Non-minimal coupling and quantum entropy of black hole

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Abstract

Formulating the statistical mechanics for a scalar field with non-minimal $\xi R \phi^2$ coupling in a black hole background we propose modification of the original ’t Hooft “brick wall” prescription. Instead of the Dirichlet condition we suggest some scattering ansatz for the field functions at the horizon. This modifies the energy spectrum of the system and allows one to obtain the statistical entropy dependent on the non-minimal coupling. For $\xi < 0$ the entropy renormalizes the classical Bekenstein-Hawking entropy in the correct way and agrees with the result previously obtained within the conical singularity approach. For a positive $\xi$, however, the results differ.

PACS number(s): 04.70.Dy, 04.62.+v

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1 Introduction

A common hope during last years is that the study of quantum entropy of black hole can shed light on the problem of obtaining a statistical meaning of the Bekenstein-Hawking entropy [1], [2]. A number of different approaches was proposed (see reviews in [3]). According to ’t Hooft [4] (see also [5]) the entropy arises from a thermal bath of quantum fields propagating just outside the horizon. This is a purely statistical calculation treating the quantum thermal bath as a system characterized by some energy spectrum. Being in equilibrium at a temperature \( T = \beta^{-1} \), the states of the system are distributed according to Gibbs. An important feature of this calculation is that the density of states becomes infinite approaching the horizon. That is why ’t Hooft introduced a “brick wall”, a fixed boundary staying at a small distance \( \epsilon \) from the horizon. Assuming that the quantum fields do not propagate within this wall, ’t Hooft imposed the Dirichlet condition at the boundary. A reformulation of this model was suggested in [6] by using the Pauli-Villars regularization scheme. Introducing a number of fictitious fields (regulators) of different statistics and masses (dependent on UV regulator \( \mu \)), it was shown that this procedure not only regularizes the UV problem in the effective action but also automatically implements a cutoff for the entropy calculation allowing one to remove the “brick wall”. Thus, the “brick wall” can be conveniently considered [7] as a fictitious boundary the role of which is just to make the calculation simpler.

In an alternate approach the entropy arises from entanglement by means of the density matrix obtained by tracing over modes of the quantum field propagating inside the horizon [8], [9], [10], [11].

The powerful method to calculate both the classical and quantum entropies of a hole is to apply the Euclidean path integral approach [12]. For an arbitrary field system it entails closing the Euclidean time coordinate with a period \( \beta = T^{-1} \) where \( T \) is the temperature of the system. This yields a periodicity condition for the quantum fields in the path integral. In the black hole case for arbitrary \( \beta \) this procedure leads to an effective Euclidean manifold which has a conical singularity at the horizon that vanishes for a fixed value \( \beta = \beta_H \). The entropy is calculated by differentiating the corresponding free energy
with respect to \( \beta \) and then setting \( \beta = \beta_H \). This procedure was consistently carried out for a static black hole and resulted in obtaining the general UV-divergent structure of the entropy [13]-[16].

For a quantum matter minimally coupled to gravity these three methods to calculate the entropy lead to the compatible results [6], [7], [10], [11], [14], [15], [16] and predict the similar structure of the UV divergences of the entropy. As was proposed by Susskind and Uglum [14], these divergences are absorbed in the renormalization of Newton’s constant and, according to [16], in the renormalization of the higher curvature couplings in the effective action. So far, however, there has not been agreement between these approaches for the quantum entropy due to a matter non-minimally coupled to gravity. This problem was considered in [17], [18], [19], [20], [21], [22].

The peculiarity of the non-minimal coupling is easily seen in the conical singularity approach [18]. Consider the Euclidean path integral for a scalar field with the non-minimal operator \(-\Box_\xi \equiv -\Box + \xi R\). For an Euclidean manifold with a conical singularity the scalar curvature behaves at the singularity as \( \delta \)-function even if the regular part of the curvature vanishes. This \( \delta \)-like potential modifies the spectrum of the operator \(-\Box_\xi\) and the resultant path integral. The similar phenomenon happens for a manifold with boundaries [21]. In that case the operator \(-\Box_\xi\) has a \( \delta \)-like potential concentrated on the boundary. In result, the conical singularity method gives rise to the following entropy due to the non-minimal quantum scalar matter [18]:

\[
S_q = \frac{1}{4} A_\Sigma (1 - 6\xi) c_1(\mu) + \left\{ -\frac{1}{8\pi} \left( \frac{1}{6} - \xi \right)^2 \int R + \frac{1}{45} \frac{1}{32\pi} \int \Sigma \left( R_{\mu\nu} n_i^\mu n_i^\nu - 2R_{\mu\nu\alpha\beta} n_i^\mu n_j^\nu n_i^\alpha n_j^\beta \right) \right\} c_2(\mu) \quad (1.1)
\]

where \( A_\Sigma \) is the area of the horizon \( \Sigma \) and \( \{ n_i, \ i = 1, 2 \} \) is a pair of vectors orthogonal to \( \Sigma \). The constants \( c_1(\mu) \) and \( c_2(\mu) \) depend on the UV (energy) cutoff \( \mu \) and for large values of \( \mu \) they grow to leading order as \( \mu^2 \) and \( \ln \mu^2 \) respectively.

The divergences in (1.1) take the correct form so that in combination with the bare entropy they are absorbed in the renormalization of Newton’s constant and the quadratic-curvature couplings in the effective action. In particular, Newton’s constant is renormal-
ized as follows [23]

\[
\frac{1}{G_R} = \frac{1}{G_B} + (1 - 6\xi)c_1(\mu) .
\] (1.2)

One can see that even in flat space there still exists dependence on \(\xi\) in the quantum entropy (1.1) and in the renormalization of Newton’s constant (1.2).

There are simple arguments showing that the quantum entropy in the ’t Hooft approach has a different dependence on \(\xi\). Consider a black hole background satisfying the constraint \(R = 0\). Then, solutions of the field equation of motion \((\Box - \xi R)\phi = 0\) are the same as for \(\xi = 0\). Consequently, the energy spectrum of the system and the entropy are not affected by the non-minimal coupling. The similar arguments are applicable for the entanglement entropy as well to argue that this entropy also does not depend on \(\xi\). This allowed some authors to make a conclusion that “entropy of a quantum matter non-minimally coupled to gravity does not have a statistical meaning”. An important circumstance missed in this sort of arguments is that a non-minimal matter possesses some non-trivial interaction with the horizon (the importance of this interaction was argued in [21]). The “brick wall” prescription is not appropriate to describe this interaction. The goal of this paper is to modify the ’t Hooft approach in order to get the correct \(\xi\)-dependence for the statistical entropy and obtain the correspondence with the conical singularity method.

An important point in our consideration is an idea that in the non-minimal case we are obliged to impose certain boundary condition the form of which directly follows from the form of the non-minimal coupling. Therefore, our strategy is to replace the Dirichlet boundary condition appearing in the original ’t Hooft calculation by some \(\xi\)-dependent condition. A motivation for this, actually, comes from the Euclidean version of the theory. Indeed, the presence in the operator \(-\Box_\xi\) of the \(\delta\)-like potential concentrated on the boundary or at the conical singularity can be precisely re-formulated as imposing an appropriate condition on the field functions on the boundary or at the singularity respectively. In the Euclidean theory these conditions are found to take a simple form [24]. To make our condition in the Minkowskian space-time more clear, note that the standard Dirichlet condition for a wave equation means that a wave is reflected by the boundary with the change of phase equal to \(\pi\). Analogously, what we propose is essentially
an ansatz for the field function near the horizon that describes scattering by the hole with some non-trivial change of phase. As a result of the scattering, in the spectrum of the system appear some low-energy modes the density of which, being proportional to \(\xi\), grows as \(g^{-1/2}(\epsilon)\) approaching the horizon. Namely the contribution of these modes to the statistical entropy is essential to obtain the correct dependence on \(\xi\) in agreement with (1.1). In principle, our calculation is quite similar to the standard statistical consideration of the non-ideal gas [25]: the corrections to the thermodynamical quantities of the ideal gas are expressed via the two-particle scattering phase shift. The only difference from our case is that in our model a particle interacts with the horizon rather than with other particles. It is important to note that we propose the scattering not just for needs of the entropy calculation. We believe that it is an actual interaction between the non-minimal matter and the horizon that is dictated by the form of the non-minimal coupling.

2 WKB field function and boundary condition

A straightforward generalization of the approach [4], [6] is to consider the more general case of static black hole with the spheri-symmetric metric written in the form:

\[
\begin{align*}
    ds^2 &= -g(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\end{align*}
\]

(2.1)

Not assuming this metric to satisfy any concrete field equations, we suppose that \(g(r)\) is a non-negative function having a simple zero at \(r = r_+\) where it behaves as \(g(r) = \frac{4\pi}{\beta_H}(r - r_+) + O((r - r_+)^2)\). This corresponds to the position of the outer event horizon at \(r = r_+\). The Hawking temperature calculated for the metric (2.1) is \(T_H = \beta_H^{-1}\). Note that the scalar curvature \(R\) for the metric (2.1) is function of only radial coordinate \(r\).

