Statistical Mechanics of Multiply Wound D-Branes

Gavin Polhemus*

Department of Physics and Enrico Fermi Institute
University of Chicago
5640 Ellis Avenue, Chicago, Illinois 60637

Abstract

The D-brane counting of black hole entropy is commonly understood in terms of excitations carrying fractional charges living on long, multiply-wound branes (e.g. open strings with fractional Kaluza-Klein momentum). This paper addresses why the branes become multiply wound. Since multiply wound branes are T-dual to branes evenly spaced around the compact dimension, this tendency for branes to become multiply wound can be seen as an effective repulsion between branes in the T-dual picture. We also discuss how the fractional charges on multiply wound branes conspire to always form configurations with integer charge.

*Electronic address: g-polhemus@uchicago.edu
The past year has seen considerable progress in understanding black hole entropy through counting states of D-brane configurations. This approach has been particularly fruitful in the study of five-dimensional black holes [1]. One of the most carefully studied models is that of black holes in the toroidal compactification of type IIB supergravity down to five dimensions [2]. When discussing the applications of the ideas below, this model will be used as an example. These five dimensional black holes are characterized by three charges, two asymptotic fields and mass. Since it is not known how to calculate the entropy of these configurations directly, the black hole is modeled by a configurations of 5-branes, 1-branes, their anti-branes and strings carrying the same charges. In certain limits (all near BPS) the entropy of the D-brane configurations can be calculated by directly by counting microstates. In these cases the entropy of the D-brane configuration has been found to agree with the Beckenstien-Hawking entropy for a black hole with the same charges, fields and mass. Further calculations have shown agreement in the absorption cross sections, and low energy emission spectra [3, 4, 5, 6].

In order for the D-brane to give the right entropy in the limit of large charges and mass the configurations must have excitations that carry fractional charges. These fractional charges also lead to the large density of string and small mass gap that are expected for a black hole. This has been commonly understood as an indication that the branes which wrap compact dimensions, rather than forming a large collection of singly wound branes, join together to form a single, long, multiply-wound brane [7, 8, 9].

The notion of D-brane winding is not as obvious as fundamental string winding. When \( Q \) D-branes coincide, the gauge theory on the brane is enlarged to a \( U(Q) \) symmetry. If the brane wraps a compact dimension (\( x_9 \) in this example), then this gauge symmetry can be broken by adding a Wilson line \( A^9 = \text{diag}\{\theta_1, \theta_2, \ldots, \theta_Q\}/2\pi R_9 \) [10]. This leads to a holonomy given by the matrix

\[
U_9 = \begin{pmatrix}
e^{i\theta_1} & 0 & \cdots & 0 \\
0 & e^{i\theta_2} & 0 & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & e^{i\theta_Q}
\end{pmatrix}.
\tag{1}
\]

In the T-dual picture (which we will refer to as the “unwound picture” to distinguish it from the “wound picture” above) this is a set of parallel branes at positions \( R_9(\theta_1, \theta_2, \ldots, \theta_Q) \) around the compact dimension.
Of course we could also add Wilson lines that have off diagonal elements. For example, one case that represents a single, multiply-wound brane is

\[
U_9 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]  

(2)

However, one can always put such a matrix in diagonal form by a change of basis. So the Wilson lines with off diagonal elements break the gauge symmetry exactly like the diagonal Wilson line with the same eigenvalues. For example the eigenvalues of (2) are the $Q^{th}$ roots of one. So a multiply wound brane is T-dual to parallel branes that are evenly spaced around the compact dimension. As described in [10, 11], this leads to fractionally charged excitations. However, the total charge of a configuration is always integer. This restriction is discussed in Section 3 below.

The issue of how branes become multiply wound so that they can carry fractional charges can be address by considering how the “unwound” D-branes spread out around the compact dimension. If the branes were non-interacting they could be expected to spread out to fill the compact dimension much the way particles of a non-interacting gas spread out to fill a container. However, we show in this paper that the branes do not behave this way and in fact spread out to fill the compact dimension with with remarkable uniformity.

