Abstract

We argue that the conjectured dark matter in the Universe may be endowed with a new kind of gravitational charge that couples to a short range gravitational interaction mediated by a massive vector field. A model is constructed that assimilates this concept into ideas of current inflationary cosmology. The model is also consistent with the observed behaviour of galactic rotation curves according to Newtonian dynamics. The essential idea is that stars composed of ordinary (as opposed to dark matter) experience Newtonian forces due to the presence of an all pervading background of massive gravitationally charged cold dark matter. The novel gravitational interactions are predicted to have a significant influence on pre-inflationary cosmology. The precise details depend on the nature of a gravitational Proca interaction and the description of matter. A gravitational Proca field configuration that gives rise to attractive forces between dark matter charges of like polarity exhibits homogeneous isotropic eternal cosmologies that are
free of cosmological curvature singularities thus eliminating the horizon problem associated with the standard big-bang scenario. Such solutions do however admit dense hot pre-inflationary epochs each with a characteristic scale factor that may be correlated with the dark matter density in the current era of expansion. The model is based on a theory in which a modification of Einsteinian gravity at very short distances can be expressed in terms of the gradient of the Einstein metric and the torsion of a non-Riemannian connection on the bundle of linear frames over spacetime. Indeed we demonstrate that the genesis of the model resides in a remarkable simplification that occurs when one analyses the variational equations associated with a broad class of non-Riemannian actions.

1. Introduction

The standard cosmological paradigm offers a fertile domain for the fusion of ideas from astrophysics and particle physics. Certain prejudices inherent in the classical Friedmann cosmologies are thought to find a more natural resolution in terms of dynamical consequences of phase transitions in the early Universe leading to a variety of inflationary scenarios. Among the claims of success are the resolution of \textit{ad hoc} fine-tuning conditions and the puzzle of the observed homogeneity and isotropy of matter. However a further problem is the strong indication that the observed matter may be but a small fraction of the gravitating matter in the Universe. In particular without a significant contribution from directly unobserved matter it is hard to reconcile Einstein’s theory of gravitation (and its Newtonian limit) with the dynamics of the Universe. Although knowledge of conditions in the early Universe by their very nature require huge extrapolations from the observed data, the conjecture that hidden matter also exists seems necessary to explain the Newtonian dynamics of both individual galaxies and galactic clusters. For a star in a circular orbit of radius $r$ with tangential speed $v$ within a spherical distribution of galactic matter with mass $M(r)$, Newtonian dynamics predicts $v^2 = G M(r) / r$ in terms of the Newtonian gravitational coupling $G$. Suppose $M_0 \equiv M(r_0)$ is the constant mass of the \textit{visible} galaxy where $r_0$ denotes the
radius of visible matter. If $r > r_0$ then in the absence of gravitating matter outside the visible galaxy $v^2(r) = \frac{GM_0}{r}$ i.e. $v(r) \sim \frac{1}{\sqrt{r}}$. This however is not observed in general. For $r > r_0$, indications are (e.g from $H_\alpha$ emission) that $v^2(r) = k$ for some constant $k$, or $M(r) \sim r$ requiring a matter density distribution $\rho(r)$ of the form $\rho(r) \sim \frac{1}{r^2}$. The material in $M(r)$ for $r > r_0$ has been called dark matter.

There have been many suggestions for the origin of these observations including a number that contemplate modifications of Newton’s law of gravity and the effects of new interactions [1, 2, 3, 4, 5, 6, 7, 8]. This paper explores the idea that they result from the existence of dark matter that is characterised by a novel gravitational interaction and that the carrier of this component of gravitation is a vector field that influences the pre-inflationary phase of the Universe (c.f. [9]). As such it may modify the inflationary period needed to address the so called “flatness problem” and may put the so called “horizon problem” into a new perspective. Depending on the sign adopted for the coupling of the Proca field in the fundamental action defining the theory this interaction may also be responsible for removing the classical curvature singularity associated with the traditional Friedmann cosmologies [10]. The coupling of the dark matter to the Proca component of gravitation is defined by a new kind of conserved charge, the value of which can be correlated with the “mass” of the Proca quantum and the behaviour of the scale factor of the Universe.

An important feature of this description is that it is rooted in recent developments in non-Riemannian descriptions of gravitation. Non-Riemannian geometries feature in a number of theoretical descriptions of the interactions between fields and gravitation. Since the early pioneering work by Weyl, Cartan, Schroedinger, Trautman and others such geometries have often provided a succinct and elegant guide towards the search for unification of the forces of nature [11]. In recent times interactions with supergravity have been encoded into torsion fields induced by spinors and dilatonic interactions from low energy effective string theories have been encoded into connections that are not metric-compatible [12, 13, 14, 15]. However theories in which the non-Riemannian geometrical fields are dynamical in the absence of matter are more elusive to interpret. We have suggested elsewhere [16] that they may play an important role in certain astrophysical contexts. Part of the difficulty in interpreting such fields is that there is little experimental
guidance available for the construction of a viable theory that can compete effectively with general relativity in domains that are currently accessible to observation. In such circumstances one must be guided by the classical solutions admitted by theoretical models that admit dynamical non-Riemannian structures [17, 18, 19, 20, 21, 22, 23, 24]. This approach is being currently pursued by a number of groups [16, 22, 23, 25, 26, 27, 28, 29, 30]. The work below continues in this vein by recognising that the essence of the non-Riemannian approach is the existence of new gravitational interactions. The nature of a particular interaction is explored in the context of modern inflationary cosmologies and the dark matter problem. Although the dark matter model below is formulated in terms of the standard Levi-Civita Riemannian connection the introduction of the Proca component of gravitation is best appreciated in terms of its genesis within non-Riemannian geometry.

2. Non-Riemannian Geometry

Einstein’s theory of gravity has an elegant formulation in terms of (pseudo-) Riemannian geometry. The field equations follow as a local extremum of an action integral under metric variations. In the absence of matter the integrand of this action is simply the curvature scalar associated with the curvature of the Levi-Civita connection times the (pseudo-) Riemannian volume form of spacetime. Such a connection $\nabla$ is torsion-free and metric compatible. Thus for all vector fields $X, Y$ on the spacetime manifold, the tensors given by:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$  \hspace{1cm} (2.1)
$$S = \nabla g$$  \hspace{1cm} (2.2)

are zero, where $g$ denotes the metric tensor, $T$ the 2-1 torsion tensor and $S$ the gradient tensor of $g$ with respect to $\nabla$. Such a Levi-Civita connection provides a useful reference connection since it depends entirely on the metric structure of the manifold. This special connection will be denoted $\hat{\nabla}$ in the following.

A general linear connection $\nabla$ on a manifold provides a covariant way to differentiate tensor fields. It provides a type preserving derivation on the
algebra of tensor fields that commutes with contractions. Given an arbitrary local basis of vector fields \( \{X_a\} \) the most general linear connection is specified locally by a set of \( n^2 \) 1-forms \( \Lambda^a_{\ b} \) where \( n \) is the dimension of the manifold:

\[
\nabla_{X_a} X_b = \Lambda^c_{\ b}(X_a) X_c.
\]

(2.3)

Such a connection can be fixed by specifying a \( (2, 0) \) symmetric metric tensor \( g \), a \( (2\text{-antisymmetric}, 1) \) tensor \( T \) and a \( (3, 0) \) tensor \( S \), symmetric in its last two arguments. If we require that \( T \) be the torsion of \( \nabla \) and \( S \) be the gradient of \( g \) then it is straightforward to determine the connection in terms of these tensors. Indeed since \( \nabla \) is defined to commute with contractions and reduce to differentiation on scalars it follows from the relation

\[
X(g(Y, Z)) = S(X, Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)
\]

(2.4)

that

\[
2g(Z, \nabla_X Y) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))
\]

\[
- g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [Y, X])
\]

\[
- g(X, T(Y, Z)) - g(Y, T(X, Z)) - g(Z, T(Y, X))
\]

\[
S(X, Y, Z) - S(Y, Z, X) + S(Z, X, Y)
\]

(2.5)

where \( X, Y, Z \) are any vector fields. The general curvature operator \( R_{X,Y} \) defined in terms of \( \nabla \) by

\[
R_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
\]

(2.6)

is also a type-preserving tensor derivation on the algebra of tensor fields. The general \( (3, 1) \) curvature tensor \( R \) of \( \nabla \) is defined by

\[
R(X, Y, Z, \beta) = \beta(R_{X,Y} Z)
\]

(2.7)

where \( \beta \) is an arbitrary 1-form. This tensor gives rise to a set of local curvature 2-forms \( R^a_{\ b} \):

\[
R^a_{\ b}(X, Y) = \frac{1}{2} R(X, Y, X_b, e^a)
\]

(2.8)

where \( \{e^c\} \) is any local basis of 1-forms dual to \( \{X_c\} \). In terms of the contraction operator \( i_X \) with respect to \( X \) one has \( i_{X_b} e^a \equiv i_b e^a = e^a(X_b) = \delta^a_{\ b} \). In terms of the connection forms

\[
R^a_{\ b} = d\Lambda^a_{\ b} + \Lambda^a_{\ c} \wedge \Lambda^c_{\ b}.
\]