In this black hole background we consider a non-minimal scalar field which satisfies the equation:

\[
    (\Box - m^2 - \xi R)\phi = 0.
\]

(2.2)

Expanding the wave function in spherical coordinates \(\phi = e^{iE\ell t}Y_{l m}(\theta, \varphi)f(r)\), we obtain the equation on the radial function

\[
    E^2 r^4 f(r) + r^2 g \partial_r (r^2 g \partial_r f(r)) - r^4 g (m^2 + \xi R + \frac{l(l+1)}{r^2}) f(r) = 0.
\]

(2.3)
The WKB approximation provides us with the following solution of the Eq.(2.3):

\[ f(r) = \rho(r) \ e^{\pm \int^r \frac{dr}{g}} k(r) \ , \quad k(r) = \sqrt{E^2 - (m^2 + \xi R + \frac{l(l+1)}{r^2})g(r) } \ , \quad (2.4) \]

which is valid in the region where \( k^2(r) \geq 0 \). It is clear that for large mass \( m \) this is space closely located to the horizon. If \( R \) takes a non-zero value at the horizon then this region can be approximated by some sort of the constant curvature space. However, the exact results for such a space show [23] that the mass parameter in the solution enters only in the combination \( (m^2 - \frac{1}{6} R) \). The WKB approximation, being well-defined for large values of \( l \), does not give this result. Correcting this, we arrive at the following wave function

\[ f(r) = \rho(r) \left( e^{\pm \int^r \frac{dk(r)}{g}} + Ae^{-\int^r \frac{dk(r)}{g}} \right) , \quad k(r) = \sqrt{E^2 - (M^2(r) + \frac{l(l+1)}{r^2})g(r) } , \quad (2.5) \]

where we denote \( M^2(r) = m^2 - (\frac{1}{6} - \xi)R \). Since the quantity \( M^2(r) \) is slowly varying near the horizon it can be considered constant there. The constant \( A \) in (2.5) is to be determined from the boundary condition. The amplitude \( \rho(r) \) is a slowly varying function of \( r \). In what follows, we neglect its derivatives with respect to \( r \) and omit writing it in the formulas.

Consider a boundary \( \Sigma_\epsilon \) staying at a small distance \( \epsilon \) from the horizon \( \Sigma \). In the limit \( \epsilon \to 0 \) it approaches the horizon \( \Sigma_\epsilon \to \Sigma \). The parameter \( \epsilon \) is assumed to be smaller† than any UV cutoff \( \mu^{-1} \) appearing under regularization of the quantum field theory [7], and such quantities as \( \mu^2g(\epsilon) \) are considered to be negligible. Therefore, the boundary \( \Sigma_\epsilon \) plays an intermediate role just to simplify the consideration. The condition, in principle, can be imposed directly on the horizon.

In order to arrive at an idea of the boundary condition to be imposed on \( \Sigma_\epsilon \) let us start with the following “simple-minded” condition:

\[ (n^\mu \partial_\mu \phi - \xi \kappa \phi)|_{\Sigma_\epsilon} = 0 \ , \quad (2.6) \]

where \( n^\mu \) is vector normal to \( \Sigma_\epsilon \) and \( \kappa \) is the extrinsic curvature of \( \Sigma_\epsilon \), \( \kappa = \nabla_\mu n^\mu \). Namely the condition of this type appears in the Euclidean version of theory [24]. For the metric

†Explicitly, this means that \( \mu(\epsilon \beta_H)^{1/2} \ll 1 \).
(2.1) and sufficiently small \( \epsilon \) the condition (2.6) reads
\[
(g(r) \partial_r f(r) - \xi^* f(r)) |_{r=r_++\epsilon} = 0 ,
\]
where \( \xi^* = 2\pi \beta_H^{-1} \xi \). For the function (2.5) it gives the expression for the constant \( A \):
\[
A = e^{i \eta(k)} .
\]
The phase \( \eta(k) \) is defined as follows:
\[
e^{i \eta(k)} \equiv \frac{ik(\epsilon) - \xi^*}{ik(\epsilon) + \xi^*} ,
\]
where \( k(\epsilon) \equiv k(r = r_+ + \epsilon) \).

