBE with spectral parameters reads as

$$\eta_{12}R_{12}(u)\eta_{13}R_{13}(u+v)\eta_{23}R_{23}(v) = \eta_{23}R_{23}(v)\eta_{13}R_{13}(u+v)\eta_{12}R_{12}(u),$$

(2.3a)

Considering the charge conservation conditions for the $R_{ab,cd}$, i.e.

$$R_{ab,cd} = 0 \quad unless \quad a + b = c + d$$

(2.4)

we can write eqs. (2.2a) and (2.3a) in the component forms as follows:

$$(-1)^{p(c)(p(d)+p(f))}R_{12}(u-v)_{ab,cd}T(u)_{ce}T(v)_{df} =$$

$$(-1)^{p(a)(p(d)+p(b))}T(v)_{be}T(u)_{ad}R_{12}(u-v)_{cd,ef}$$

(2.2b)

$$(-1)^{p(d)(p(b)+p(e))}R(u)_{ab,cd}R(u+v)_{ce,fg}R(v)_{dh,ij} =$$

$$(-1)^{p(d)(p(h)+p(j))}R(v)_{be,df}R(u+v)_{ah,cf}R(u+v)_{ah,cf}$$

(2.3b)

where the repeated indices is understood to take summation. Note that, in Eqs. (2.2) and (2.3) the grading property is taken into account by introducing the factor $\eta$. If we set $\eta = 1$, then Eqs. (2.2) and (2.3) reduce to the usual RTT relation and YBE respectively.

3. Super-Yangian $Y(gl(1|1))$ and Its Oscillator Realization

It is well known that

$$R_{12}(u) = u + P_{12}$$

(3.1)
satisfies GYBE (2.3), where
\[ \mathcal{P}_{12} = \eta_{12} \mathcal{P}_{12}, \]  
(3.2)
P stands for the usual permutation operator, i.e. \( P(u \otimes v) = v \otimes u \). Substituting eq (3.1) into eq (2.2) and introducing the notation

\[ [T(u)_{ab}, T(v)_{cd}] = T(u)_{ab}T(v)_{cd} - (-1)^{(p(a)+p(b))}(p(c)+p(d)) T(v)_{cd}T(u)_{ab} \]  
(3.3)
we obtain the following relations:

\[ (u-v)[T(u)_{ab}, T(v)_{cd}] + (-1)^{(p(a)p(c)+p(b)p(c))} (T(u)_{cb}T(v)_{ad} - T(v)_{cb}T(u)_{ad}) = 0 \]  
(3.4)

Let
\[ T(u)_{ab} = \sum_{n=0}^{\infty} u^{-n} T_{ab}^{(n)} \]  
(3.5)
then from eq (3.4), we have
\[ [T_{ab}^{(0)}, T_{cd}^{(n)}] = 0 \]  
(3.6)
\[ [T_{ab}^{(n+1)}, T_{cd}^{(m)}] - [T_{ab}^{(n)}, T_{cd}^{(m+1)}] + (-1)^{(p(a)p(c)+p(b)p(b))} (T_{cb}^{(n)}T_{ad}^{(m)} - T_{cb}^{(m)}T_{ad}^{(n)}) = 0 \]  
(3.7a)

Similar to the discussion for the Yangian, eq (3.7a) can be rewritten into the following equivalent form:

\[ [T_{ab}^{(n)}, T_{cd}^{(m)}] = (-1)^{1+p(a)p(c)+p(b)p(b)} \sum_{i=0}^{\min(n,m)-1} (T_{cb}^{(i)}T_{ad}^{(m+n-i-1)} - T_{cb}^{(m+n-i-1)}T_{ad}^{(i)}) \]  
(3.7b)
In particular, for the case of \( a = c, b = d \) in the above equation, we have
\[ [T_{ab}^{(n)}, T_{ab}^{(m)}] = (-1)^{1+p(a)p(a)} \sum_{i=0}^{\min(n,m)-1} [T_{ab}^{(i)}, T_{ab}^{(m+n-i-1)}] \]  
(3.8)
this shows that \( T_{ab}^{(n)} \), with \( a \neq b \) and different \( n (n > 1) \) will neither commute nor anti-commute.

