Toward a Definition of Chaos for General Relativity

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Abstract. General relativity exhibits a unique feature not represented in standard examples of chaotic systems; it is a spacetime diffeomorphism invariant theory. Thus many characterizations of chaos do not work. It is therefore necessary to develop a definition of chaos suitable for application to general relativity. This presentation will present results towards this goal.

1 Introduction

Recent work in chaos has lead to quantitative studies of chaotic behavior in different mathematical and physical systems. Such results have led to useful insight into the nature of these systems. Clearly, a characterization of whether or not general relativity exhibits chaos would be valuable. However, there are well-known difficulties in applying the current characterizations of chaos to general relativity; they are not diffeomorphism invariant. Thus when applied to general relativity, they give different results in different coordinate systems (See for example, summary and references in Rugh [1994]). In particular, one test for chaos commonly used by physicists is to search for sensitive dependence to initial conditions in a system \( \dot{x}(t) = F(x) \) by computing Liapunov exponents via \( \lambda = \lim_{t \to \infty} \frac{1}{t} \ln |z(t)| \) where \( z(t) \) is the linearized perturbation of \( x(t) \). Positive Liapunov exponent is taken as indicating chaotic behavior. Application of this test to general relativity is fundamentally flawed; a coordinate change rescales a positive Liapunov exponent to zero. For example, consider computing the Liapunov exponents for the one parameter model corresponding to expanding de Sitter spacetime; one finds \( \lambda = H \) in the coordinate chart \( ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \) whereas \( \lambda = 0 \) in the chart \( ds^2 = -\frac{dt^2}{H^2} + a^2(t)(dx^2 + dy^2 + dz^2) \). Clearly, as the physics of the solution is independent of coordinate choice, there is a problem with this characterization.

It is clear that a coordinate independent definition of chaos is needed for the study of general relativity. This paper discuss work toward developing such a definition. A longer version of this paper with details is in preparation (Schleich and...
Witt [1995]). The starting point will be a discussion of a rigorous definition of chaos for standard dynamical systems. Next crucial aspects of general relativity that distinguish it from these standard systems will be outlined. Finally a generalized definition of chaos that addresses these crucial aspects will be proposed.

2 Chaos in Standard Dynamical Systems

The literature contains many characterization of chaos, often adapted to the type of the system being studied. However, it is well known that testing for chaos simply by testing for positive Liapunov exponent is flawed. For example, the system with evolution \( \dot{x} = cx \) where \( c \) is a positive constant has positive Liapunov exponent. Clearly, this system is not chaotic. Rigorous characterizations of chaos require ergodicity in addition to tests of sensitive dependence on initial conditions such as Liapunov exponents. (See for example, Wiggins [1990], p. 608 and Pollicott [1993] in the context of non-uniformly hyperbolic diffeomorphisms.)

While developing the machinery for a rigorous definition of chaos, it is useful to keep in mind two standard examples of dynamical systems: One corresponds to two linear coupled oscillators; for fixed initial momenta, this system is isomorphic to the translational flow on the flat torus \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \), \( f_\omega : T^2 \to T^2 \) given by \( f_\omega(x_1, x_2) = (x_1 + \omega_1 t, x_2 + \omega_2 t) + K, \, K \in \mathbb{Z}^2 \). The other is the cat map on the flat torus \( T^2 \) given by \( f(x_1, x_2) = (2x_1 + x_2, x_1 + x_2) + K, \, K \in \mathbb{Z}^2 \).

A general dynamical system is a measurable map: A measure space \((X, \mathcal{B})\) consists of a set \( X \) and a \( \sigma \)-algebra \( \mathcal{B} \). A map \( T : (X_1, \mathcal{B}_1) \to (X_2, \mathcal{B}_2) \) between two measure spaces is called measurable if and only if \( T^{-1}(B) \in \mathcal{B}_1 \) for all \( B \in \mathcal{B}_2 \). A measure \( m \) on \((X, \mathcal{B})\) is a function \( m : \mathcal{B} \to \mathbb{R}^+ \cup \{ \infty \} \) which satisfies \( m(\emptyset) = 0 \) and \( m(\bigcup_k B_k) = \sum_k m(B_k) \) for any countable disjoint collection \( \{B_k\} \). Then

**Definition 2.1** A measure \( m \) on \((X, \mathcal{B})\) is a probability measure if \( m(X) = 1 \).

Note that \( X \) need not be a compact space; for example the measure induced by \( \frac{1}{\sqrt{2\pi}} \exp(-x^2) \) is a probability measure on \( \mathbb{R} \). For simplicity of notation, the term measure will refer to probability measure in this paper. Next

**Definition 2.2** Given a measurable map \( T \) on \((X, \mathcal{B})\), \( m \) is \( T \)-invariant if and only if \( T^*m(B) = m(B) \) for all \( B \in \mathcal{B} \) where \( T^*m(B) = m(T^{-1}(B)) \).