2 How Branes Repel

The most obvious reason for the branes to spread out evenly around the compact dimension would be to minimize some sort of repulsive potential. However, there is absolutely no way for a potential to be generated—Wilson lines do not break the supersymmetries so, in the absence of excitations, all Wilson lines must have zero energy. Likewise, in the unwound picture there are no forces pushing the branes apart because any distribution of parallel branes preserves some of the supersymmetries and therefore has zero energy.

Since all Wilson lines are zero energy, we assume that all Wilson lines are equally probable when doing D-brane thermodynamics. It may be that the process of creating the D-brane configurations leads to certain Wilson lines more frequently than others, but that would be a rather exceptional situation. Certainly, one can carefully prepare a specific configuration,
but in a realistic scenario we anticipate that interactions during preparation lead to every zero energy configuration with equal probability. Note that, since all of the states are zero energy, the temperature of the system does not matter.

While all Willson lines are equally probable, not all positions of the branes are equally probable! There is only one element of $U(Q)$ with all its eigenvalues equal to 1, but there are many whose eigenvalues are, for example, the $Q^{th}$ roots of one. The probability that the branes will have positions $(\theta_1, \theta_2, \ldots, \theta_Q)$ around the compact dimension is proportional to the volume of $U(Q)$ which eigenvalues $(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_Q})$.

### 2.1 An Example with Two Branes (Not 2-Branes)

As an example, consider the case of two coincident branes. In this case the gauge group is $U(2) = U(1) \times SU(2)$. The $U(1)$ subgroup is the center of mass position in the unwound picture, so the branes relative position is given by the eigenvalues of the $SU(2)$. The elements of $SU(2)$ can be parameterized by $\xi^j$ using the three Pauli matrices, $\sigma_j$, as generators:

$$U = e^{i\xi^j\sigma_j}.$$  \hfill (3)

The eigenvalues are then $e^{\pm i|\xi|}$, resulting in a separation between the branes $\theta_1 - \theta_2 = 2|\xi|$. Notice that $SU(2)$ is topologically $S^3$ and the surfaces of constant $|\xi|$ are like the spheres of constant latitude running from $|\xi| = 0$ at the north pole to $|\xi| = \pi$ at the south pole on $S^3$. The radius of each sphere is $r = \sin |\xi|$ so the probability that the two branes are at positions $\theta_1$ and $\theta_2$ is

$$\mathcal{P}(\theta_1, \theta_2) = \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right).$$  \hfill (4)

The probability density actually vanishes for the branes being coincident. The peak of the probability distribution is at maximum separation in the unwound picture, which corresponds to a doubly wound brane in the wound picture.
2.2 Many Branes

The probability of a randomly chosen element of $SU(Q)$ having eigenvalues $(\theta_1, \theta_2, \ldots, \theta_Q)$ has been exactly solved for all $Q$ [12],

$$P(\theta_1, \theta_2, \ldots, \theta_Q) = C \prod_{k<l}^{Q} |e^{i\theta_k} - e^{i\theta_l}|^2.$$ (5)

The maximum is clearly at equal spacing. Fixing the origin at $\langle \theta_Q \rangle = 0$ and putting the remaining $\theta_j$ in increasing order gives $\langle \theta_j \rangle = 2\pi j/Q$.

To gain an intuitive understanding of the branes’ behavior it is useful to consider another physical system that has the same probability distribution: a collection of $Q$ particles, each with charge $q$, living in two dimensions on a circle of radius $R$ [12, 13]. The two dimensional world can be thought of as the complex plane. The two-dimensional Coulomb potential for a pair of charges at $z_k$ and $z_l$ is $W_{kl} = -q^2 \ln |z_k - z_l|$. Restricting the charges to the unit circle, $z_j = Re^{i\theta_j}$. The energy of the system is then

$$W = -q^2 \sum_{k<l}^{Q} \ln |e^{i\theta_k} - e^{i\theta_l}|.$$ (6)

In a thermal ensemble of such systems at inverse temperature $\beta$ the probability distribution will be

$$P(\theta_1, \theta_2, \ldots, \theta_Q) = \frac{C e^{-\beta W}}{C} = \prod_{k<l}^{Q} |e^{i\theta_k} - e^{i\theta_l}|^\beta q^2.$$ (7)

For $\beta = 2/q^2$ this is exactly the probability distribution in (5). Note that the inverse temperature $\beta$ has nothing to do with the temperature of the branes or the black hole. This is the temperature of the model system of electric charges that has the same probability distribution as the D-brane configuration at any temperature.