From eq (3.7b), we know that the following property holds
\[ [T_{ab}^{(n)}, T_{cd}^{(m)}] = [T_{ab}^{(m)}, T_{cd}^{(n)}] \quad (n, m \geq 1) \]  
(3.9)
Here we notice that in the non-graded case, eq (3.9) will give the relation
\[ [T_{ab}^{(n)}, T_{cd}^{(m)}] = 0 \quad (n, m \geq 1) \]  
(3.10)
For the case of superalgebra $gl(1|1), a = 1, 2$ and $\mathcal{P}$ takes the form

\[
\mathcal{P}_{12} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

$T(u)$ is a $2 \times 2$ matrix. Because of the relation (3.5), we can choose $T^{(0)}$ to be of the form

\[
T^{(0)} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

up to a constant factor. Here we should stress that (3.11) is only a choice, which different from the non-graded case in that there it is the result of the Schur’s lemma. From eq. (3.5), we see that eq. (3.11) is equivalent to impose asymptotic condition $T(u) \to 1$ for $u \to \infty$. With eqs (3.6) (3.7) and (3.11), we obtain the following relations:

\[
\begin{aligned}
&\begin{cases}
[T_3^{(n)}, T_{12}^{(1)}] = [T_3^{(1)}, T_{12}^{(n)}] = 0, \\
[T_3^{(n)}, T_{21}^{(1)}] = [T_3^{(1)}, T_{21}^{(n)}] = 0, \\
[T_0^{(n)}, T_{12}^{(1)}] = [T_0^{(1)}, T_{12}^{(n)}] = -2T_{12}^{(n)}, \\
[T_0^{(n)}, T_{21}^{(1)}] = [T_0^{(1)}, T_{21}^{(n)}] = 2T_{21}^{(n)}, \\
[T_{12}^{(n)}, T_{21}^{(1)}] = -T_3^{(n)}
\end{cases} \\
&\begin{cases}
[T_0^{(n)}, T_3^{(2)}] + 2(T_{21}^{(1)} T_{12}^{(n)} - T_{21}^{(2)} T_{12}^{(1)}) = 0 \\
[T_3^{(n)}, T_{12}^{(2)}] - T_{12}^{(1)} T_3^{(n)} + T_{12}^{(n)} T_3^{(1)} = 0 \quad (n \geq 1) \\
[T_3^{(n)}, T_{21}^{(2)}] + T_{21}^{(1)} T_3^{(n)} - T_{21}^{(n)} T_3^{(1)} = 0 \quad (n \geq 1)
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
&\begin{cases}
-T_{12}^{(n+1)} = \{[T_0^{(n)}, T_{12}^{(2)}] + T_{12}^{(n)} T_0^{(1)} - T_{12}^{(1)} T_0^{(n)}\} / 2 \\
T_{21}^{(n+1)} = \{[T_0^{(n)}, T_{21}^{(2)}] + T_{21}^{(1)} T_0^{(n)} - T_{21}^{(n)} T_0^{(1)}\} / 2 \\
T_3^{(n+1)} = -\{T_{12}^{(1)}, T_{21}^{(2)}\} + T_{22}^{(1)} T_{11}^{(n)} - T_{22}^{(n)} T_{11}^{(1)} \quad (n \geq 2)
\end{cases}
\end{aligned}
\]

where

\[
T_3^{(n)} = T_{22}^{(n)} - T_{11}^{(n)}, T_0^{(n)} = T_{22}^{(n)} + T_{11}^{(n)}
\]

From the recurrence relations (3.14), we see that only $T_{ab}^{(1)}, T_{ab}^{(2)}$ are basic operators. Now, if we make the following correspondence.

\[
\begin{aligned}
&T_3^{(1)} = -\gamma_0 z_0, \quad T_3^{(2)} = -\gamma_1 z_1 \\
&T_{12}^{(1)} = \alpha_0 \epsilon_0, \quad T_{12}^{(2)} = \alpha_1 \epsilon_1 \\
&T_{21}^{(1)} = \beta_0 f_0, \quad T_{21}^{(2)} = \beta_1 f_1 \\
&T_0^{(1)} = -2h_0, \quad T_0^{(2)} = \delta h_1
\end{aligned}
\]

and take the choice

\[
\alpha_0 \beta_0 = \gamma_0, \quad \alpha_0 \beta_1 = \alpha_1 \beta_0 = \gamma_1, \quad \alpha_0 \delta = -2 \alpha_1
\]
then from eqs (3.12)-(3.14) we obtain following algebraic relations

\[
\begin{align*}
&\left\{ e_0^2 = f_0^2 = 0, \left[ h_0, e_0 \right] = e_0, \quad \left[ h_0, f_0 \right] = -f_0, \\
&\left[ z_0, e_0 \right] = \left[ z_0, f_0 \right] = \left[ h_0, z_0 \right] = 0, \quad \left\{ e_0, f_0 \right\} = z_0
\end{align*}
\]

and

\[
\begin{align*}
&\left[ z_1, e_0 \right] = \left[ z_1, f_0 \right] = \left[ z_1, z_0 \right] = \left[ z_1, h_0 \right] = 0 \\
&\left[ f_1, z_0 \right] = 0, \\
&\left\{ f_1, e_0 \right\} = z_1, \\
&\left\{ e_1, e_0 \right\} = z_1, \\
&\left[ e_1, z_0 \right] = 0, \\
&\left[ h_1, z_0 \right] = \left[ h_1, h_0 \right] = 0 \\
&\left[ h_1, e_0 \right] = e_1,
\end{align*}
\]

