Probing Large Scale Structure using Percolation and Genus Curves

Varun Sahni 1, B.S. Sathyaprakash 2 and Sergei F. Shandarin 3

ABSTRACT

We study topological properties of large scale structure in a set of scale free N-body simulations using the genus and percolation curves as topological characteristics. Our results show that as gravitational clustering advances, the density field shows an increasingly pronounced departure from Gaussianity reflected in the changing shape of the percolation curve and the changing amplitude and shape of the genus curve. Both genus and percolation curves differentiate between the connectedness of overdense and underdense regions if plotted against the density. When plotted against the filling factor the percolation curve alone retains this property. The genus curve shows a pronounced decrease in amplitude caused by phase correlations in the non-linear regime. Both genus and percolation curves provide complementary probes of large scale structure topology and can be used to discriminate between models of structure formation and the analysis of observational data such as galaxy catalogs and MBR maps.

Subject headings: Subject Headings: (cosmology:) large-scale structure of universe–galaxies: clusters: general–methods: numerical

1. Introduction

It is reasonably well established that, far from being randomly distributed, galaxies are strongly clustered and form a roughly cellular three-dimensional structure consisting of filaments, sheets and clusters separated by large voids (de Lapparent, Geller, & Huchra 1991). More than a decade ago Zel’dovich and Shandarin advocated using percolation theory to explore the topological properties of the large scale distribution of galaxies (Zel’dovich 1982, Shandarin 1983, Shandarin & Zel’dovich 1983, 1989). Percolation theory is also a useful means with which to understand morphology viz. the filamentarity/planarity/clumpiness of a distribution (Sathyaprakash, Sahni, & Shandarin 1996a). A related topological approach complementary to percolation and involving the genus characteristic was suggested in Gott, Melott, & Dikinson 1986. In the present study we, for the first time, apply techniques based both on percolation analysis and the genus curve to the same set of N-body simulation data.

It is well known that, as an initially random distribution of matter evolves under self gravity, it rapidly develops non-Gaussian features (e.g. Sahni & Coles 1995). We assess the relative merits of the genus curve and the percolation curve in providing measures of non-Gaussian features in such a distribution. Both genus and percolation complement conventional indicators of clustering by providing an estimate of

1Inter-University Centre for Astronomy & Astrophysics, Post Bag 4, Pune 411007, India;
2Theoretical Astrophysics, California Institute of Technology, Pasadena, CA 91106 and Department of Physics and Astronomy, UWCC, Cardiff, CF2 3YB, United Kingdom;
3Department of Physics and Astronomy, University of Kansas, Lawrence, KS 66045.
the ‘connectedness’ of structure missed by standard estimators such as the two-point correlation function (which lacks phase information vital to an understanding of large scale coherence).

To this end we have studied N-body simulations in an Ω = 1 universe. We examined models with scale invariant initial spectra $P_{in}(k) \equiv \langle |\delta_k|^2 \rangle \propto k^n$, $n = -2, -1, 0$ and 1. The simulations are studied at several epochs, each characterized by the scale of nonlinearity $k_{NL}^{-1}$ at that epoch measured in units of the fundamental mode $k_f \equiv 2\pi/L$, $L$ being the length of the simulation box. For the sake of brevity we present results for $n = -2$ and $n = 0$ models, at epochs when the nonlinear scale is: $k_{NL} = 64, 16$ and 4. These spectral indices can be considered as the lower and upper limits of the slope of the initial spectrum on the galaxy and supercluster scales. N-body simulations are performed on a grid of size $128^3$ employing a particle-mesh algorithm (Melott & Shandarin 1993). Density fields are constructed on a reduced grid of size $64^3$ and subsequent studies of percolation and genus are carried out on these fields.

2. Percolation Analysis

The starting point for percolation analysis is a rule or a criteria by which to define structures. For instance, given the density field of matter in the Universe and a density threshold, we can define a cluster (or a void) as a connected overdense (underdense) region, connectivity being defined using a friends-of-friends or ‘nearest neighbors’ algorithm. The aim of percolation analysis is to study the connectedness of structure as a function of the density threshold. In an infinite medium varying the density threshold leads to a ‘percolation transition’ as the volume fraction in the largest cluster changes rapidly from almost zero to unity when the density threshold crosses a critical value. (In what follows we use density contrast $\delta = (\rho - \bar{\rho})/\bar{\rho}$, instead of density $\rho$, to define the percolation transition.)