Clearly the charges are going to have a very strong tendency to spread out evenly around the circle, forming a sort of crystal. In the case of only two particles it is easy to see that the fluctuations in the spacing are roughly the same as the average spacing, $\Delta(\theta_1 - \theta_2) \approx \langle \theta_2 - \theta_1 \rangle = \pi$. It turns out that as the number of particles gets large and the average spacing
gets smaller, the fluctuations in nearest neighbor spacing get proportionally smaller so that 
\[ \Delta(\theta_j - \theta_{j+1}) \approx \langle \theta_j - \theta_{j+1} \rangle = 2\pi/Q \] [12].

Although the spacings between adjacent particles varies significantly, long wavelength
fluctuations in the average spacing are minuscule. These long wavelength fluctuations can
be studied in the continuum limit. In this case the discrete index, \( j \), is replaced by a
continuous parameter, \( \xi = 2\pi j/Q \). The positions can be expanded about their means in
Fourier modes,

\[ \theta_\xi = \langle \theta_\xi \rangle + \sum_{n\neq 0} a_n e^{in\xi}, \] (9)

where \( a_{-n} = a_n^* \) and \( \langle \theta_\xi \rangle = \xi \). The energy of the system is the ground state energy plus the
energies of these modes,

\[ W = W_0 + \sum_{n=-(Q-1)}^{Q-1} W_n. \] (10)

Calculating these energies of these fluctuations is a simple exercise in two dimensional elec-
trostatics:

\[ W_n = 2\pi^2 \rho_0^2 n |a_n|^2, \] (11)

where \( \rho_0 \) is the average charge density, \( \rho_0 = Qq/2\pi \).

As the number of charges gets large all of the long wavelength fluctuations actually
freeze out. This is best seen by fixing the average charge density of the system, \( \rho_0 \). Since
we are interested in the temperature at which the charged particles are distributed like the
“unwound” D-branes, \( \beta = 2/q^2 = Q^2/2\pi^2\rho_0^2 \), the temperature decreases like \( 1/Q^2 \).

To calculate the average energy in these modes at \( \beta = 2/q^2 \) we use the partition function,

\[ Z_n = \int_0^\infty da_n e^{-\beta W_n} \]
\[ = \frac{1}{2\rho_0 \sqrt{2\pi n\beta}}. \] (13)

The mean energy in these modes is then

\[ \langle W_n \rangle = -\frac{\partial}{\partial \beta} \ln Z_n = \frac{1}{2\beta}. \] (14)
From this we can easily find the average (root mean squared) amplitude of these macroscopic oscillations,

\[
\langle |a_n|^2 \rangle = \frac{1}{(2\pi \rho_0)^2 n \beta} = \frac{1}{2Q^2 n}.
\]  

(15)

(16)

This can be used to find the fluctuations in the spacing between remote particles,

\[
\langle (\theta_\xi - \theta_\zeta)^2 \rangle = \langle \left[ \xi - \zeta + \sum_{n \neq 0} a_n \left( e^{in\xi} - e^{in\zeta} \right) \right]^2 \rangle
\]

\[
= (\xi - \zeta)^2 + \sum_{n > 0} \langle |a_n|^2 \rangle \left( e^{in\xi} - e^{in\zeta} \right) \left( e^{-in\xi} - e^{-in\zeta} \right)
\]

\[
= \langle \theta_\xi - \theta_\zeta \rangle^2 + \frac{1}{Q^2} \sum_{n > 0} \frac{1}{n} \sin^2 \frac{n(\xi - \zeta)}{2}.
\]

(17)

(18)

(19)

The infinite sum is divergent and needs to be cut off at \( n = Q/2 \). The sum can be approximated by an integral; the sine cuts off the integral in the infrared at \( n = \pi/(\xi - \zeta) \).

\[
\sum_{n > 0} \frac{1}{n} \sin^2 \frac{n(\xi - \zeta)}{2} \approx \frac{1}{2} \int_{\pi/(\xi - \zeta)}^{Q/2} \frac{dn}{n} = \frac{1}{2} \ln(k - l)
\]

(20)

where we have used \( \xi = 2\pi k/Q \) and \( \zeta = 2\pi l/Q \). So the size of the fluctuations in spacing for \( k - l \gg 1 \) is

\[
\Delta(\theta_k - \theta_l) \approx \frac{1}{Q} \sqrt{\frac{1}{2} \ln(k - l)}.
\]

(21)

As we would expect, the fluctuations in spacing get bigger as we consider particles that are farther apart. What is surprising is how slowly the fluctuations increase.