The condition we are looking for is a \( \xi \)-dependent modification of the Eq.(2.7). Curiously enough, an appropriate condition takes the form
\[
\left( g(r) \partial_r f(r) - \xi^* \right)_{\nu} f(r) |_{r=r_++\epsilon} = 0 , \nu = 2\xi ,
\]
where \( (g(r) \partial_r - \xi^*)_{\nu} \) is a pseudo-differential operator. It acts as follows
\[
(g(r) \partial_r - \xi^*)_{\nu} e^{\pm i \int^r d r \frac{d k(r)}{d r}} = (\pm i k - \xi^*)_{\nu} e^{\pm i \int^r d r \frac{d k(r)}{d r}} ,
\]
where we neglected the derivatives of \( k(r) \). The condition (2.9) leads to the following constant
\[
A = e^{i \eta(k)} e^{i \pi (\nu + 1)} ,
\]
where \( \eta(k) = \arctan \left( \frac{2k(\epsilon)\xi^*}{k^2(\epsilon) - \xi^* \xi^*} \right) \) is defined as in (2.8). Remarkably, for \( \nu = 0 \) (\( \xi = 0 \)) the Eq.(2.9) coincides with the Dirichlet condition.

With the condition (2.9) imposed, the WKB field function takes the form
\[
f(r) = \left( e^{i \int^r d r \frac{d k}{d r}} e^{i \eta(k)} e^{i \pi (\nu + 1)} e^{-i \int^r d r \frac{d k}{d r}} \right) .
\]
This is that ansatz which we propose for the field function near the horizon. The function (2.11) is valid in the region \( r_+ + \epsilon \leq r \leq r_E \), where \( r_E \) is determined by the equation \( k^2(r_E) = 0 \). The precise meaning of the condition appears when we consider the field function at the horizon. Introduce new coordinate \( z = \int^r g^{-1} d r \). Then, at the horizon (\( z \) goes to \( -\infty \)) the function (2.11) becomes
\[
f(z) = e^{i E(z-z_0)} + e^{i \eta(E)} e^{i \pi (\nu + 1)} e^{-i E(z-z_0)} ,
\]
\[
e^{i \eta(E)} = \left( \frac{i E - \xi^*}{i E + \xi^*} \right)^\nu ,
\]
where \( z_0 \) is a constant, and describes scattering by the hole with change of phase \( \nu \eta(E) + \pi (\nu + 1) \).
3 Density of states and entropy calculation

In order to discretize the energy spectrum and simplify counting of states we impose also the Dirichlet condition $\phi = 0$ for $r = r_E$. This gives the quantization condition

$$2 \int_{r_{+}+\epsilon}^{r_E} \frac{dr}{g(r)} k(r) = \nu \eta(k) + 2\pi n + \pi \nu \quad ,$$

where $n$ is an integer number. Inverting this, we get the equation

$$n = n(E, l) \equiv n_0(E, l) + n_1(E, l) \quad ,$$

$$n_0(E, l) = \frac{1}{\pi} \int_{r_{+}+\epsilon}^{r_E} \frac{dr}{g(r)} k(r) \quad ,$$

$$n_1(E, l) = -\frac{\nu}{2\pi} \left( \arctan \left( \frac{2k(\epsilon)\xi^*}{k^2(\epsilon) - \xi^*} \right) + \pi \right) \quad ,$$

solution of which gives the energy levels $\{E_n,l\}$ at a fixed $l$.

Near the horizon the metric function can be approximated as $g(r) \simeq \frac{4\pi}{\beta_H}(r - r_+)$. In this approximation the integral in (3.2) is calculated exactly. We find in result that

$$n_0(E, l) \simeq \frac{\beta_H}{2\pi^2} \left( E \arctanh \sqrt{1 - \frac{E_{\min,l}^2}{E^2}} - \sqrt{E^2 - E_{\min,l}^2} \right) \quad ,$$

$$n_1(E, l) = -\frac{\nu}{2\pi} \left( \arctan \left( \frac{2\sqrt{E^2 - E_{\min,l}^2}}{E^2 - E_{\min,l}} \right) + \pi \right) \quad ,$$