In reality one deals with finite systems and by an ‘infinite’ structure one means a structure that spans the entire simulation box or observational sample. In this paper we will only consider finite systems and shall use the term simulation box as a generic term which can refer to an observational catalog or an N-body simulation box. For illustrative purposes we shall discuss the percolation of overdense regions keeping in mind that our arguments are equally valid for underdense regions (voids).

Gaussian random fields percolate at the critical filling factor $FF_C \simeq 16\%$ regardless of the spectrum (filling factor – henceforth $FF$ – is the total volume in all clusters/voids above/below the density contrast threshold divided by the simulation volume). Density fields that have evolved under gravitational instability typically percolate at lower levels of $FF_C$ depending upon the spectral index and the extent of non-linear evolution (Yess & Shandarin 1996). Similar conclusions can be made in the case of point like distributions although the numbers are different and the natural reference is the Poisson distribution (Klypin & Shandarin 1993).

3. Effect of Dynamical evolution on Genus and Percolation Curves

---

4 We use six nearest neighbors on the cubic grid to identify structures. Although our results do depend on details of the nearest neighbors scheme, our main conclusions regarding the relative discriminating power of genus & percolation curves do not.

5 $FF$ is the cumulative probability distribution function: $FF = P(\delta > \delta_T)$. 
Traditional applications of percolation to gravitational clustering focused primarily on the percolation threshold as a diagnostic measure (Shandarin 1983, Klypin 1987, Dominik & Shandarin 1992, Klypin & Shandarin 1993). Although $FF_C$ (or $\delta_C$) is a useful characteristic of evolved density fields, it has certain drawbacks which make it unsatisfactory for the study of a realistic distribution of galaxies. For instance, the percolation threshold is sensitive to resolution, number of particles, etc. In addition $FF_C$ is sensitive to the sample geometry and in cases when the observational sample is wedge-shaped it becomes difficult to provide an unambiguous definition of the percolation threshold (Dekel & West 1985).

We investigate a powerful new statistic which does not face most of the above limitations – the Percolation Curve. The percolation curve (henceforth PC) describes the volume fraction $v_{\text{max}}$ in the largest structure (cluster/void) as a function of the density contrast threshold $\delta_T$ or $FF$. Formally, $v_{\text{max}}$ is defined as the ratio of the volume in the largest cluster/void to the total volume in all clusters/voids lying above/below a density contrast threshold: $v_{\text{max}} = V_{\text{max}} / FF$. ($V_{\text{max}}$ is the fraction of the total volume in the largest cluster/void.)

In order to illustrate some salient features of the percolation curve we first consider evolved density fields from $n = -2$ and $n = 0$ models at an epoch when the scale of nonlinearity is $k_{\text{NL}} = 64k_f, 16k_f, 4k_f$. In Fig. 1 a,b we have plotted percolation curves for clusters (thick solid lines) and voids (thick dashed lines) as a function of density contrast $\delta$. Starting from a high density threshold (small $FF$) we find initially only one cluster in our field which corresponds to the highest density peak, as a result $v_{\text{max}} = 1$. However, this is a finite system effect and we shall ignore it. Lowering the threshold (increasing $FF$) we find several isolated clusters, all of roughly comparable size so that the volume fraction in the largest cluster is very small $v_{\text{max}} \ll 1$. Lowering the density threshold still further results in the merger of clusters leading to a rapid growth in $v_{\text{max}}$ and to the onset of percolation. A further lowering of the threshold to very small values results in the merger of almost all clusters so that $v_{\text{max}} \to 1$. An identical procedure followed for underdense regions by gradually increasing the density contrast threshold (increasing $FF$) results in a similar functional form for the volume fraction in the largest void. In our samples the largest cluster percolates between opposite faces of the cube when its filling factor is about half the total $FF$. In most cases percolation also coincides with the highest jump in the volume of the largest cluster which was used by de Lapparent, Geller, & Huchra 1991 as a working definition of the percolation threshold. The solid/dashed vertical line in Fig. 1 represents the density contrast threshold below/above which clusters/voids percolate. We note that both clusters and voids percolate over a range of overlapping density contrasts — a feature that is only possible in three or more dimensions — and corresponds to what is commonly called a sponge topology for the density distribution.