(2.9)
In a similar manner the torsion tensor gives rise to a set of local torsion 2-forms $T^a$:

$$T^a(X, Y) \equiv \frac{1}{2} e^a(T(X, Y)) \quad (2.10)$$

which can be expressed in terms of the connection forms as

$$T^a = de^a + \Lambda^a_b \wedge e^b. \quad (2.11)$$

Since the metric is symmetric the tensor $S$ can be used to define a set of local non-metricity 1-forms $Q_{ab}$ symmetric in their indices:

$$Q_{ab}(Z) = S(Z, X_a, X_b). \quad (2.12)$$

3. Non-Riemannian Actions

To exploit the geometrical notions above in a gravitational context one must construct field equations that determine the torsion and metric gradients. These are most naturally derived from an action principle in which the metric, components of the connection and matter fields are the configuration variables. One requires that an action be stationary with respect to suitable variations of such variables. If the action 4-form $\Lambda(g, \nabla, \cdots)$ in spacetime contains the Einstein-Hilbert form

$$\Lambda_{EH}(g, \nabla) = R \star 1 \quad (3.1)$$

where $\star$ denotes the Hodge map on forms and $\frac{\delta \Lambda_{EH}}{g}$ denotes the variational derivative of $\Lambda_{EH}(g, \nabla)$ with respect to $g$, then

$$\frac{\delta \Lambda_{EH}}{g} = -h^{ab} \text{Ein}(X_a, X_b) \star 1 \quad (3.2)$$

where $h_{ab} \equiv \dot{g}(X_a, X_b)$ and

$$\text{Ein} \equiv \dot{\text{Ric}} - \frac{1}{2} gR \quad (3.3)$$
is given in terms of the Ricci tensor $\text{Ric}(X, Y) = R(X_a, X, e^a)$ by

$$\widetilde{\text{Ric}}(X, Y) = \frac{1}{2} (\text{Ric}(X, Y) + \text{Ric}(Y, X)) \quad (3.4)$$

and $R = \text{Ric}(X_a, X^a)$. Further details of these variational calculations may be found in [16]. Unlike the Einstein tensor associated with the Levi-Civita connection, the Einstein tensor $\text{Ein}$ defined in (3.3) is associated with the non-Riemannian connection $\nabla$ and is not in general divergenceless:

$$(\nabla \cdot \text{Ein})(X_b) \equiv (\nabla_{X_a} \text{Ein})(X^a, X_b) \neq 0. \quad (3.5)$$

It is therefore instructive to decompose the non-Riemannian Einstein tensor $\text{Ein}$ into parts that depend on the Levi-Civita connection $\text{\nabla}$. For this purpose introduce the tensor $\lambda$ by

$$\lambda(X, Y, \beta) = \beta(\nabla_X Y) - \beta(\text{\nabla}_X Y) \quad (3.6)$$

for arbitrary vector fields $X, Y$ and 1-form $\beta$. In terms of the exterior covariant derivative [31] $\text{\tilde{D}}$ and the Ricci tensor $\text{Ric}$ associated with the Levi-Civita connection, one may write:

$$\text{Ric}(X_a, X_b) = \text{\tilde{Ric}}(X_a, X_b) + i_a i_c (\text{\tilde{D}}\lambda^c_b + \lambda^c_d \wedge \lambda^d_b) \quad (3.7)$$

where $\lambda^a_b \equiv \lambda(-, X_b, e^a)$ is a set of local 1-forms. In terms of these forms

$$T^a = \lambda^a_c \wedge e^c \quad (3.8)$$

$$Q_{ab} = -\lambda_{ab} - \lambda_{ba}. \quad (3.9)$$

It follows from (3.6) that $\text{Ein}$ differs from the Levi-Civita Einstein tensor $\text{Ein}$ by terms involving the tensor $\lambda$ and its derivatives. However for a large class of actions containing the torsion and metric gradient fields one finds that these terms can be dramatically simplified.

To see this simplification most easily it is preferably to change variables from $g, \nabla$ to $g, \lambda$ in the total action 4-form and write $\Lambda(g, \nabla, \cdots) =$
\[ \mathcal{L}(g, \lambda, \cdots). \] Since \( \dot{\nabla} \) depends only on the metric it follows from (3.6) that \( \dot{\lambda} = \dot{\nabla} \) and \( \dot{\lambda} = -\dot{\nabla} \). Hence the variational field equations are

\[
\dot{\lambda} = \dot{\nabla} = 0 \tag{3.10}
\]

\[
\dot{\lambda} = \dot{\nabla} + \dot{\lambda} \dot{\nabla} = 0. \tag{3.11}
\]

From (3.10) one sees that the term \( \dot{\nabla} \) in (3.11) does not contribute. To evaluate these variations one first expresses the action in terms of \( g \) and the components of \( \lambda \) and its derivatives. To this end it is convenient to introduce the (traceless) 1-forms \( \hat{\lambda}_a^b \equiv \lambda_a^b - \frac{1}{4} \delta^a_b \lambda^d_d \), and the (traceless) 0-forms \( \hat{\lambda}_c^b \equiv \lambda_c^b - \frac{1}{4} \delta^b_c \lambda^d_d \). If we express the total action 4-form \( \mathcal{L}(g, \lambda, \cdots) \) as

\[
\mathcal{L}(g, \lambda, \cdots) = \mathcal{L}_{EH}(g, \lambda) + \mathcal{L}_1(g, \lambda, \cdots) \tag{3.12}
\]

for some form \( \mathcal{L}_1(g, \lambda, \cdots) \) where

\[
\mathcal{L}_{EH} = \hat{\nabla}^c \hat{\lambda}_c^a \wedge \hat{\lambda}_b^c \wedge \ast (e^b \wedge e_a) - d(\hat{\lambda}_b^c \wedge \ast (e^b \wedge e_c)) \tag{3.13}
\]

in terms of the Levi-Civita scalar curvature \( \hat{\nabla} \) then

\[
\dot{\mathcal{L}}_{\mathcal{L}_{EH}} = -h^{ab} \mathbf{Ein}(X_a, X_b) \ast 1 - h^{ab} \mathcal{E}_{ab} \pmod{d} \tag{3.14}
\]

where

\[
\mathcal{E}_{ab} = \frac{1}{2} \hat{\lambda}_d^a \wedge \hat{\lambda}_p^d \{ g_{ab} \ast (e^p \wedge e_q) - \delta^p_a \ast (e_b \wedge e_q) - \delta^p_b \ast (e_a \wedge e_q) \}. \tag{3.15}
\]

Thus the Einstein field equation is

\[
\mathbf{Ein}(X_a, X_b) \ast 1 + \mathcal{E}_{ab} + \mathcal{T}_{ab} = 0
\]

where \( \dot{\mathcal{L}}_1 = -h^{ab} \mathcal{T}_{ab} \). Next the variations with respect to \( \lambda \) yield
\[ \mathcal{L}_{\text{EH}} = \hat{\lambda}^a_b \wedge \{ \hat{\lambda}^c_{a \wedge} \wedge (e^b \wedge e_c) - \hat{\lambda}^b_{c \wedge} \wedge (e^c \wedge e_a) \} \quad (\text{mod } d) \]

\[ \mathcal{L}_1 = \hat{\lambda}^a_b \wedge \mathcal{F}_a^b. \]

By splitting off the trace part of the resulting field equation one may write:

\[ \mathcal{F}^a_a = 0 \quad (3.16) \]

\[ \hat{\lambda}^c_{a \wedge} \wedge (e^b \wedge e_c) - \hat{\lambda}^b_{c \wedge} \wedge (e^c \wedge e_a) + \mathcal{F}_a^b = 0 \quad (3.17) \]

where \( \mathcal{F}_a^b \equiv \mathcal{F}_a^b - \frac{1}{4} \delta_a^b \mathcal{F}_c^c \). To illustrate how the field equations (3.16) and (3.17) can greatly simplify the terms in (3.15) consider the eight parameter class of models in which the torsion and metric gradient fields enter the action according to:

\[ \mathcal{L}_1 = 4 \kappa R^a_a \wedge \star R^b_b - 2 \ell \star 1 + \alpha_1 Q \wedge \star Q + \alpha_2 u \wedge \star u + \alpha_3 v \wedge \star v + \alpha_4 Q \wedge \star u + \alpha_5 Q \wedge \star v + \alpha_6 u \wedge \star v \quad (3.18) \]

where \( \kappa, \alpha_k \) are arbitrary coupling constants and \( \ell \) is a cosmological constant. The 1-forms \( u \) and \( v \) may be expressed in terms of the torsion forms \( T^a \) and non-metricity forms \( Q_{ab} \) as follows:

\[ u \equiv \lambda^c_{ac} e^a = T - \frac{1}{2} Q \]

\[ v \equiv \lambda^c_{ac} e^a = \frac{1}{2} Q - \frac{1}{2} Q - T \]