where $E_{\min,l}^2 \equiv (M^2 + \frac{l(l+1)}{r_+^2})g(\epsilon)$. We see that $E = E_{\min,l}$ is the minimal possible energy at a fixed $l$. For $E \simeq E_{\min,l}$ the behavior of the function $n(E, l)$ (3.2) is mainly determined by the component $n_1(E, l)$ while for large enough $E$ the component $n_0(E, l)$ becomes the leading one. For $\nu = 2\xi$ the $n(E, l)$ is monotonically increasing (both for $\xi > 0$ and $\xi < 0$) function\footnote{In general the behavior of the function $n_1(E, l)$ depends on relative sign of $\nu$ and $\xi$. Namely, if $\text{sign}(\nu) = \text{sign}(\xi)$ the function $n_1(E, l)$ is increasing for $E \geq E_{\min,l}$ while in opposite case, $\text{sign}(\nu) = -\text{sign}(\xi)$, it is decreasing. This affects the behavior of the whole function $n(E, l)$ which is monotonically increasing in one case ($\text{sign}(\nu) = \text{sign}(\xi)$) and has one minimum in other case ($\text{sign}(\nu) = -\text{sign}(\xi)$). Here we assume that $\nu = 2\xi$ and discuss the possible choice $\nu = -2|\xi|$ below in the Discussion section.} taking value $n(E, l) = \xi$ for $E = E_{\min,l}$. The total number of modes with energy less than $E$ can be determined by taking the quantity $(n(E, l) - n(E_{\min,l}))$ and summing over the degeneracy of the angular modes:

$$n(E) \equiv \int dl (2l + 1)(n(E, l) - n(E_{\min,l})) = n_0(E) + n_1(E) \quad ,$$

‡}
where the sum over \( l \) has been approximated by an integral, and this integration runs over non-negative values of \( l \) for which the square roots \( k(r) \) and \( k(\epsilon) \) in the integrand are real. After the integration we have

\[
n_0(E) = \frac{2}{3\pi} \int_{r_+ + \epsilon}^{r_E} dr \frac{r^2}{g^2(r)} (E^2 - M^2 g(r))^{3/2}
\]

(3.5)

and

\[
n_1(E) = -\frac{\nu}{\pi} \frac{r^2_+}{g(\epsilon)} \int_0^{\sqrt{E^2 - M^2 g(\epsilon)}} dk k \arctan\left( \frac{2k \xi^*}{k^2 - \xi^{*2}} \right)
\]

\[
= -\frac{\nu}{\pi} \frac{r^2_+}{g(\epsilon)} \left\{ \frac{k^2(\epsilon)}{2} \arctan\left( \frac{2k \xi^*}{k^2 - \xi^{*2}} \right) + \xi^* k(\epsilon) - \xi^{*2} \arctan\left( \frac{k(\epsilon)}{\xi^*} \right) \right\}
\]

(3.6)

where \( k(\epsilon) = \sqrt{E^2 - M^2 g(\epsilon)} \).

One can think about this system of particles as consisting of two components: the ordinary particles with the number \( n_0(E) \) as in the original 't Hooft model and the scattering particles with the number \( n_1(E) \). Remarkably, \( n_1(E) \) is proportional to the horizon area \( A_+ = 4\pi r^2_+ \).

For \( E \approx E_{\text{min}} = Mg^{1/2}(\epsilon) \) the functions \( n_0(E) \) and \( n_1(E) \) behaves as follows

\[
\begin{align*}
n_0(E) &\approx 4 \frac{\beta_H}{15} \frac{M^2}{r^2_+ E_{\text{min}}^4} (E^2 - E_{\text{min}}^2)^{5/2} \\
n_1(E) &\approx \text{sign}(\nu\xi) \frac{\beta_H}{2\pi^2} r^2_+ \frac{M^2}{E_{\text{min}}^2} (E^2 - E_{\text{min}}^2)^{3/2}.
\end{align*}
\]

(3.7)

If \( \text{sign}(\nu\xi) = +1 \) the total number of modes \( n(E) \) is the monotonically increasing function of \( E \) and \( E = E_{\text{min}} \) is actual minimum of energy.

To determine the thermodynamics of this system, we consider the free energy of a thermal ensemble of scalar particles with an inverse temperature \( \beta \)

\[
\beta F = \int_{E_{\text{min}}}^{\infty} dE \frac{dn}{dE} \ln(1 - e^{-\beta E}) = \beta F_0 + \beta F_1,
\]

(3.8)

where we separate the contributions due to the ordinary modes and the scattering modes.