We clearly see that the percolation threshold $\delta_C$ (cluster) is higher for $n = -2$ models relative to $n = 0$. As the simulation evolves $\delta_C$ increases monotonically for $n = -2$, as structures form and align on increasingly larger scales. For $n = 0$, $\delta_C$ initially increases but later begins to decrease signaling the formation of small, isolated clumps. It must be noted that for $n = 0$ the behavior of $\delta_C$ (cluster) is non-monotonic whereas $FF_C$ (cluster) is monotonic (see Fig. 2), the reason for this lies in the non-linear relationship between density threshold and filling factor (Yess & Shandarin 1996). From Fig. 1 we see

---

6The critical exponents remain almost unchanged as clustering evolves and are therefore poor indicators of non-Gaussianity (Klypin & Shandarin 1993).

7The smallness of the filling factor in simulations has been used to argue that the shape of overdense regions is likely to be filamentary (Sathyaprakash, Sahni, & Shandarin 1996a). Note that error bars in Fig. 1 & 2 — representing the dispersion computed with the aid of four different realizations of a given power spectrum — are very small.
that voids find it easier to percolate as the simulation evolves, as a result the range in densities when both phases percolate initially increases, enhancing the extent of sponge-like topology in the distribution. We thus see that percolation analysis not only provides us with an appreciation of different aspects of gravitational clustering but is sensitive enough to differentiate between models with differing sets of initial conditions.

The percolation curve (PC) is in some respects similar to the ‘genus curve’ (GC) which can be formally expressed as an integral over the Gaussian curvature $K$ of the iso-density surfaces $S_\nu$ lying above/below a density threshold $\nu = \delta/\sigma_0$ by the Gauss-Bonnet theorem: $4\pi G(\nu) = \int_{S_\nu} K dA$ (Gott, Melott, & Dikinson 1986). Multiply connected surfaces have $G \geq 0$ while simply connected have $G < 0$. For Gaussian Random fields the genus curve has a ‘bell shaped’ form: $G(\nu) = A(1 - \nu^2)\exp(-\nu^2/2)$ (Hamilton et al. 1986, Gott et al. 1987, Gott et al. 1989). (A corresponding analytical form for PC has so far eluded researchers.) It should be borne in mind that realistic Gaussian fields are defined over a finite interval, so that discreteness effects must be taken into account when determining $G(\nu)$ even for the Gaussian case. Genus curves for models $n = -2, 0$ are shown as functions of $\delta$ in the right hand panels of Fig. 1. We see that GC shows a rapid departure from its original symmetric ‘bell shape’ reflecting growth of non-Gaussianity in the density distribution. Plotting GC against $\delta$ has one major drawback: when plotted in this manner GC shows a difference in topological properties of distributions related to each other by transformations $\delta \rightarrow f(\delta)$ (such as the log-normal) when in fact no such difference exists. A physically more relevant quantity to consider is the genus curve plotted against the filling factor, which we discuss in the next section.

We would like to stress that, whereas properties of the largest cluster are used to determine PC, all clusters lying above/below a given threshold are used to determine GC.

4. Results, Discussion and Conclusions

Percolation and genus curves are plotted as functions of $FF$ in Fig. 2 for $n = -2$ (Fig. 2a) and $n = 0$ (Fig. 2b), the scale of nonlinearity increasing from top to bottom ($k_{NL} = 64, 16$ and $4$). For PC solid/dashed curves correspond to the volume fraction in the largest cluster/void. Vertical solid/dashed lines correspond to the filling factor at percolation $FF_C$ for overdense/underdense regions. The lightly dashed vertical line corresponds to $FF_C$ for Gaussianized fields constructed by randomizing the phases of the N-body particle distributions (Yess & Shandarin 1996). For GC the solid line corresponds to overdense, and the dashed to underdense regions.