where \( T \equiv i_a T^a, \, Q \equiv Q^a_a = -2 \lambda^a_a \) and \( Q \equiv e^a \wedge Q_{ab} \). Furthermore \( R^a_a = -\frac{1}{2} dQ \) is proportional to the Weyl field 2-form \( dQ \). Computing the variational derivatives above one finds that (3.16) yields:

\[ d \star dQ + \frac{16 \alpha_1}{16 \kappa} - \frac{2 \alpha_2 - \alpha_6}{16 \kappa} \star u + \frac{8 \alpha_5 - 2 \alpha_3 - \alpha_6}{16 \kappa} \star Q = 0 \quad (3.19) \]

while (3.17) implies:

\[ u = \beta_1 Q \quad (3.20) \]
\[ v = \beta_2 Q \] (3.21)

and

\[
\hat{\lambda}^a_{bc} = -\frac{4\beta_1 + 4\beta_2 + 1}{24} \delta^a_b \iota_c Q + \frac{10\beta_2 - 2\beta_1 + 1}{12} (g_{bc} \iota^a Q + \delta^a_c \iota_b Q)
\] (3.22)

where

\[
\beta_1 = \frac{6 \alpha_3 + 3 \alpha_4 - 21 \alpha_5 - 3 \alpha_6 + 54 \alpha_3 \alpha_4 - 27 \alpha_5 \alpha_6 - 2}{16 - 6 \alpha_2 - 6 \alpha_3 + 42 \alpha_6 + 27 \alpha_6^2 - 108 \alpha_2 \alpha_3}
\]

\[
\beta_2 = \frac{6 \alpha_2 - 21 \alpha_4 + 3 \alpha_5 - 3 \alpha_6 + 54 \alpha_2 \alpha_5 - 27 \alpha_4 \alpha_6 - 2}{16 - 6 \alpha_2 - 6 \alpha_3 + 42 \alpha_6 + 27 \alpha_6^2 - 108 \alpha_2 \alpha_3}.
\]

Substituting (3.20) and (3.21) into (3.19) one now obtains a Proca-type equation in the form

\[ d \star d Q + \beta_3 \star Q = 0 \] (3.23)

where

\[
\beta_3 = \frac{16 \alpha_1 - \alpha_4 - \alpha_5}{16 \kappa} + \frac{16 \alpha_4 - 2 \alpha_2 - \alpha_6}{16 \kappa} \beta_1 + \frac{16 \alpha_5 - 2 \alpha_3 - \alpha_6}{16 \kappa} \beta_2.
\]

From the metric variation of \( L_1 \) one finds

\[ T_{ab} = \ell g_{ab} \star 1 + \kappa \tau(X_a, X_b) \star 1 + \hat{T}_{ab} \]

where

\[ \tau = \star^{-1} \{ \beta_3 (i_a Q \wedge \iota_b Q - \frac{1}{2} g_{ab} Q \wedge \star Q) + i_a dQ \wedge \iota_b dQ - \frac{1}{2} g_{ab} dQ \wedge \star dQ \} e^a \otimes e^b \]

and

\[ \hat{T}_{ab} = \zeta_1 g_{ab} Q \wedge \star Q + \zeta_2 i_a Q \wedge \iota_b Q \]

in terms of

\[
\zeta_1 = \frac{13 \beta_1}{324} + \frac{7 \beta_2}{324} + \frac{29 \beta_1 \beta_2}{162} + \frac{23 \beta_1^2}{324} - \frac{\beta_2^2}{324} + \frac{5}{1296}
\] (3.24)
\[
\zeta_2 = -\frac{4}{81} \beta_1 + \frac{2}{81} \beta_2 - \frac{37}{162} \beta_1^2 + \frac{5}{81} \beta_1 \beta_2 + \frac{11}{162} \beta_2^2 - \frac{1}{648}.
\] (3.25)

Remarkably \( \tilde{T}_{ab} \) exactly cancels \( E_{ab} \) leaving the Einstein equation in the form
\[
E_{\tilde{ab}} + \ell g + \kappa \tau = 0.
\] (3.26)

Thus the solutions to the field equations derived from the action \( \mathcal{L}_{EH}(g, \lambda) + \mathcal{F} \) where \( \mathcal{F} \) is given by (3.18) may be generated from solutions to the Levi-Civita Einstein-Proca system (3.23), (3.26) with arbitrary cosmological constant \( \ell \). Once \( Q \) is determined the full connection follows from
\[
\lambda^a_b = \tilde{\lambda}^a_b - \frac{1}{8} \delta^a_b Q
\] (3.27)

where \( \tilde{\lambda}^a_b \) is given by (3.22). The torsion and metric gradient then follow from (3.8) and (3.9).

The above discussion has concentrated on the gravitational sector of the theory. The inclusion of matter into \( \mathcal{F} \) is straightforward in principle. For example if the matter is minimally coupled to the Weyl form \( Q \), the action \( \mathcal{F} \) will contain a term \( Q \wedge j \) where the 3-form \( j \) is the Proca charged matter current. Then (3.23) will contain an extra source term proportional to \( j \) while (3.26) will contain an addition stress tensor from the variation of the matter action with respect to the metric tensor.

4. Divergence Conditions

The previous section dealt with a general framework for deriving the variational equations for matter interacting with non-Riemannian gravitation. In the following we construct models in which the new component of gravitation is described in terms of a Proca vector field with the matter fields considered as thermodynamic fluids. Action principles for relativistic fluid models interacting with gravitation require the imposition of sufficient constraints to
maintain consistency. Fortunately it is possible to write down the coupled field equations for an Einstein-Proca-fluid system without much effort by exploiting the Bianchi identities to ensure that the fluids satisfy the required relativistic continuum field equations. This procedure relies on recognising how the metric variations of the components of the action are correlated with the matter variations.

Suppose the underlying theory of gravitational and matter fields (with compact support) is defined by an action 4-form $\Lambda$ on spacetime $M$ as a sum of 4-forms $\sum_r \Lambda_r$. Then in general there exists a set of symmetric second degree tensors $\{T_s\}$, each member of which is constructed from an element $\Lambda_s$ belonging to a subset $\{\Lambda_s\}$ contained in $\{\Lambda_r\}$, satisfying $\hat{\nabla} \cdot T_s = 0$ in terms of the Levi-Civita connection $\hat{\nabla}$. The subset $\{\Lambda_s\}$ is defined so that each member $\Lambda_s$ depends on a set of tensor fields $\{\Phi^s_k\}$ (and possibly their derivatives) where the intersection of the sets $\{\Phi^s_k\}$ contains only the metric tensor on $M$.

The essence of this result is well known (although perhaps not widely appreciated) and relies on the behaviour of the action functional $S$ in response to variations of fields induced by local diffeomorphisms of the manifold $M$. For any set of 4-forms $\{\Lambda_r\}$ the action

$$S = \int_M \Lambda = \sum_r \int_M \Lambda_r$$

where $\Lambda$ is constructed from the metric tensor $g$ and tensor fields of arbitrary type $\{\Phi_1, \Phi_2, \cdots\}$ on $M$. For any local diffeomorphism on $M$ generated by a vector field $V$

$$\int_M \mathcal{L}_V \Lambda_r = \int_M d (i_V \Lambda_r) = \int_{\partial M} i_V \Lambda_r = 0 \quad (4.1)$$

since the Lie derivative $\mathcal{L}_V = i_V d + d i_V$ in terms of the contraction operator $i_V$, $d\Lambda_r = 0$ and all fields in $\Lambda_r$ are assumed to have compact support. To explicitly construct the set $\{T_s\}$ write the variations of $\Lambda_r$ with respect to $\Phi_k \in \{\Phi^s_k\}$ as

$$\dot{\Lambda}_r |_{\Phi_k} = \dot{\Phi}_k \cdot F_k \quad \text{(mod } d)$$

for some tensor $F_k$ contracted with $\dot{\Phi}_k$. In an arbitrary local frame $\{X_a\}$ with dual coframe $\{e^b\}$ such that $e^b(X_c) = \delta^b_c$, the symmetric “stress” tensor,
$\mathcal{T}_r = \mathcal{T}_{r ab} e^a \otimes e^b$, associated with $\Lambda_r$ is obtained from the metric variation:

$$\frac{\Delta \Lambda_r}{\mathcal{G}} = \mathfrak{g}(X_a, X_b) \mathcal{T}_r^{ab} \star 1 \quad (\text{mod } d).$$

Since the Lie derivative is a derivation on tensors, $\mathcal{L}_V \Lambda_r$ may be expressed in terms of the field and metric variations:

$$\mathcal{L}_V \Lambda_r = (\mathcal{L}_V \mathfrak{g})(X_a, X_b) \mathcal{T}_r^{ab} \star 1$$

$$+ (\mathcal{L}_V \Phi_1) \cdot \mathcal{F}_1 + (\mathcal{L}_V \Phi_2) \cdot \mathcal{F}_2 + \cdots \quad (\text{mod } d)$$