Applying Pauli-Villars regularization scheme for the present four-dimensional scalar field theory, one introduces five regulator fields \( \{\phi_i, i = 1, \ldots, 5\} \) of different statistics and masses \( \{m_i, i = 1, \ldots, 5\} \) dependent on the UV cut-off \( \mu \) [6]. Together with the original scalar \( \phi_0 = \phi \ (m_0 = m) \) these fields satisfy two constraints: \( \sum_{i=0}^5 \Delta_i = 0 \) and \( \sum_{i=0}^5 \Delta_i m_i^2 = \ldots \)
where $\Delta_i = +1$ for the commuting fields, and $\Delta_i = -1$ for the anticommuting fields. Additionally, we assume that all the fields have the same non-minimal coupling $\xi_i = \xi$, $i = 0, ..., 5$. Not deriving the exact expressions for $m_i$, we just quote here the asymptotes

$$
\sum_{i=0}^{5} \Delta_i m_i^2 \ln m_i^2 = \mu^2 b_1 + m^2 \ln \frac{m^2}{\mu^2} + m^2 b_2 ,
$$

$$
\sum_{i=0}^{5} \Delta_i \ln m_i^2 = \ln \frac{m^2}{\mu^2} ,
$$

(3.9)

where $b_1$ and $b_2$ are some constants, valid in the limit $\mu \to \infty$. With contribution of each field added the free energy (3.8) becomes

$$
\beta F = \sum_{i=0}^{5} \Delta_i \beta F^i = \beta F_0 + \beta F_1 .
$$

(3.10)

For the scattering modes of a single field we have

$$
\beta F_1 = \int_{E_{\text{min}}}^{\infty} dE \frac{dn_1}{dE} \ln(1 - e^{-\beta E})
$$

$$
= -\nu \int_{E_{\text{min}}}^{\infty} dE E \arctan \left( \frac{2 \sqrt{E^2 - E_{\text{min},l}^2 \xi^*}}{E^2 - E_{\text{min},l}^2 - \xi^2} \right) \ln(1 - e^{-\beta E}) .
$$

(3.11)

A remarkable property of the expression (3.11) is that for $\nu = 2\xi$ it depends only on the absolute value of $\xi$. Integrating over $E$ in (3.11) and focusing only on the divergent for small $\epsilon$ terms, we find

$$
\beta F_1 \simeq -|\xi| \frac{r^2}{g(\epsilon)} (E_{\text{min}}^2 \ln(\beta E_{\text{min}}) - \frac{1}{2} E_{\text{min}}^2) - |\xi| \frac{r^2 \chi}{\beta^2 g(\epsilon)} ,
$$

(3.12)

where $\chi$ is an independent on $M$ constant. Summing contributions of each field $\phi_i$, we obtain

$$
\beta \bar{F}_1 \simeq -\frac{|\xi|}{2} r^2 \sum_{i=0}^{5} \Delta_i M_i^2 \ln M_i^2 ,
$$

(3.13)

where $M_i^2 = m_i^2 - (\frac{1}{6} - \xi)R$.

Our calculation of a part of the free energy due to the ordinary modes essentially repeats for more general static metric (2.1) the calculation presented in [6]. In the expression for the free energy of a single field we can integrate by parts and get

$$
\beta F_0 = -\beta \int_{E_{\text{min}}}^{\infty} \frac{n_0(E)}{e^{\beta E} - 1} dE .
$$

(3.14)
Focusing on the divergences at the horizon in the expression for the number of ordinary modes \( n_0(E) \) (3.5) we find
\[
n_0(E) \simeq -\frac{r_+^2}{\pi} \left( \frac{E}{\beta H} \right)^3 C + M^2 \left( \frac{E}{\beta H} \right) \ln \frac{E^2}{E_{\text{min}}^2} + \frac{2}{3\pi} \left( \frac{\beta H}{4\pi} \right) r_+^2 E^3 M^2 \left( E_{\text{min}}^{-2} - E^{-2} \right),
\]
where \( C = (R_{\mu\nu} n^\mu_i n^\nu_i - 2R_{\mu\nu\alpha\beta} n^\mu_i n^\nu_i n^\alpha_j n^\beta_j)_{\Sigma} \) with notions as in (1.1).

Substitution of this in (3.14), integration over \( E \) and summation for all fields \( \phi_i \) give us the total free energy due to the ordinary modes:
\[
\beta \bar{F}_0 \simeq -\frac{1}{24} \frac{\beta H}{\beta} r_+^2 + \sum_{i=0}^{5} \Delta_i M_i^2 \ln M_i^2 - \frac{1}{45} \frac{\beta H}{\beta} r_+^2 C \sum_{i=0}^{5} \Delta_i \ln M_i^2.
\]
(3.16)

It is a manifestation of the mechanism discovered in [6] that the expressions (3.13) and (3.16) are regular in the limit \( \epsilon \to 0 \). There is a precise cancelation of the divergences between the original scalar and regulator fields. The resultant expressions, however, become dependent on the UV regulator \( \mu \).