The eq (3.18) is just the defining relation of the Lie superalgebra \( gl(1|1) \). Taking the correspondence

\[
z_0 \rightarrow N + M, e_0 \rightarrow x, f_0 \rightarrow y, h_0 \rightarrow N
\]

eq (3.18) will give the same result as that of Liao and Song \(^9\) in the limit \( q \rightarrow 1 \). Eq (3.19) shows that \( e_1, f_1, h_1, z_1 \) form a representation of eq (3.18). \( e_i, f_i, h_i, z_i(i = 0, 1) \) also satisfies Serre relations:

\[
\begin{align*}
&\left[ z_1, \left\{ e_1, f_1 \right\} \right] = C_0 z_0 \left( f_0 e_1 - f_1 e_0 \right) \\
&2\left\{ e_1, \left[ h_1, e_1 \right] \right\} + C_0 \left[ e_0, e_1 \right] + 2C_1 \left[ h_1, e_1 \right] e_0 = 0 \\
&2\left\{ f_1, \left[ h_1, f_1 \right] \right\} + C_0 \left[ f_0, f_1 \right] + 2C_1 \left[ h_1, f_1 \right] f_0 = 0 \\
&\left\{ e_1, \left[ z_1, e_1 \right] \right\} + C_0 e_1 e_0 z_0 = 0 \\
&\left\{ f_1, \left[ z_1, f_1 \right] \right\} + C_0 f_1 f_0 z_0 = 0 \\
&\left\{ e_1, \left[ e_1, f_1 \right] \right\} + \left[ z_1, \left[ h_1, e_1 \right] \right] = C_1 \left( \left[ e_1, h_1 \right] z_0 + z_1 e_1 \right) + C_0 e_0 \left( f_0 e_1 - f_1 e_0 \right) \\
&\left[ f_1, \left[ e_1, f_1 \right] \right] - \left[ z_1, \left[ h_1, f_1 \right] \right] = C_1 \left( \left[ f_1, h_1 \right] z_0 - z_1 f_1 \right) + C_0 f_0 \left( f_0 e_1 - f_1 e_0 \right) \\
&\left[ h_1, \left[ e_1, f_1 \right] \right] - C_1 \left( f_0 \left[ h_1, e_1 \right] + \left[ h_1, f_1 \right] e_0 \right) - C_0 \left( f_0 e_1 - f_1 e_0 \right) = 0.
\end{align*}
\]

where

\[
C_0 = \gamma_0/\alpha_1 \beta_1, C_1 = \gamma_1/\alpha_1 \beta_1
\]

The operators \( \left\{ e_i, f_i, h_i, z_i \right\}; i = 0, 1 \) and the relation (3.18) (3.19) and (3.21) constitute an infinite-dimensional algebra called super-Yangian of the Lie superalgebra \( gl(1|1) \) and denoted by \( Y(gl(1|1)) \). \( Y(gl(1|1)) \) is a Hopf algebra with the comultiplication \( \Delta \), co-unit \( \epsilon \) and antipode \( S \) defined, respectively, by

\[
\begin{align*}
&\Delta(T(u)_{ab}) = \sum_c T(u)_{ac} \otimes T(u)_{cb} \\
&\epsilon(T(u)) = 1 \\
&S(T(u)) = T(u)^{-1}
\end{align*}
\]
If writing the Hopf structure in terms of operators \( \{e_i, f_i, h_i, z_i\}_{i=0,1} \), we get the following forms:

\[
\begin{align*}
\Delta(X) &= 1 \otimes X + X \otimes 1 \\
\Delta(e_1) &= 1 \otimes e_1 + e_1 \otimes 1 - \frac{c_0}{c_1}(h_0 \otimes e_0 + e_0 \otimes h_0) + \frac{c_0\gamma_0}{2c_1}(z_0 \otimes e_0 - e_0 \otimes z_0) \\
\Delta(f_1) &= 1 \otimes f_1 + f_1 \otimes 1 - \frac{c_0}{c_1}(h_0 \otimes f_0 + f_0 \otimes h_0) + \frac{c_0\gamma_0}{2c_1}(-z_0 \otimes f_0 + f_0 \otimes z_0) \\
\Delta(z_1) &= 1 \otimes z_1 + z_1 \otimes 1 + \frac{c_0}{c_1}(e_0 \otimes f_0 - f_0 \otimes e_0) - \frac{c_0\gamma_0}{4c_1}(z_0 \otimes h_0 + h_0 \otimes z_0) \\
\Delta(eh_1) &= 1 \otimes h_1 + h_1 \otimes 1 - \frac{c_0\gamma_0}{2c_1}(f_0 \otimes e_0 + e_0 \otimes f_0) - \frac{c_0}{c_1}h_0 \otimes h_0 - \frac{c_0\gamma_0}{4c_1}z_0 \otimes z_0 \\
S(X) &= -X \\
S(e_1) &= -e_1 - \frac{c_0}{c_1}(h_0e_0 + e_0h_0) \\
S(f_1) &= -f_1 - \frac{c_0}{c_1}(h_0f_0 + f_0h_0) \\
S(z_1) &= -z_1 - \frac{c_0}{c_1}(f_0e_0 - e_0f_0 + 2z_0h_0) \\
S(h_1) &= -h_1 - \frac{c_0\gamma_0}{2c_1}(f_0e_0 + e_0f_0) - \frac{c_0\gamma_0}{4c_1}z_0z_0 - \frac{c_0}{c_1}h_0h_0 \\
\epsilon(1) &= 1, \epsilon(X) = \epsilon(Y) = 0
\end{align*}
\]

(3.22b)

where \( X = e_0, f_0, z_0, h_0, Y = e_1, f_1, z_1, h_1 \).

Now we introduce a set of bosonic oscillators \( b_i, b_i^\dagger \) and a set of fermionic oscillators \( a_i, a_i^\dagger \) satisfying

\[
\begin{align*}
\{a_i, a_j\} &= [b_i, b_j^\dagger] = \delta_{ij} \\
\{a_i, b_j\} &= [a_i^\dagger, b_j^\dagger] = [b_i^\dagger, b_j^\dagger] = 0 \\
[a_i, b_j] &= [a_i^\dagger, b_j] = [a_i^\dagger, b_j^\dagger] = 0
\end{align*}
\]

(3.23)

Identifying

\[
\begin{align*}
e_0 &= \sum_i b_i^\dagger a_i, & f_0 &= \sum_i a_i^\dagger b_i \\
z_0 &= \sum_i (a_i^\dagger a_i + b_i^\dagger b_i), & h_0 &= \sum_i b_i^\dagger b_i, \\
e_1 &= \sum_{i,j} A_{ij} b_i^\dagger a_j + \sum_{i,j} B_{ij} b_i^\dagger a_i (a_j^\dagger a_j + b_j^\dagger b_j) \\
f_1 &= \sum_{i,j} A_{ij} a_i^\dagger b_j - \sum_{i,j} B_{ij} a_i^\dagger b_i (a_j^\dagger a_j + b_j^\dagger b_j) \\
z_1 &= \sum_{i,j} A_{ij} (a_i^\dagger a_i + b_i^\dagger b_i) \\
h_1 &= \frac{1}{2} \sum_{i,j} A_{ij} (-a_i^\dagger a_i + b_i^\dagger b_i) + \sum_{i,j} B_{ij} b_i^\dagger a_i a_j^\dagger b_j
\end{align*}
\]

(3.24)

where \( A_{ij}, B_{ij} \) are parameters and \( B_{ij} + B_{ji} = 0 \). We can prove that eqs (3.24) reproduce the commutation relations given in eqs (3.18) and (3.19). Substituting eqs (3.24) into Serre relations (3.21), then there are some constrains on \( A_{ij}, B_{ij} \) and they will be related to parameters \( C_0, C_1 \).

4. Remarks and Discussions

In this letter, we only discuss the super-Yangian of the Lie superalgebra \( gl(1|1) \) and its oscillator realization. The question we should answer is how to generalize the discussion to the case of superalgebra \( gl(m|n) \) and other superalgebras. However, this is connected with physical problems, i.e., whether there exist integrable models with \( R \) matrix associated with Lie superalgebras. As a first step, we wish find a model with the super-Yangian symmetry we have discussed. This problem asks for a further study of super-Yangian and its representation theory.
From section 3, we see that (super-)Yangian is related to the (graded) RTT relation. Actually, there are dual relations to the (graded) RTT relation, their corresponding algebras is not contained in the (super-)Yangian. Yangian double considers all algebraic information contained in RTT relation and its dual relations. The Yangian double for simple Lie algebras become an interesting research object recently\textsuperscript{11,12}. Naturally, the super-Yangian double and the related problems also need to be studied. The work in this respect is under investigation.

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