If the set $\{\Phi_1, \Phi_2, \ldots\}$ is unique to the 4-form $\Lambda_r$ then the equations $\{\mathcal{F}_k = 0\}$ constitute a set of variational field equations. If these are satisfied then

$$\mathcal{L}_V \Lambda_r = (\mathcal{L}_V \mathfrak{g})(X_a, X_b) \mathcal{T}_r^{ab} \star 1 \quad (\text{mod } d).$$

Since

$$(\mathcal{L}_V \mathfrak{g})(X_a, X_b) = \mathfrak{i}_a \nabla X_b \mathfrak{V} + \mathfrak{i}_b \nabla X_a \mathfrak{V}$$

where $\mathfrak{V} = \mathfrak{g}(V, -)$ and $\mathfrak{g}$ is symmetric one may write

$$\frac{1}{2} \int_M \mathcal{L}_V \Lambda_r = \int_M \mathcal{T}_r^{ab} \mathfrak{i}_a \nabla X_b \mathfrak{V} \star 1.$$

For any $\mathcal{T}_r$ it is convenient to introduce the associated 3-form $J_{rV}$ by

$$J_{rV} = \star \mathcal{T}_r(V, -).$$

Now

$$d(J_{rV}) = d(\mathcal{T}_{r ab} V^a \star e^b)$$

$$= \mathfrak{i}_c \mathfrak{D} \mathcal{T}_{r ab} \wedge V^a \star e^b + \mathcal{T}_{r ab} \mathfrak{D} V^a \wedge \star e^b$$

$$= e^c \mathfrak{i}_c \mathfrak{D} \mathcal{T}_{r ab} \wedge V^a \star e^b + \mathcal{T}_{r ab} e^c \mathfrak{i}_c \mathfrak{D} V_a \wedge \star e_b$$

$$= (\nabla_{X_c} \mathcal{T}_r)(X_a, X_b) V^a g^{bc} \star 1 + \mathcal{T}_{r ab} \mathfrak{i}_a \nabla X_c \mathfrak{V} \delta^c \delta^b \star 1$$

$$= (\nabla \cdot \mathcal{T}_r(V) \star 1 + \mathcal{T}_{r ab} \mathfrak{i}_a \nabla X_b \mathfrak{V} \star 1$$
where $\mathring{\mathcal{D}}$ above denotes the exterior covariant derivative associated with $\mathring{\nabla}$. Thus for fields with compact support

$$\frac{1}{2} \int_M \mathcal{L}_V \Lambda = \int_M \mathcal{T}_r^{ab} i_a \mathcal{\mathring{\nabla}}_{X_b} V \ast 1 = \int_M dJ_V - \int_M (\mathring{\nabla} \cdot \mathcal{T})(V) \ast 1 = \int_{\partial M} J_V - \int_M (\mathring{\nabla} \cdot \mathcal{T}_r)(V) \ast 1.$$

Since $J_V = 0$ on $\partial M$ and $V$ is arbitrary it follows that

$$\mathring{\nabla} \cdot \mathcal{T}_r = 0.$$

This result follows from the definition of $\mathcal{T}_r$ and the imposition of the field equations and should not be confused with a “conservation law” in an arbitrary spacetime. In general, “conservation laws” owe their existence to further conditions. For example if there exists a vector field $V = K$ where $K$ satisfies the Killing condition ($\mathcal{L}_K g = 0$) then both terms on the right hand side of (4.2) are zero and $J_{rK}$ defines a genuine conserved current:

$$dJ_{rK} = 0.$$

Thus we have shown that every component 4-form $\Lambda_r$ that gives rise, by variation, to a set of field equations $\{\mathcal{F}_k = 0\}$ for any of the dynamical tensor fields in the action, excluding the metric $g$, also gives rise to a divergenceless second rank tensor $\mathcal{T}_r$.

### 5. The Einstein-Proca-Matter System

Based on the reduction of the non-Riemannian action to a theory of gravity in terms of the standard Levi-Civita torsion free, metric compatible connection $\mathring{\nabla}$, we now construct a model of gravity and matter that includes the Proca field in the gravitational sector. As befits its origin in terms of purely geometrical concepts the Proca field is regarded as a gravitational vector field that is expected to modify the gravitational effects produced by the tensor
nature of Einsteinian gravity. The model is based on (3.26) in section 3 with \( \ell = 0 \) and the right hand side replaced by the stress tensors for fluid matter. We may normalise the Proca field \( Q \) so that the term \( \kappa \tau \) becomes \( \sigma \Sigma \) where \( \Sigma \) is the Proca stress tensor, \( \sigma = \pm 1 \) and the constant \( \beta_3 \) is replaced by the square of the (real) Proca field mass \( m_\alpha \). In the following we explore the consequences of both signs for \( \sigma \). With \( \sigma = -1 \) and \( m_\alpha = 0 \) the model is analogous to the classical Einstein-Maxwell-fluid system. With \( \sigma = +1 \) an interpretation of \( \sigma \Sigma \) as a stress tensor for matter would be “unphysical”. In the absence of other matter and in a weak field limit it might be expected to lead to stability problems analogous to fermionic stresses in the absence of quantisation. However whatever criteria are needed to maintain the stability of the Minkowski background in the presence of gravitational (metric) waves might be similarly invoked in the presence of gravitational Proca fields. We shall not immediately discard the \( \sigma = +1 \) possibility since it has interesting consequences for cosmology. Henceforth we restore the physical constants \( G, c, \hbar \) and adopt MKS units when comparing with observation.

In general we consider matter to be composed of ordinary matter defined to have zero coupling to the Proca field and “dark matter” defined to have a non-zero coupling. We call this coupling Proca charge and denote the basic unit of Proca charge by \( q \). For our discussion of cosmology we model both types of matter by fluids with standard stress tensors \( T_0 \) and \( T_q \) respectively. Denoting the standard Levi-Civita Einstein tensor by \( \hat{E}^{\text{in}} \) and the contribution of the Proca field \( \alpha \) to the Einstein equation by \( \sigma \Sigma \), \( \sigma = \pm 1 \) we have

\[
\hat{E}^{\text{in}} + \sigma \Sigma = \frac{8 \pi G}{c^4} (T_0 + T_q) \tag{5.1}
\]

where

\[
T_0 = (c^2 \rho_0 + P_0) V_0 \otimes V_0 + P_0 g
\]

\[
T_q = (c^2 \rho_q + P_q) V_q \otimes V_q + P_q g
\]

\[
g(V_0, V_0) = -1
\]

\[
g(V_q, V_q) = -1.
\]
The mass densities $\rho_f$, $f = 0, q$ of ordinary and dark matter respectively are functions of the particle densities $n_f$ and the entropies $s_f$ per particle:

$$\rho_0 = \rho_0(n_0, s_0)$$

$$\rho_q = \rho_q(n_q, s_q).$$

The pressure $P_f$ and temperature $T_f$ may be derived from Gibb’s relation

$$c^2 \, d\rho_f = c^2 \mu_f \, d\, n_f + n_f \, T_f \, d\, s_f$$

(5.2)

where $\mu_f = \frac{\rho_f + P_f/c^2}{n_f}$ is the associated chemical potential. In terms of the notation introduced in Section 3 we have rescaled the Weyl 1-form $Q$ to the 1-form $\alpha$ and $\tau$ to

$$\Sigma = \left( \frac{c \, m_\alpha}{\hbar} \right)^2 \left( \alpha \otimes \alpha - \frac{1}{2} \alpha(\tilde{a}) \, g \right) + \left( i_c F \otimes i^c F - \frac{1}{2} \star^{-1} (F \wedge \star F) \, g \right)$$

where $F = d\alpha$ is the Proca field strength. Equation (5.1) for the metric must be supplemented by field equations for the Proca field $\alpha$ and the fluid variables together with their equations of state. In view of the comments in Section 4 we adopt as matter field equations

$$\nabla \cdot \mathcal{T}_0 = 0$$

(5.3)

and

$$\nabla \cdot \left( \frac{8 \pi G}{c^4} \mathcal{T}_q - \sigma \Sigma \right) = 0$$

(5.4)

which are certainly compatible with the Bianchi identity $\nabla \cdot \mathbf{Ein} = 0$ and give rise, as we shall demonstrate below, to the expected Lorentz forces on charged matter due to vector fields. Since the Proca field couples to a current $j_q$ of Proca charged matter we have

$$d \star F + \left( \frac{c \, m_\alpha}{\hbar} \right)^2 \star \alpha + \sigma j_q = 0.$$

(5.5)

The Proca charge current will be assumed to take the convective form

$$j_q = q \, n_q \star \tilde{V}_q$$

(5.6)
with constant Proca charge, \( q \) (of dimension length).

In a similar manner we assume that the Proca neutral particle current is
given by

\[
j_0 = n_0 \star \widetilde{V}_0.
\]

We postulate conservation of Proca charged particles

\[
d(n_q \star \widetilde{V}) = 0 \quad \Leftrightarrow \quad d\, \mathbf{j}_q = 0 \quad \text{(since} \ q \ \text{is constant).} (5.7)
\]

If the neutral Proca matter is also conserved (as befits behaviour in the late
post inflationary epoch)

\[
d\mathbf{j}_0 = 0 \quad (5.8)
\]

and one may then interpret the matter field equations as equations of motion
for the fluid flows.