When the map is understood, a \( T \)-invariant measure is called an invariant measure. In general, a map \( T \) on a given space may have more than one invariant measure. Especially note that \( m \) need not have support on the entire space; for example \( m = \delta(x) \) is an invariant measure for the map \( T : \mathbb{R} \to \mathbb{R} \) given by \( T(x) = ax \). At this point a key concept can be introduced:

**Definition 2.3** An invariant measure \( m \) is called an ergodic measure if whenever \( T^{-1}B = B \) for some \( B \in \mathcal{B} \), then either \( m(B) = 0 \) or \( m(B) = 1 \).

Note that in general, the space of ergodic measures is not equal to the space of invariant measures. One can show that when \( T \) is a homeomorphism, there exists at least one ergodic measure on any compact space. Both examples have ergodic measures; the measure induced by \( dx_1 dx_2 \) is the unique invariant and ergodic measure for the cat map and for the translational flow when \( \frac{2\pi}{\omega_2} \notin \mathbb{Q} \). For \( \frac{2\pi}{\omega_2} \in \mathbb{Q} \), there are an infinite number of ergodic measures for the translational flow; these correspond to delta functions that concentrate support on one periodic orbit. A rigorous definition of chaos is then
**Definition 2.4** A system \( T : (X, B) \to (X, B) \) is chaotic if and only if it has an ergodic measure \( m \) and exhibits sensitive dependence on initial conditions with respect to this measure.\(^1\)

When applied to the examples, (2.4) distinguishes chaotic behavior. The translational flow is not chaotic for any values of the parameters, even incommensurate ones; it is easy to show that trajectories do not diverge by any test of sensitive dependence as they are simply straight lines. In contrast, the cat map is chaotic as it has positive Liapunov exponent; it illustrates the importance of sensitive dependence in distinguishing chaotic behavior.

Note especially that the use of ergodic measure implies that chaotic behavior can be concentrated on a subset of the measure space. This feature is important as dissipative systems such as the Henon map exhibit chaotic behavior on such a subset. Thus the use of the ergodic measure make (2.4) a very general definition of chaos. Moreover, other characterizations of chaos such as self-similarity and dense sets of periodic orbits can be recovered under certain conditions. It is this definition that will be the starting point for a more general definition of chaos suitable both to the standard examples and to general relativity.

### 3 Important Characteristics of General Relativity

There are two issues that must be addressed by any formulation of chaos appropriate for application to general relativity: its diffeomorphism invariance and its lack of a physically motivated choice of measure.

The most discussed difficulty associated with diffeomorphism invariance is concentrated in time reparameterization invariance; time is no longer an absolute quantity but is tied to the coordinate system. The coordinate systems given for the de Sitter space model clearly indicate this fact. Note that one can always associate an observer (though not necessarily a freely falling observer) to any choice of coordinate system; thus any attempt to tie chaos to observers is simply making physical a particular coordinate choice.

This problem is closely linked to the fact that diffeomorphism invariance is a form of gauge invariance; the metric degrees of freedom are not all physical. However, it is key to the correct identification of chaos that it be carried out on physical degrees of freedom alone. For example, consider the lifts of the translational flow and the cat map from \( T^2 \) to the covering space \( \mathbb{R}^2 \). The measure induced by \( dx_1 dx_2 \) is not ergodic for either map on \( \mathbb{R}^2 \); there are obviously invariant sets for both the translational flow and the cat map. Therefore neither system is ergodic on the covering space in this measure. Clearly the global topology of the space plays a key role in the nature of the dynamical system!\(^1\)

Such global topology also arises in general relativity; removing the large gauge transformations will change the global structure of the phase space. An example is provided by Bianchi I (Bhushan et. al. [1995]). Spatial gauge is fixed locally by writing the metric as \( ds^2 = -N^2(t)dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2 \). Residual

\(^1\)One test for sensitive dependence is now given by positive Liapunov exponents subject to computation in the ergodic measure \( m \) (See for example Pollicott [1993] for application to nonuniformly hyperbolic diffeomorphisms). This definition differs significantly from the standard physicist’s use in that Liapunov exponents defined w.r.t. an ergodic measure are constant over the space. Alternately Wiggins [1990] presents a definition that does not presume exponential divergence.
transformations corresponding to interchanges of spatial coordinates are not fixed.
Removing these modifies the nature of the flow; the fully reduced phase space is no
longer $\mathbb{R}^2$ but a wedge and the flows are no longer straight lines but include bounces
at the boundaries of this wedge. This change in behavior in a such a simple model
stresses the need to fully reduce to physical degrees of freedom.