For a Gaussian distribution overdense and underdense regions have identical topological properties. This fact is reflected in the uppermost panels of Fig. 1b and 2b which show an early epoch of the $n = 0$ model when the difference between the solid and dashed curves is small. In the $n = -2$ model the departure from Gaussianity is evident even at this early stage. Our method of plotting the genus curve is different from convention which usually shows this curve plotted as a function of the density contrast. Plotted in that manner the curve has a bell-shaped form for Gaussian initial conditions. Our method ‘folds the bell’ and shows GC as a function of the filling factor. When plotted in this manner GC appears to have a prescribed shape which it maintains irrespective of the evolutionary epoch at which it is measured. The shape of GC appears to be sensitive to the primordial spectral index: for small $n$ the separation between cluster and void curves in GC is large, for large $n$ it is small (see Fig. 2). The fact that the shape of the genus curve is very sensitive to the primordial spectral index and shows little, if any, evolution leads us to feel that it may

---

8For discrete sets $4\pi G(\nu) = -\sum_{i=1}^{N} D_i$, $D_i$ is the deficit angle at the ith vertex of a polyhedral surface having $N$ vertices.
be a useful indicator of the spectrum of primordial density fluctuations.

Since for spectra with substantial small scale power (such as $n = 0$), the asymmetry between clusters and voids in GC is small, one might have expected the evolved genus curve to be a scaled version of GC for a Gaussian random field. To check this we generated Gaussian fields (GC$_{\text{RAN}}$) by randomizing the phases of the N-body particle distributions. Generating Gaussian fields in this manner allows some degree of control over finite grid effects which now contribute equally to Gaussian and non-Gaussian fields (Yess & Shandarin 1996). Comparing GC with GC$_{\text{RAN}}$ we found that GC has a smaller amplitude than GC$_{\text{RAN}}$ (see Table I). Thus the decreasing amplitude of GC cannot be attributed to the (non-linear) evolution of the power spectrum. The difference in amplitude between GC and GC$_{\text{RAN}}$ (more pronounced for spectra with small scale power) is caused by non-linear mode coupling and phase correlations during advanced gravitational clustering. In principle this effect could be used to probe the extent of non-linear evolution in the Universe by comparing red-shift data with numerical simulations at identical smoothing scales (see also Melott et al. 1988, Melott 1990).

Turning now to percolation, consider the percolation curves (PC) showing volume fraction in the largest cluster/void $v_{\text{max}}$ as a function of filling factor $FF$ (solid/dashed line) in Fig. 2. The rapid growth in $v_{\text{max}}$ with increasing $FF$ indicates the onset of percolation. As the distribution evolves the percolation threshold $FF_C$ becomes different for clusters and voids, the difference: $FF_C(\text{clusters}) - FF_C(\text{voids})$ increases with epoch indicating the growth of non-Gaussianity.¹ Divergence of the solid and dashed lines and the increase of the area under the ‘hysteresis-like curve’ in Fig. 2 are also good measures of the increase in non-Gaussianity. It is worth noting that the vertical thin dashed line marking the percolation threshold in Gaussian fields (constructed by randomizing phases) remains virtually unchanged.

The relative ease with which a distribution percolates relative to a Gaussian has been used to categorize some of its topological properties. For instance a distribution in which overdense regions (clusters) percolate at lower $FF$ than Gaussian is said to possess a ‘network’ topology, whereas the reverse signals a ‘meat-ball’ topology. For voids, percolation at greater $FF$ than Gaussian, implies a bubble-topology consisting of isolated underdense regions surrounded by overdense ‘cluster walls’. Similar conclusions have also been drawn for the genus curve (Melott 1990). Our results show easier percolation for overdense regions indicating a shift towards a network-like topology. Underdense regions on the other hand find it harder to percolate and so have a bubble-like topology. ¹⁰

Comparing GC and PC at identical epochs we find the degree of asymmetry between clusters and voids to be much more pronounced for PC. As a result the increasing difference in topological properties of overdense and underdense regions – a clear indicator of non-Gaussianity – appears to be better encapsulated in PC than GC when both are plotted against the filling factor.