To explicitly develop these equations we must compute the divergence of
\( \Sigma \). To facilitate this we introduce the tensors

\[
\Sigma_\alpha \equiv \alpha \otimes \alpha - \frac{1}{2} \alpha(\bar{\alpha}) \mathbf{g}
\]

(5.9)

and

\[
\Sigma_F \equiv i_\text{c} F \otimes i_\text{c} F - \frac{1}{2} \star^{-1} (F \wedge \star F) \mathbf{g} \quad (5.10)
\]

so that

\[
\Sigma = \left( \frac{c m_o}{\hbar} \right)^2 \Sigma_\alpha + \Sigma_F.
\]

It is a standard result that

\[
\nabla \cdot \Sigma_F = i_U F \quad (5.11)
\]

where the vector field \( U \) is defined by \( \star \widetilde{U} = d \star F \). To calculate the divergence
of \( \Sigma_\alpha \) consider

\[
\nabla \cdot (\alpha \otimes \alpha) = \nabla_{X_\alpha} (\alpha \otimes \alpha)(X^\alpha)
\]

\[
= (\nabla \cdot \alpha) \alpha + \alpha^a \nabla_{X_a} \alpha = (\star^{-1} d \star \alpha) \alpha + \alpha^a \nabla_{X_a} \alpha \quad (5.12)
\]
where $\alpha^a = \alpha(X^a)$ and the relation $\hat{\nabla} \cdot \beta = i_{X^a} \hat{\nabla}_{X^a} \beta = (-1)^{p+1} \star^{-1} d \star \beta$ for any $p$-form $\beta$ has been used. Furthermore, for any 0-form $f$

$$\hat{\nabla} \cdot (f \ g) = df$$

(5.13)

since $\hat{\nabla}$ is metric compatible. Hence

$$\hat{\nabla} \cdot (\alpha(\tilde{\alpha}) \ g) = d(\alpha(\tilde{\alpha})) = e^a \wedge \hat{\nabla}_{X^a}(\alpha(\tilde{\alpha}))$$

$$= 2(\hat{\nabla}_{X^a} \alpha)(\tilde{\alpha}) e^a = 2 \alpha^a \hat{\nabla}_{X^a} \alpha - 2 i_{\tilde{\alpha}} d \alpha$$

(5.14)

where the vector field $\tilde{\alpha}$ is the metric dual of the form $\alpha$. It follows from (5.9), (5.12) and (5.14) that

$$\hat{\nabla} \cdot \Sigma_{\alpha} = (\star^{-1} d \star \alpha) \alpha + i_{\tilde{\alpha}} F.$$  

Together with (5.11) the divergence of $\Sigma$ becomes

$$\hat{\nabla} \cdot \Sigma = i_Z F + \left(\frac{cm\alpha}{\hbar}\right)^2 (\star^{-1} d \star \alpha) \alpha$$

(5.15)

where the vector field $Z$ is defined by

$$\star \tilde{Z} = d \star F + \left(\frac{cm\alpha}{\hbar}\right)^2 \star \alpha.$$  

(5.16)

Comparing (5.16) and (5.5) one has

$$Z = \sigma q n_q V_q$$

(5.17)

while the exterior derivative of both sides of (5.5) yields

$$\left(\frac{cm\alpha}{\hbar}\right)^2 d \star \alpha + \sigma d j_q = 0.$$  

Therefore the divergence of $\Sigma$ becomes

$$\hat{\nabla} \cdot \Sigma = \sigma q n_q i_{V_q} F$$
using (5.7), (5.15) and (5.17).

For completeness we derive the divergence of the generic fluid stress:

$$
\mathcal{T} = (c^2 \rho + P) V \otimes V + P \mathbf{g}
$$

where $\mathbf{g}(V, V) = -1$ and $\rho = \rho(n, s)$. It follows from (5.13) that

$$
\hat{\nabla} \cdot \mathcal{T} = (c^2 \rho + P) \hat{\nabla}_V \tilde{V} + dP + \tilde{V} \star^{-1} d((c^2 \rho + P) \star \tilde{V}).
$$

(5.18)

The third term on the right hand side of (5.18) may be expanded as

$$
\star^{-1} d((c^2 \rho + P) \star \tilde{V}) = \star^{-1}\{d(c^2 \rho + P) \wedge \star \tilde{V} + (c^2 \rho + P) d \star \tilde{V}\}
\equiv i_V \{d(c^2 \rho + P) + (c^2 \rho + P) \star^{-1} d \star \tilde{V}\}
$$

(5.19)

by using the relation

$$
\beta \wedge \star \omega = (i_{\beta} \omega) \star 1
$$

for any 1-forms $\beta$ and $\omega$. Furthermore, the general conservation equation

$$
d(n \star \tilde{V}) = dn \wedge \star \tilde{V} + nd \star \tilde{V} = 0
$$

implies that

$$
d \star \tilde{V} = -\frac{1}{n} dn \wedge \star \tilde{V} = -\frac{1}{n} (i_V dn) \star 1
$$

and so

$$
\star^{-1} d \star \tilde{V} = \frac{1}{n} (i_V dn).
$$

Thus (5.19) becomes

$$
\star^{-1} d((c^2 \rho + P) \star \tilde{V}) = i_V \left(d(c^2 \rho + P) - \frac{(c^2 \rho + P)}{n} dn\right).
$$

Applying the thermodynamic relation (5.2) to $\rho(n, s)$ gives the divergence of $\mathcal{T}$ finally as:

$$
\hat{\nabla} \cdot \mathcal{T} = (c^2 \rho + P) \hat{\nabla}_V \tilde{V} + \Pi_V dP + i_V (n T ds) \tilde{V}
$$

where $\Pi_V dP \equiv dP + (i_V dP) \tilde{V}$ is the transverse part of $dP$ with respect to $V$. Thus the vanishing of the tangential components of the field equations (5.3) and (5.4) gives rise to the isentropic conditions:

$$
V_0(s_0) = 0
$$

(5.20)
while the vanishing of the transverse components yields the appropriate relativistic Navier-Stoke’s type equations for neutral and charged fluids respectively:

\[(c^2 \rho_0 + P_0) \nabla_{V_0} \tilde{V}_0 = -\Pi_{V_0} dP \]

\[(c^2 \rho_q + P_q) \nabla_{V_q} \tilde{V}_q = -\Pi_{V_q} dP + \frac{q n_q c^4}{8 \pi G} \text{i}_{V_q} F. \]

In the case of negligible pressure \((P_f/c^2 \ll \rho_f, f = 0, q)\) the relation (5.2) yields \(\rho_f \approx \mu_f n_f\) where the constant \(\mu_f\) is identified as the Newtonian mass of the neutral and Proca charged particle respectively. In the limit where Einstein gravity is negligible we may introduce coordinates \(\{t, x^1, x^2, x^3\}\) and write the metric \(g \approx -c^2 dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3\). The Proca equation (5.5) reduces to

\[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \left(\frac{cm_\alpha}{\hbar}\right)^2\right) A = \sigma q \hat{n}_q \]

with the static ansätze \(n_q = \hat{n}_q(x^k)\) and \(\alpha = c A(x^k) dt\) for \(k = 1, 2, 3\). The Green function associated with (5.24) is given by

\[A = -\frac{1}{4 \pi r} e^{-\frac{c m_\alpha}{\hbar} r} \]

where \(r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\). It follows that \(\sigma q A\) represents the Proca field produced by a single Proca charged source whose number density is approximated by the Dirac delta distribution \(\hat{n}_q(x^k) = \delta^3(x^k)\). If the world line of another particle has spatial velocity \(\dot{x}^k(t) (x^k/c \ll 1)\) then

\[\nabla_{V_q} V_q \approx \dot{x}^1 \frac{\partial}{\partial x^1} + \dot{x}^2 \frac{\partial}{\partial x^2} + \dot{x}^3 \frac{\partial}{\partial x^3}. \]

If this particle also has Proca charge \(q\) and mass \(\mu_q\) then in the low energy and non-relativistic limit (5.23) and (5.25) imply that

\[\mu_q \ddot{x}^k = -\frac{\sigma c^4 q^2}{8 \pi G} \frac{\partial}{\partial x^k} A. \]
Thus the “Yukawa potential energy” $V(r)$ between these two Proca charged particles with separation $r$ is

$$V(r) = -\frac{\sigma c^4}{32 \pi^2 G} \frac{q^2}{r} e^{-\frac{cm_0}{\hbar} r}.$$  \hspace{1cm} (5.27)

Particles carrying the same sign of Proca charge attract or repel each other according as $\sigma = +1$ or $-1$ respectively, with a force of range $\frac{\hbar}{e c m_0}$. We give arguments below why it is of interest to consider the possibility that dark matter may interact with the Proca field having the polarity $\sigma = +1$.