Unfortunately, this problem is not confined to large gauge transformations; note
that the usual Hamiltonian formulation of gravity in terms of superspace is also not
in terms of physical degrees of freedom. Although the spatial diffeomorphisms have
been removed, those corresponding to timelike diffeomorphisms have not. Therefore
the study of flows on this space includes redundant information unless a further
reduction is carried out. In particular as different time reparameterizations correspond
to different gauge choices, studying the dynamics of the system without
further reduction imparts physical meaning to different gauges. Thus gauge effects
can mask the underlying physics of the system. Unfortunately there are no known
ways to reduce to physical degrees of freedom explicitly; one obvious approach is
to isolate a preferred timelike Killing vector or conformal Killing vector for superspace. However, Kuchár [1981] has shown no such vectors exist in general. It is
thus unlikely that explicit reductions can be carried out.

The second issue is the lack of a physically motivated choice of measure for
general relativity. It is well known that there is no preferred metric on superspace.
Therefore, one cannot begin with the corresponding measure as a candidate for an
invariant or ergodic measure. This problem is also complicated by the fact that
superspace is not in terms of physical degrees of freedom. Thus it is not clear what
role any choice of metric would play in defining any such measure.

4 Toward a Generalized Definition of Chaos

The above issues clearly demonstrate the futility of testing for chaos in general
relativity by (2.4) by studying flows on superspace. However, what should this
definition be replaced by? First observe that general relativity is a hamiltonian
system in terms of its physical degrees of freedom, even if these variables cannot be
identified explicitly. It is therefore natural to first ask the question in the context of
the fully reduced theory, even if it cannot be explicitly constructed. Given such a
definition on the physical degrees of freedom, one can then attempt to parameterize
it to be applicable to more tractable forms of the theory.

Observe that a key element in (2.4) is ergodicity, intuitively that orbits in
the phase space are dense in some measurable set. Topological generalizations of
ergodicity are well known (See for example Peterson [1990]). Therefore, a natural
approach is to generalize the condition of ergodic measure to a topological condition.

Given any invariant measure $m$ and a map $T$ one can define measurable entropy
$0 \leq h_{meas}(T, m)$ (See Peterson [1990] sect. 5.1). The measurable entropy is related
to the (rigorous) Liapunov exponents by the Pesin-Reuelle inequality under certain
restrictive conditions on the map and measure $h_{meas} \leq \sum_{\lambda_i > 0} \lambda_i$ where the sum
is over positive Liapunov exponents (Pollicott [1993] p. 46). Therefore $h_{meas} > 0$
implies chaos for ergodic measures. Next

**Definition 4.1** The topological entropy is $h_{top}(T) = \sup_m h_{meas}(T, m)$ where
$m$ is an invariant measure.

Clearly if there is any measure with $h_{meas} > 0$, then $h_{top} > 0$. This is independent
of whether or not the measure has support on the entire space. The utility of this
definition of entropy is given in the following theorem

**Theorem 4.2** If the topological entropy of a map $T$ is positive, then there
exists an ergodic measure such that the measurable entropy is positive.

Thus topological entropy encompasses both ergodicity and sensitive dependence
in one quantity. The translational flow has zero topological entropy by explicit
calculation. The cat map has positive topological entropy; it is a special case of
a family of chaotic systems with positive topological entropy given by maps on
a 2-manifold of genus $> 1$, $T_* : H_1(M^2; \mathbb{R}) \to H_1(M^2; \mathbb{R})$ such that $T_*$ has all
eigenvalues $> 1$. The difficulty with topological entropy has been in its application.
However, numerical estimates of this quantity are possible (Gribble [1995]).

Given the above, a natural generalized definition of chaos is

**Definition 4.3** A dynamical system is chaotic if and only $h_{top} > 0$.

This definition is especially suited to systems without known ergodic measure such
as general relativity.

### 4 Conclusions

Definition (4.3) can be taken as a working definition of chaos for general rela-
tivity; it is not a final definition but a plausible starting point. Its difficulty is that
it must be applied to the theory in terms of physical degrees of freedom. A key task
is to see if it can be recast into parameterized form. Such a hope is not unreason-
able given that topological entropy is closely related to partition functions which
have obvious analogs in gauge theories. Furthermore quantization of gauge theories
in general proceeds along a very similar track of reparameterization of quantities
formally given in terms of physical degrees of freedom. Finally, infinite dimensional
systems such as general relativity only have finite evolutions, that is singularities
and caustics can arise in the future evolution of regular initial data. These effects
do not appear in finite dimensional systems. Although such effects appear in other
infinite dimensional theories, they are an essential part of the nature of general
relativity. Thus it must be verified that (4.3) correctly handles these features.

### References

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Springer-Verlag, New York.