Altogether, (3.13) and (3.16) give the total free energy (3.10) of the system. Calculation of the entropy \( S = \beta^2 \partial_\beta \bar{F} \) at the Hawking temperature \( \beta^{-1} = \beta_H^{-1} \) gives
\[
S = \frac{1}{2} \left( \frac{1}{6} + |\xi| \right) r_+^2 \sum_{i=0}^{5} \Delta_i M_i^2 \ln M_i^2 + \frac{1}{45} \frac{\beta H}{\beta} r_+^2 C \sum_{i=0}^{5} \Delta_i \ln M_i^2.
\]
(3.17)

Using the definitions of \( C \) and \( M_i^2 = m_i^2 - (\frac{1}{6} - \xi) R \), and assuming that the value of the scalar curvature \( R \) at the horizon is much smaller than each \( m_i^2 \), we arrive at the expression
\[
S = \frac{1}{4} A_{\Sigma} \frac{1}{12\pi} (1 + 6|\xi|) \sum_{i=0}^{5} \Delta_i m_i^2 \ln m_i^2 + \left\{ -\frac{1}{8\pi} \left( \frac{1}{6} + |\xi| \right) (\frac{1}{6} - \xi) \right\} \int_{\Sigma} R
\]
\[+ \frac{1}{45} \frac{1}{32\pi} \left( R_{\mu\nu} n^\mu_i n^\nu_i - 2R_{\mu\nu\alpha\beta} n^\mu_i n^\nu_i n^\alpha_j n^\beta_j \right) \sum_{i=0}^{5} \Delta_i \ln m_i^2.
\]
(3.18)

This is our main result. In minimal case (\( \xi = 0 \)) and for the Reissner-Nordstrom background Eq.(3.18) coincides with the result of [6].

4 Discussion of the result

It should be noted that the expression (3.18) is not exactly that we were going to obtain for the entropy anticipating the complete agreement with the result (1.1) found within the
conical singularity method. The dramatical difference of our result is that the statistical entropy (3.18) is not analytic with respect to the non-minimal coupling $\xi$. For negative $\xi$ it exactly reproduces the conical expression (1.1) while for positive $\xi$ the both quantities differ. This is well illustrated in the leading order. The conical entropy (1.1) then takes the form $S_q = \frac{1}{4}A_\Sigma(1 - 6\xi)c_1(\mu)$ and becomes negative for $\xi > \frac{1}{6}$. This puzzling behavior has been discussed in [19], [18] and more recently in [26]. In particular, it was noted that a statistical entropy defined as $S = -Tr\rho \ln\rho$ for a density matrix $\rho$ can not behave in this way being automatically positive. Therefore, may be it is not so surprising that our semi-classical computation of the statistical entropy gives rise to the expression $S = \frac{1}{4}A_\Sigma(1 + 6|\xi|)c_1(\mu)$ which is always positive.

However, can we modify in some way our calculation and reach the complete agreement with (1.1)? At first sight, it seems possible if we assume that $\nu = -2|\xi|$ instead of $\nu = 2\xi$. Then the free energy $\beta F_1$ (3.11) looks becoming dependent on sign of $\xi$ and the entropy does too. However, in this case the function $n(E,l)$ (3.2), (3.3) is not monotonically increasing for $\xi > 0$. Instead, it develops a minimum in a point $E_{min,l}$ different than $E_{min,l}$. One observes the same behavior for the function $n(E)$ (3.4) having sense of the total number of modes with energy less than $E$. As is seen from (3.7), for $sign(\nu\xi) = -1$ it decreases at $E = E_{min}$, takes minimal value at some point $E_{min}$ and then monotonically increases. This behavior means that for $\nu = -2|\xi|$, $\xi > 0$ we have to re-count the number of modes which is no more given by the expression (3.4). In result, we should get new function $n(E)$ which is monotonic and leads for $\xi > 0$ to a positive entropy as well.

The computation we present in this work sheds some light on the origin of the $\xi$-dependent part of the entropy of the non-minimal scalar field. It is important to note that the main contribution to $\beta F_1$ comes from modes with energy close to $E_{min} = Mg^{1/2}(\epsilon)$.