6. Implications from Galactic Dynamics

The basic equations for our dark matter model are the Einstein equation (5.1), the matter field equations (5.3), (5.4), the Proca equation (5.5), supplemented by the conservation equation (5.7) for the Proca current (5.6) and the imposition of equations of state for thermodynamic variables satisfying the Gibb’s relation (5.2). The dark matter is currently assumed to form an all pervading environment in which the less abundant ordinary matter moves. Since the ordinary matter is considered to have zero Proca charge its interaction with the dark matter is purely Newtonian in the non-relativistic low energy domain. We examine the extent to which this simple picture is sufficient to reconcile the galactic rotation curves with observation. For simplicity we shall treat the stars in a galaxy to be test particles that experience Newtonian forces from a locally spherical distribution of low pressure ($P_q \ll \rho_q c^2$) dark matter gas at some temperature $T$ and pressure $P_q$.

Since the gas is charged one expects that its equation of state take the imperfect gas form [32]

$$\frac{P_q}{k_B T} = n_q + B_2 n_q^2 + O(n_q^3)$$  \hspace{1cm} (6.1)

where $k_B$ is Boltzmann’s constant and $B_2$ is the second virial coefficient given in terms of the Yukawa type potential (5.27).
Under approximately adiabatic conditions (5.2) yields \( \rho_q \approx \mu_q n_q \) with constant \( \mu_q \). The Newtonian dynamics of the self gravitating dark matter gas sphere gives rise to the Lane-Emden equation

\[
4 \pi r^2 \left( \frac{\partial P_q}{\partial r} \right) = -4 G \rho_q \dot{M} \quad (6.2)
\]

where \( \dot{M} = \dot{M}(r) \) is the total mass within the sphere of radius \( r \). If \( q \neq 0 \) then in terms of the dimensionless quantities \( b, \eta, u \) and \( \hat{N} = \hat{N}(u) \) defined by

\[
\eta = \frac{4 \sqrt{2} \pi G \mu_q}{c^2 q} \quad (6.3)
\]

\[
u = \frac{32 \pi^2 G k_B T}{c^4 q^2} r \quad (6.4)
\]

\[
\hat{N} = \frac{1}{\mu_q} \hat{M}(r(u)) \quad (6.5)
\]

and

\[
b = \frac{32768 \pi^5 G^3 k_B^3 T^3 B_2}{c^{12} q^6} \quad (6.6)
\]

(6.2) becomes

\[
u^3 \left( \frac{\partial^2 \hat{N}}{\partial u^2} \right) + (\eta^2 u \hat{N} - 2 u^2) \left( \frac{\partial \hat{N}}{\partial u} \right) + b \left( \frac{u}{2} \frac{\partial^2 \hat{N}}{\partial u^2} - \frac{\partial \hat{N}}{\partial u} \right) = 0. \quad (6.7)
\]

For \( u \gg 1 \) (6.7) has the asymptotic solution \( \hat{N}(u) \approx \frac{2u}{\eta^2} \), corresponding to

\[
\dot{M}(r) \approx \frac{2 k_B T}{\mu_q G} r \quad (6.8)
\]

irrespective of the sign of \( \sigma \). This gives rise to the asymptotic dark matter mass density distribution

\[
\rho_q(r) = \frac{k_B T}{2 \pi G \mu_q r^2} \quad (6.9)
\]
and the asymptotic dark matter pressure distribution

\[ P_q(r) = \frac{k_B^2 T^2}{2 \pi G \mu_q^2 r^2}. \]  \hspace{1cm} (6.10)

Thus \( \frac{P_q/c^2}{\rho_q} = \frac{k_B T}{\mu_q c^2} \) and so the low pressure condition is satisfied if \( k_B T \ll \mu_q c^2 \) (cold dark matter). Within such a dark matter gas, the rotation speed for stars, treated as test particles with zero Proca charge, following circular orbits of radius \( r \) under Newtonian gravity is approximately \textit{constant} with value

\[ v_c = \sqrt{\frac{G \dot{M}(r)}{r}} \approx \sqrt{\frac{2 k_B T}{\mu_q}}. \]  \hspace{1cm} (6.11)

This relation may be used to eliminate \( \frac{k_B T}{\mu_q} \) in (6.9) yielding the approximate dark matter density

\[ \rho_q(r) \approx \frac{v_c^2}{4 \pi G r^2}. \]  \hspace{1cm} (6.12)

Taking a typical observational value \[ v_c \approx 220 \text{Km/sec} \] Equation (6.12) gives a value \( \rho_q(r) \approx 6 \times 10^{-28} \text{Kg/m}^3 \) of the dark matter density for \( r = 10 \text{kpc} \).

Although there is no fundamental reason why the dark matter should have been in thermal equilibrium with the microwave background radiation at any time we may bound our estimates by assuming that its temperature now is no hotter than \( T = 2.7 \text{K} \). For \( v_c \approx 220 \text{Km/sec} \) (6.11) implies that

\[ \mu_q \approx 1.5 \times 10^{-23} \text{Kg} \]  \hspace{1cm} (6.13)

or \( \mu_q c^2 \approx 10 \text{GeV} \).

### 7. Cosmological Models

We now turn attention to the implications for cosmology by considering the class of homogeneous and isotropic Robertson-Walker type metrics

\[ g = -c^2 dt \otimes dt + S^2 \left( d\chi \otimes d\chi + f^2 (d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) \right) \]  \hspace{1cm} (7.1)
in local coordinates \( \{t, \chi, \theta, \phi\} \), where \( S = S(t) \) denotes the scale factor and the function \( f = f(\chi) \) is specified by the curvature parameter \( k \) according to

\[
\begin{align*}
  f(\chi) &= \sin \chi, \quad \text{if} \quad k = 1, \quad \text{(spatially closed universe)} \\
  f(\chi) &= \chi, \quad \text{if} \quad k = 0, \quad \text{(spatially flat universe)} \\
  f(\chi) &= \sinh \chi, \quad \text{if} \quad k = -1, \quad \text{(spatially open universe)}.
\end{align*}
\]

Both neutral and Proca charged fluids are supposed to have the common velocity:

\[
V_0 = V_q = \frac{1}{c} \frac{\partial}{\partial t}
\]

to a first approximation. Furthermore suppose that each fluid maintains the same entropy so that \( s_0 \) and \( s_q \) are constants. This is consistent with the (weaker) isentropic conditions (5.20) and (5.21). The densities thus reduce to \( \rho_0 = \rho_0(n_0) \) and \( \rho_q = \rho_q(n_q) \) where \( n_0 = n_0(t) \) and \( n_q = n_q(t) \).

We adopt as ansatze for the Proca 1-form

\[
\alpha = A \, dt
\]  

(7.2)

where \( A = A(t) \). It follows that \( F = d\alpha = 0 \) and the Proca field equation

\[
d \star F + \left( \frac{cm_\alpha}{\hbar} \right)^2 \star \alpha + \sigma \, j_q = 0
\]

(7.3)

gives

\[
A = \frac{\sigma \, q \, n_q \, h^2}{c^2 \, m_\alpha^2}.
\]  

(7.4)

The Proca charge conservation equation (5.7) yields the general solution:

\[
n_q = \frac{N_q}{S^3}
\]  

(7.5)

where \( N_q \) is a dimensionless constant.

The Einstein equation (5.1) yields the differential equations

\[
\left( \frac{\partial S}{\partial t} \right)^2 - \frac{8 \pi G}{3} (\rho - \zeta_\alpha) S^2 + c^2 \, k = 0
\]

(7.6)
and

\[ 3 \frac{\partial^2 S}{\partial t^2} + 4 (\rho + 3 P - 4 \zeta_\alpha) = 0 \]  \hspace{1cm} (7.7)

where the total mass density and total pressure are defined by

\[ \rho \equiv \rho_0 + \rho_q \]
\[ P \equiv P_0 + P_q \]

and

\[ \zeta_\alpha \equiv \frac{\sigma c^4 m_\alpha^2 A^2}{16 \pi G \bar{h}^2}. \]

Since \( d s_f = 0 \) for \( f = 0, q \), (5.2) becomes

\[ d \rho_f = \mu_f d n_f \]  \hspace{1cm} (7.8)

where

\[ \mu_f = \frac{\rho_f + P_f/c^2}{n_f}. \]  \hspace{1cm} (7.9)

It is instructive to write (7.6) and (7.7) in terms of the Hubble parameter

\[ H(t) \equiv \frac{1}{S(t)} \frac{\partial S(t)}{\partial t} \]  \hspace{1cm} (7.10)

and the deceleration parameter

\[ q_d(t) \equiv -\frac{1}{H(t)^2 S(t)} \frac{\partial^2 S(t)}{\partial t^2}. \]  \hspace{1cm} (7.11)

Then (7.6) and (7.7) become

\[ \rho = \zeta_\alpha + \frac{3 H^2}{8 \pi G} + \frac{3 c^2 k}{8 \pi G S^2} \]  \hspace{1cm} (7.12)

\[ P = \frac{(2 q_d - 1) c^2 H^2}{8 \pi G} - \frac{c^4 k}{8 \pi G S^2} + c^2 \zeta_\alpha. \]  \hspace{1cm} (7.13)
Furthermore in terms of the critical density:

\[ \rho_c(t) = \zeta_\alpha + \frac{3 H^2}{8 \pi G} \]  
(7.14)

we define

\[ \Omega \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{\rho}{\zeta_\alpha - \frac{3 c^2 k}{8 \pi G S^2}}. \]  
(7.15)

Using (7.12) one may eliminate \( \zeta_\alpha \) in (7.15) to get

\[ \Delta \equiv \Omega - 1 = \frac{3 c^2 k}{8 \pi G \rho S^2 - 3 c^2 k} \]  
(7.16)

This parameter plays an important role in the discussion below.