Since the probability distribution for the particles is the same as that for the unwound branes, all of the above statements about deviations from equal spacing applies to the brane case as well. However, the mechanism is different. The particles spread out to minimize their energy while the branes spread out to find larger volumes of phase space. Because of this difference, the charged particles cannot be used to model all aspects of the branes’ behavior—for example, the momentum distribution of the branes in different.
All interesting D-brane models of black holes have more than one compact dimension, each with its own (possibly trivial) Wilson line. These Wilson lines all transform together under the $U(Q)$ symmetry, allowing a trace term in the Lagrangian that gives non-zero energy to configurations with Wilson lines that cannot be simultaneously diagonalized [14]:

$$V = \frac{T^2}{2} \sum_{\mu, \nu=D}^{9} \text{Tr} [X^\mu, X^\nu]^2,$$

where $T$ is the fundamental string tension and $D$ is the dimension of the non-compact space-time. Both the additional Wilson lines and the potential that come from the additional compact dimensions may be important in understanding D-brane models. However, D-brane calculations of black hole entropy have only required that the branes become multiply wound around a single $S^1$ [8]. Based on this we expect that the remaining four compact dimensions do not affect this winding.

### 3 Integer Winding and Momentum of D-Brane Configurations

Non-extremal configurations have non-zero energy, allowing excitations away from zero potential in (22). These excitations correspond to loops of unexcited fundamental string connecting the various branes [14]. Since winding is easier to visualize than momentum, we start by looking at the fractional winding states that connect the branes in the unwound picture. T-Duality allows us to apply this understanding to the fractional momentum states in the wound D-brane picture.

In the unwound picture the excitations are strings connecting the D-branes. Since there are a lot of branes around, the strings don’t have to go all the way around the compact direction—they are allowed to have fractional winding. However, the entire configuration must have integer winding. This is a result of charge conservation on the branes. The charge on the brane that comes from the end of a string must be canceled by an opposite charge from another string. In terms of the orientation of the strings: each brane must have as many strings beginning on it as it has ending on it. This guarantees that the net winding of any configuration will be zero.

T-duality turns the fractional winding that appeared in the unwound picture into fractional momentum in the wound picture. The excitations of wound branes are free to move
independently. However, their ends are still charged, so the excitations trail field lines in the D-brane as they move. Moving a single excitation around the compact direction wraps the field lines, changing the energy of the system. Since moving a single excitation around the compact dimension does not restore the system to its original configuration, single excitations don’t see the periodicity of the compact dimension and there is no reason for their momentum to be quantized in units of $1/R$.

To avoid trailing any field lines, excitations must band together into parties that are neutral in the D-brane theory. These parties can then circumnavigate the compact dimension and discover its periodicity. It is the momentum of these parties which is quantized in units of $1/R$. It is easy to verify that the sum of the fractional momenta that make up a neutral party is always integer.

4 Discussion

In the models of five-dimensional black holes the excitations are strings connecting D-1-branes to D-5-branes. We expect that the “repulsion” described above encourages both the 1-branes and the 5-branes to link up into long, multiply-wound branes, allowing fractionally charged excitations. (The total charge will still be integer for the reasons given above)

It may be possible to understand models of near extremal black holes entirely in the language D-branes (without any fundamental strings attached) by replacing the fundamental strings with excitations away from $V = 0$ in (22). However, since the fundamental strings in black hole models connect branes of different dimensionality, it is less clear how to proceed.

It would also be interesting to see if these ideas could be extended to the branes of M-theory.

5 Acknowledgments

I would like to thank Jeff Harvey for many enlightening discussions on many matters and Juan Maldacena for a helpful conversation about charge conservation on branes. This work was supported by an Office of Naval Research Graduate Fellowship.
References