For this energy we may substitute $\ln(1-e^{\beta E}) \simeq \ln(\beta E)$ in (3.11) and this is the divergence (if $E \to 0$) that appears in (3.12). It is interesting that one may give an alternative simple calculation of $\beta F_1$ assuming existence in the spectrum (3.1) exactly$^5$ $n = |\xi|$ modes with the minimal energy $E_l = E_{min,l} \equiv (M^2 + \frac{l(l+1)}{r^2})^{(1/2)}g(\epsilon)$. Indeed, summing then over $l$ we

$^5$Since $n$ is an integer it is better to suppose that $n = [\xi]$ where $[\xi]$ is integer part of $\xi$. For a big $\xi$ this is, however, not important.
find

\[ \beta F_1 = \int dl (2l + 1) |\xi| \ln (1 - e^{\beta E}) = \frac{2|\xi|r_+^2}{g(\epsilon)} \int_{E_{\text{min}}=E_L=0}^{\infty} dEE \ln (1 - e^{\beta E}) \]

\[ \simeq -\frac{|\xi|r_+^2}{g(\epsilon)} E_{\text{min}}^2 \ln (\beta E_{\text{min}}) \] (4.1)

that is exactly the expression (3.12). So, namely, these low-energy modes are responsible for the \( \xi \)-dependent part of the entropy. The role of the boundary condition we propose in the Section 2 is just to provide us with necessary number of these modes. Unfortunately, it is not clear that we have a well-posed problem with those boundary conditions and, for example, the Hamiltonian is obviously self-adjoint (though we deal with apparently positive energy spectrum). Therefore, it may happen that there exists a more strict way of getting the low-energy behavior we mentioned above within a well-formulated self-adjoint extension of the scalar field Hamiltonian.

The following conclusions are in order.

1. The quintessence in understanding the entropy of the non-minimal scalar field is existence of the low-energy modes number of which is governed by the non-minimal coupling \( \xi \). In our approach these modes appear due to the non-trivial scattering condition which we impose on the scalar field at the horizon. In the limit \( \epsilon \to 0 \) (“brick wall” removed) they presumably become zero modes of the Hamiltonian which are infinitely degenerate due to the angle dependence. However, their status in the well-defined Hamiltonian picture is not clear.

2. For negative \( \xi \) the entropy (1.1) may have a statistical explanation within the procedure described in this paper. However, we still lack this explanation for (1.1) for positive \( \xi \). Moreover, the validity of the expression (1.1) for \( \xi > 0 \) is under question. More careful analysis [24] shows that the conical method is not unambiguous and one may be needed to re-consider the way one obtains the expression (1.1) within this method.

3. For \( \xi < 0 \) the divergences of the statistical entropy may be renormalized by the renormalization of Newton’s constant according to (1.2). However, for \( \xi > 0 \) even after such a renormalization we still have a divergence in the entropy behaving as \( S_{\text{div}} = 3|\xi| c_1(\mu) \). It is not clear in a renormalization of which quantity it may be absorbed.

4. Our result may be considered as a confirmation (for the non-minimally coupled
scalar field) of the point that any physical quantum field (bosonic or fermionic) must have a positive entropy. This, however, may have dramatrical consequences for the understanding the origin of the black hole entropy in theories of the induced gravity. In this kind of theories [27] the Einstein gravity arises in the low-energy regime by averaging over the constituent matter fields interacting with a background (classical) metric. The set of these fields is specially arranged to make the induced Newton’s constant $G_{\text{ind}}$ UV finite. It is hoped [28], [22] that the finite black hole entropy $S_{\text{bh}} = \frac{1}{4G_{\text{ind}}} A_\Sigma$ can be induced in a similar way as a statistical entropy of the constituents. This hope is essentially based on the possibility to make the black hole entropy finite by making finite Newton’s constant. However, this assumes that among the constituents there should present fields carrying negative divergent entropy which compensates the positive divergent entropy of other constituents. In a concrete realization [22] of the induced gravity the role of those particles is played by the scalar fields non-minimally coupled to gravity with positive $\xi$. However, if each constituent has a positive divergent entropy the total induced entropy is also positive and divergent even if the induced Newton’s constant is finite. Thus, we have an obvious problem with inducing the Bekenstein-Hawking entropy. Possibly, the approached developed in [22] can be useful in resolving this problem.

One certainly needs understand better all these issues and we hope to return to them in further publications.

Acknowledgments

I wish to thank V.P.Frolov, D.Fursaev, R.C.Myers and A.Zelnikov for helpul comments. I would like to thank the North Atlantic Treaty Organization (NATO) for support. In part this work is also supported by the Natural Sciences and Engineering Research Council of Canada.
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