8. Dynamical Modifications to Friedmann Cosmologies

To study the effect of \( \Sigma \) on cosmology we first consider equations of state for both Proca charged and neutral fluids to be approximated by pressure-free dusts with conserved particle currents. It then follows from \( P_0 = 0, P_q = 0 \) and (7.8) that

\[ \rho_0 = \mu_0 n_0 \]

\[ \rho_q = \mu_q n_q \]

with the constant chemical potentials \( \mu_q, \mu_0 \) interpreted as the masses of the Proca charged and neutral particles. From the particle conservation equation (5.8) it follows that

\[ n_0 = \frac{N_0}{S^3} \]  
(8.1)

where \( N_0 \) is a dimensionless constant. Hence from (8.1) and (7.5) one has

\[ \rho_0 = \frac{M_0}{S^3} \]  
(8.2)
\[ \rho_q = \frac{M_q}{S^3} \]  

(8.3)

and (7.4) becomes

\[ A = \frac{\sigma q \hbar M_q}{\mu_q c^2 m_\alpha^2 S^3} \]

where \( M_0 = N_0 \mu_0, M_q = N_q \mu_q \). Using (7.8) it may be shown that (7.7) is satisfied provided \( S \) is not constant and the Equations (8.1), (7.5) and (7.6) are satisfied. Now (7.6) becomes

\[
\left( \frac{\partial S}{\partial t} \right)^2 - \frac{8 \pi G}{3} \left( \frac{M S}{S^3} - \Theta \right) + c^2 k = 0
\]

(8.4)

where

\[ M \equiv M_0 + M_q. \]  

(8.5)

In the following it is convenient to introduce the constant \( \Theta \) by

\[ \Theta = \frac{\sigma \hbar N_q}{16 \pi G \left( \frac{\mu_q N_q}{m_\alpha} \right)^2}. \]  

(8.6)

Note that the sign of \( \Theta \) is determined by the coupling polarity \( \sigma \). In terms of the dimensionless quantities defined by

\[ x = \frac{3 c^3 t}{8 \pi G M} \]

\[ \vartheta = \frac{27}{512} \frac{c^6 \Theta}{\pi^3 G^3 M^4} \]

and

\[ Y(x) = \frac{3 c^2 S(t(x))}{8 \pi G M} \]

(8.4) may be expressed as

\[
\left( \frac{\partial Y}{\partial x} \right)^2 - \frac{1}{Y} + \frac{\vartheta}{Y^4} + k = 0
\]

(8.7)
The solutions of this equation for various choices of $\vartheta$ (and $k = 0$) are illustrated in Figures 1, 2, 3 around $x = 0$ where the solution with small $\vartheta$ is insensitive to the choice of $k$.

For $\sigma = -1$ where $\Theta < 0$ one finds solutions for $Y$ exhibiting zeroes where the Levi-Civita geometry is singular as in the standard model of cosmology. For the case $\sigma = +1$ where $\Theta > 0$ any positive valued solution to (8.7) has a non-zero minimum $Y_{\min}$ given by $\frac{\partial Y}{\partial x}|_{Y = Y_{\min}} = 0$. If $0 < \vartheta \ll 1$ then

$$Y_{\min} \approx \vartheta^{1/3}. \quad (8.8)$$

For $Y_{\min}$ non-zero the scale factor never vanishes and the geometry is non-singular. For example the Levi-Civita quadratic curvature invariant

$$\hat{R} \equiv \ast^{-1}(\hat{R}^a_{\ b} \wedge \ast \hat{R}^b_{\ a}) \quad (8.9)$$

has the value

$$\hat{R} = -\frac{243}{512 \, \pi^4 \, G^4 \, M^4} \left( \frac{5}{16} \frac{\vartheta}{Y^6} - \frac{\vartheta}{Y^9} + \frac{5}{4} \frac{\vartheta^2}{Y^{12}} \right) \quad (8.10)$$

which is clearly finite for $Y > 0$.

The invariant $\hat{R}$ has been scaled to

$$\Xi = -\frac{512 \, \pi^4 \, G^4 \, M^4}{243 \, c^8} \hat{R} \quad (8.11)$$

for display in Figures 1, 2, 3 in the vicinity of $x = 0$. We also show that in the domain $Y(x) < 4^{1/3} \vartheta^{1/3}$, $Y(x)$ has an acceleration phase ($\frac{\partial^2 Y}{\partial x^2} > 0$, $q_d < 0$ obtained by differentiating (8.7)). In Figure 4 the nature of the scale factor is exhibited over a larger region of $x$. The eternal nature of the cosmology is apparent for $\vartheta$ positive although depending on the size of $S_{\min}$ the model can readily accommodate hot dense phases where quantum conditions may be relevant.
9. Dynamical Modifications to Inflationary Friedmann Cosmologies

In the above we have shown that the presence of the Proca field in the Einstein equations with $\sigma = 1$ can modify the singular nature of the geometry associated with the Friedmann cosmologies. It is therefore of interest to examine how this field modifies these cosmologies when they are preceded by an inflationary expansion era.

We recall the motivation for an inflationary phase starting at some $t = t_i$ after the standard singularity at $t = 0$. If $t_i$ is chosen to be the Planck time $t_{pl}$ then from (7.16) a Friedmann radiation era predicts

$$\tau \equiv \frac{\Delta_0}{\Delta_i} \approx \frac{t_0}{t_{pl}} \approx 10^{60}$$ (9.1)

where $\Delta_0 = \Delta(t_0)$ for $t_0 \approx 3.5 \times 10^{17}$ sec and $t_{pl} \approx 5.3 \times 10^{-44}$ sec. It is accepted wisdom that unnatural constraints are required to maintain this ratio over such a long period. The standard inflationary model [34, 35, 36, 37, 38, 39, 40] attempts to bring the ratio $\tau$ to order 1 by postulating a pre-Friedmann era from $t_i$ to $t_r$ in which the scale factor grows exponentially. Thus if the inflationary epoch ends at $t = t_r$ with an $e$-folding $\epsilon$ at an energy scale $\sim 10^{15}$ GeV: [41]

$$\tau = \frac{\Delta_0}{\Delta_i} \approx e^{2(60-\epsilon)}$$ (9.2)

By choosing $\epsilon > 60$ the fine tuning puzzle is considered to be solved.

We first note that the parameters of the purely inflationary models enable one to gain some initial estimates of $\Theta$. Standard inflationary models postulate that during the inflationary phase ($t > t_i$) the Universe is dominated by the constant mass density $\rho = \rho_v \approx 1 \times 10^{77}$ Kg/m$^3$ which drives a rapid expansion of $S$ until $S = S_r \approx 10$ m [42]. Thus for the pure inflation phase we prescribe the equation of state

$$P = -c^2 \rho$$ (9.3)

so by differentiating (7.6) and using (7.7) the Proca modified Einstein equations yields

$$\rho = \frac{\Lambda}{8 \pi G} = \rho_v$$ (9.4)
and
\[ \left( \frac{\partial S}{\partial t} \right)^2 - \frac{\Lambda}{3} S^2 + \frac{8\pi G}{3} \frac{\Theta}{S^4} + c^2 k = 0 \]  
(9.5)

where the constant \( \Lambda \) is the “effective cosmological constant”. (Standard inflation is described by (9.5) with \( \Theta = 0 \).) With \( \frac{\partial S}{\partial t} = 0 \) at \( S = S_{\text{min}} \) then
\[ \Theta = \frac{1}{8\pi G} \left( \Lambda S_{\text{min}}^6 - 3 k c^2 S_{\text{min}}^4 \right). \]  
(9.6)

Thus with \( \log \left( \frac{S}{S_{\text{min}}} \right) > 60 \)
\[ S_{\text{min}} \equiv S(t_i) < 10^{-25} \text{ m} \]  
(9.7)

and (9.6) implies that
\[ \Theta < 5 \times 10^{-73} \text{ Kg m}^3 \]  
(9.8)

for \( k = 0, 1, -1 \). If the current Friedmann era of the Universe with \( t = t_0 \approx 3.5 \times 10^{17} \text{ sec} \) is dominated by dark matter with negligible pressure (\( \rho_q(t_0) \ll P_q(t_0)/c^2 \)), then the current scale factor may be estimated [42] to be \( S(t_0) \approx 3 \times 10^{10} \text{ light years} = 3 \times 10^{26} \text{ m} \). Furthermore,
\[ M_q \approx M \approx \rho_c(t_0) S(t_0)^3 = 2 \times 10^{54} \text{ Kg} \]  
(9.9)

where the critical density has been chosen to be \( 6 \times 10^{-26} \text{ Kg/m}^3 \). It then follows from (9.9), (6.13) and
\[ N_q = \frac{M_q}{\mu_q} \approx 1.3 \times 10^{77} \text{ m}^{-3} \]  
(9.10)

that
\[ \frac{q^2}{m_{\alpha}^2} = 16\pi G \left( \frac{1}{\hbar N_q} \right)^2 \Theta < 10^{-168} \text{ m}^2/\text{Kg}^2. \]  
(9.11)