The fact that the percolation curve is a sensitive diagnostic of non-Gaussianity could be used for the analysis of galaxy distributions and maps of the Cosmic Microwave Background (CMB). CMB maps derived from string/texture models of structure formation are likely to be significantly non-Gaussian on small angular scales and in a future study we shall use percolation analysis to compare string/texture MBR maps with CDM maps (Moessner et al. 1996).

---

¹ We feel that the quantity $FF_C(\text{clusters}) - FF_C(\text{voids})$ may be less sensitive to sample geometry than $FF_C(\text{clusters/voids})$.

¹⁰ Note: The above definition of topology depends upon whether one uses $FF$ or $\delta$ to measure percolation. For instance $\delta_C$ always decreases for voids whereas $FF_C$ increases. Thus percolation of voids turns out to be either easier or more difficult than Gaussian depending upon whether $\delta$ or $FF$ has been used to quantify percolation.
Finally, our results clearly imply that gravitational clustering cannot be described by a mapping between initial and final states such as: $\delta_{\text{final}} = f(\delta_{\text{initial}})$. Such a mapping would leave the percolation curve unaltered. This is in conflict with our results which unambiguously show that the percolation curve evolves during gravitational clustering.

**Acknowledgments:** Acknowledgments are due to the Smithsonian Institution, Washington, USA, for International travel assistance under the ongoing Indo-US exchange program at IUCAA, Pune, India. We thank Adrian Melott for useful discussions, one of us (BSS) would like to thank him and the Department of Physics and Astronomy, University of Kansas at Lawrence for hospitality where this work was initiated. SS acknowledges NSF grant AST-9021414, NASA grant NAGW-3832, and the University of Kansas GRF-95 grant. BSS would like to thank Kip Thorne for hospitality and encouragement and acknowledges Caltech NSF Grant AST-9417371.

**REFERENCES**

Klypin A.A. 1987, Soviet Astron., 31, 8
Melott, A.L. 1990, Physics Reports, 193, 1
Sahni V. & Coles P., 1995, Physics Reports, 262, 1
Fig. 1.— Percolation (left panels) and genus (right panels) curves are plotted as functions of the density contrast $\delta$ for scale free models of gravitational clustering (a) $n = -2$ and (b) $n = 0$. Solid and dashed curves in the left panels correspond to $v_{\text{max}}$ for the largest cluster and void respectively. Vertical solid/dashed lines mark the threshold describing percolation between opposite faces of the cube for clusters/voids respectively.

Fig. 2.— Percolation (left panels) and genus (right panels) curves are plotted as functions of the filling factor for N-body simulations of scale invariant spectra with index (a) $n = -2$ and (b) $n = 0$. Panels illustrating percolation have solid/dashed lines showing $FF$ in the largest cluster/void ($v_{\text{max}}$) as a function of the total $FF$. Heavy vertical solid/dashed lines mark the onset of percolation in the overdense/underdense phases; thin dashed lines mark percolation in Gaussian fields with identical spectra. Panels illustrating genus have solid/dashed lines showing the genus curve for overdense/underdense regions plotted against the total $FF$.

Table 1: Evolution of amplitude of the genus curve (at $FF = 0.5$) relative to Gaussianized fields ($\text{GC}_{\text{RAN}} - \text{GC}$)/$\text{GC}_{\text{RAN}}$) is tabulated for $n = 0$ and $-2$ models at three expansion epochs.

<table>
<thead>
<tr>
<th>$k_{\text{NL}}$</th>
<th>$n = 0$</th>
<th>$n = -2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.00</td>
<td>0.33</td>
</tr>
<tr>
<td>16</td>
<td>0.32</td>
<td>0.58</td>
</tr>
<tr>
<td>4</td>
<td>0.63</td>
<td>0.67</td>
</tr>
</tbody>
</table>