If \( m_{\alpha} \) is less than the Planck mass \( m_{\text{pl}} = \sqrt{\frac{h c}{G}} = 2.18 \times 10^{-8} \text{ Kg} \) then \( |q| < 2 \times 10^{-92} \text{ m} \) which is much smaller than the Planck length \( \ell_{\text{pl}} = \sqrt{\frac{h c}{G \hbar}} = 1.6 \times
$10^{-35}$ m and somewhat unnatural. Furthermore the dimensionless coupling
\[ \frac{\hat{q}^2}{\hbar c} \]
associated with the effective coupling
\[ \hat{q} \equiv \frac{c^2 q}{\pi \sqrt{32 G}} \]  
(9.12)
is many orders of magnitude smaller than the fine structure constant. The origin of these values is the high $e$-folding $\epsilon$ needed in the standard inflationary model to accommodate the flatness problem associated with the standard Friedmann curvature singularity. However in the model with Proca fields there need be no such singularity and one may adjust the size of the minimum scale factor to accommodate the flatness problem with a smaller $\epsilon$. We estimate the necessary parameters by requiring that $q \approx \ell_{\text{pl}}$ and $m_\alpha \approx m_{\text{pl}}$. In this case the dimensionless coupling strength between two Proca charges is
\[ \frac{\hat{q}^2}{\hbar c} = \frac{1}{32 \pi^2} \approx \frac{1}{315} \]  
(9.13)
which is of the order of fine structure constant.

For simplicity we arrange that at the end of the inflationary epoch the Universe proceeds to the current time in a Friedmann era dominated by dark matter modelled as a Proca charged fluid with negligible pressure. From (9.6) the minimal value of the scale factor $S_{\text{min}} = S(0)$ may be determined by $\Theta$ and $\Lambda$
\[ S_{\text{min}} = \sqrt{2} \left( \frac{\pi G \Theta}{\Lambda} \right)^{1/6} \]  
(9.14)
neglecting $k$. If the inflationary era ends at time $t_r$ with a scale factor $S_r = S(t_r)$ and
\[ \rho_{\text{inflation}}(t_r) = \rho_{\text{Friedmann}}(t_r) \]  
(9.15)
one has
\[ \Lambda = \frac{8 \pi G M_q}{S_r^3}. \]  
(9.16)
With the value of $N_q$ from (9.10) and $q \approx \ell_{\text{pl}}$, $m_\alpha \approx m_{\text{pl}}$, (8.6) gives
\[ \Theta = 3 \times 10^{24} \text{ Kg m}^3. \]  
(9.17)
It follows from (7.16), (8.3), (9.9) and (9.15) that

\[ \tau = \frac{\Delta_0}{\Delta_r} \frac{\Delta_r}{\Delta_i} = \frac{S_0}{S_r} \left( \frac{S_{\text{min}}}{S_r} \right)^2 \]  

(9.18)

Eliminating \( S_{\text{min}} \) using (9.14) and assuming the desired value \( \tau < 1 \) we may solve (9.18) for \( S_r \):

\[ S_r > 2 \times 10^8 \text{ m} \]  

(9.19)

Inserting this value in (9.14) yields the value of the minimal scale factor

\[ S_{\text{min}} > 10 \text{ cm} \]  

(9.20)

This corresponds to \( \epsilon = \log \left( \frac{S_r}{S_{\text{min}}} \right) > 20 \) which is a considerably smaller bound than the traditional inflationary expansion based on Einsteinian gravitation alone.

10. Conclusion

Motivated by the structure of a class of actions that involve (in addition to the generalised Einstein-Hilbert action) terms including the torsion and metric gradient of a general connection on the bundle of linear frames over spacetime, the consequences of Einstein-Proca gravitation coupled to matter have been examined. This theory is written entirely in terms of the traditional torsion free, metric compatible connection where all the effects of torsion and non-metricity reside in a single vector field satisfying the Proca equation. In such a theory the weak field limit admits both massless tensor gravitational quanta (traditional gravitons) and massive vector gravitational quanta. The mass of the Proca field is determined by the coupling constants in the parent non-Riemannian action. The interaction mediated by the new Proca component of gravitation is expected to modify the traditional gravitational interaction on small scales. In order to confront this expected modification with observation we have constructed an Einstein-Proca-Fluid model in which the matter is regarded as a perfect thermodynamic fluid.
We have suggested that in addition to ordinary matter that couples gravitationally through its mass the conjectured dark matter in the Universe may couple gravitationally through both its mass and a new kind of gravitational charge. The latter coupling is analogous to the coupling of electric charge to the photon where the analogue of the Maxwell field is the Proca field strength (the curl of the Proca field). If one assumes that the amount of dark matter dominates over the ordinary matter in the later phase of evolution of the Universe, that the Proca field mass is of the order of the Planck mass and the appropriate coupling to the dark matter is of the same order as the fine structure constant then one finds that such hypotheses are consistent with both the inflationary scenario of modern cosmology as well as the observed galactic rotation curves according to Newtonian dynamics. The latter follows by assuming that stars, composed of ordinary (as opposed to dark matter), interact via Newtonian forces to an all pervading background of massive gravitationally charged cold dark matter in addition to ordinary matter. The novel gravitational interactions are predicted to have a significant influence on pre-inflationary cosmology. For attractive forces between dark matter charges of like polarity the Einstein-Proca-matter system exhibits homogeneous isotropic eternal cosmologies that are free of cosmological curvature singularities thus eliminating the horizon problem associated with the standard big-bang scenario. Such solutions do however admit dense hot pre-inflationary epochs each with a characteristic scale factor that may be correlated with the dark matter density in the current era of expansion.

The Einstein-Proca-Fluid model offers a simple phenomenological description of dark matter gravitational interactions. It has its origins in a geometrical description of gravitation and the theory benefits from a variational formulation in which the connection is a bona fide dynamical variable along with the metric. The simplicity of the model is a consequence of the structure of a class of non-Riemannian actions whose dynamical consequences imply that the new physics resides in a component of gravitation mediated by a Proca field. It will be of interest to confront the theory with other aspects of astrophysics such as localised gravitational collapse, the nature of the inflation mechanism and the origin of dark matter.
11. Acknowledgement

The authors are grateful to J Schray and D H Lyth for valuable interaction. RWT is grateful to the Human Capital and Mobility Programme of the European Union for partial support. CW is grateful to the Committee of Vice-Chancellors and Principals, UK for an Overseas Research Studentship and to Lancaster University for a School of Physics and Chemistry Studentship and a Peel Award.

References


The solution to (8.7) with a small positive value of $\vartheta$ has an asymptotic form $Y(x) \approx \frac{1}{3} + \frac{3}{4} \vartheta^{-2/3} x^2$ around $x = 0$. Such a $Y(x)$ has a minimum at $x = 0$, where the corresponding dimensionless curvature invariant $\Xi(x)$ defined in (8.11) has a maximal value. Note that $Y(x)$ has an acceleration phase ($\ddot{Y}(x) > 0$) within the region $-2 \left(\vartheta/3\right)^{1/2} < x < 2 \left(\vartheta/3\right)^{1/2}$.

Figure 1: The Early Universe for $\vartheta > 0$
The solution $Y(x) \approx \frac{|3x/2|^{2/3}}{3}$ for the standard Friedmann model is obtained by choosing $\vartheta = 0$ in (8.7). The "big-bang" or "big-crunch" singularity follows from $Y \to 0$ and $\Xi \to \infty$ as $x \to 0$. $\ddot{Y}(x)$ is negative definite throughout spacetime.

Figure 2: The Early Universe for $\vartheta = 0$
With a choice of negative $\vartheta$ (8.7) yields the asymptotic solution $Y(x) \approx |\vartheta|^{1/6} |3x|^{1/3}$ which approaches zero as $x \to 0$ more rapidly than $Y(x)$ described by the Friedmann solutions, thus leading to a more severe singular behaviour of the spacetime curvature.
Figure 4: Eternal and Oscillating Cosmologies

The global behaviour of $Y(x)$ for $\vartheta = 0.1$ exhibits three types of eternal cosmology. If the curvature parameter $k = 1$ an oscillating universe is depicted by a periodic $Y(x)$. If $k = 0$ or $-1$ the universe contracts to a minimal scale and then expands to infinity. In all three cases the Levi-Civita curvature is regular throughout spacetime.