Abstract

In this paper the Maxwell field theory is considered on the $\mathbb{Z}_n$ symmetric algebraic curves. As a first result, a large family of nondegenerate metrics is derived for general curves. This allows to treat many differential equations arising in quantum mechanics and field theory on Riemann surfaces as differential equations on the complex sphere. The examples of the scalar fields and of an electron immersed in a constant magnetic field will be briefly investigated. Finally, the case of the Maxwell equations on curves with $\mathbb{Z}_n$ group of automorphisms is studied in details. These curves are particularly important because they cover the entire moduli space spanned by the Riemann surfaces of genus $g \leq 2$. The solutions of these equations corresponding to nontrivial values of the first Chern class are explicitly constructed.
1 INTRODUCTION

The two dimensional gauge field theories on a manifold and in particular on Riemann surfaces have recently been considered by various authors [1], [2]. The abelian case is particularly interesting because of the presence of topologically nontrivial sectors labelled by the nonzero values of the first Chern class [3]. The problem of deriving topologically nontrivial solutions of the Maxwell equations on a manifold is relevant in string theories [4], in Quantum Mechanics [5] and in the theory of the Quantum Hall effect [6]. The classical solutions of the Maxwell equations on Riemann surfaces have been derived in ref. [7] for particular metrics and in ref. [8] for conformally flat metrics. More recently, the results of [8] have been generalized to any metric in [9]. In the works mentioned above the gauge field configurations corresponding to nonvanishing values of the first Chern class have been expressed in terms of the prime form and theta functions [10]. In this paper, we use instead an alternative approach, which has already provided physically relevant results in string theory for its explicitness [11], [12]. Namely, the Riemann surface will be represented here as an $n$-sheeted covering of the complex sphere, i. e. as an affine algebraic curve [13]. One advantage of this representation is that any differential equation defined on a Riemann surface, such as those arising for instance in Quantum Field Theory or Quantum Mechanics, becomes a differential equation on the sphere. A second advantage is that, at least in the case of algebraic curves with $Z_n$ symmetry group of automorphisms, the dependence on the moduli is explicit. The latter are in fact given by the branch points and enter in the equation of the curve as simple complex parameters. Despite of these advantages, the analytic solution of the physically relevant field or wave equations is very difficult on an algebraic curve because of the presence of multivalued differential operators. Until now, only the theory of the socalled $b-c$ systems has been fully solved on any algebraic curve using the operator formalism of refs. [11]–[12]. The Maxwell field theory is more complicated than the $b-c$ systems because it is not conformally invariant and the metric is present in the equations of motion. Unfortunately, on an algebraic curve the powerful tools offered by the theory of theta functions and exploited for instance in refs. [7] and [9], are not very helpful for computational purposes. For instance, the already existing formulas of the prime form [10] are somewhat cumbersome and un-explicit for physical applications.
Despite of these problems, we will derive in this paper all the nontrivial classical solutions of the Maxwell equations at least in the case of the $\mathbb{Z}_n$ symmetric curves. This is an important class of curves, which covers the entire moduli space spanned by the Riemann surfaces of genus $g \leq 2$. We will see that the topologically nontrivial gauge fields obtained here have a relatively simple expression, similar to their analogues on the complex sphere and thus can provide new insights in two dimensional Quantum Mechanics and FQHE on a manifold. Moreover, we derive a big family of tensors describing nondegenerate metrics on the $\mathbb{Z}_n$ symmetric curves. This allows to write explicitly in terms of multivalued differential operators on the sphere not only the Maxwell equations, but also any other equation of motion that involves particles of integer spin living on a Riemann surface.

Concerning the Maxwell equations, the greatest difficulty in solving them is provided by the lack of a compact expression for the prime form. In fact, apart from the case of the $C\theta M$ metric of [7], the nontrivial gauge field configurations are expressed by means of the prime form [9]. On the other side, on algebraic curves it is difficult to construct the $C\theta M$ metric because the canonically normalized differentials are not known (see for instance ref. [14] for an attempt to construct these differentials). To circumvent these difficulties, we give here particular kinds of nondegenerate metrics which are singlevalued, i.e. they are invariant under arbitrary permutations of the sheets composing the algebraic curve. Using these metrics the Maxwell equations simplify considerably and the computation of their classical solutions is made possible.

The material presented in this paper is divided as follows. In Section 2 the Maxwell field theory on Riemann surfaces and algebraic curves is briefly introduced. The gauge fields are decomposed in their exact, coexact and harmonic components using the Hodge decomposition theorem. The harmonic components are explicitly derived. In Section 3 we construct nondegenerate metrics on the $\mathbb{Z}_n$ algebraic curves. For some of these metrics the corresponding Ricci tensor is computed. We check using the Poincaré–Lelong equation [13], that different curvature tensors yield the same Euler characteristics as expected. In Section 4 the topologically nontrivial solutions of the Maxwell equations are derived. We verify that the magnetic fluxes generated by these gauge fields satisfy the Dirac quantization condition. Finally, in the Conclusions we discuss the possible applications and generalizations of our results. In particular, it is shown how to generalize the metrics of
Section 3 to any affine algebraic curve. Moreover, the equations of motion of the scalar fields on algebraic curves are treated with some details and the Hamiltonian of a massive electron immersed in a constant magnetic field is explicitly constructed.

2 THE MAXWELL FIELD THEORY ON ALGEBRAIC CURVES

In this paper we consider the Maxwell field theory on a Riemann surface $\Sigma$ of genus $h > 1$ and with $Z_n$ group of symmetry. The action is given by:

$$S_{\text{Maxwell}} = \int_\Sigma d\xi^1 \wedge d\xi^2 \sqrt{g} F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu, \nu = 1, 2$ and $\xi^1, \xi^2$ form a system of real coordinates on $\Sigma$. Finally, $g_{\mu\nu}$ is the Euclidean metric on $\Sigma$ with determinant $g$. The only nonvanishing components of the field strength are given by $F_{21} = - F_{12}$. On a Riemann surface it is always possible to assume that the metric $g_{\mu\nu}$ is conformally flat. In the latter case, it is convenient to choose on $\Sigma$ complex coordinates $\xi = \xi^1 + i \xi^2$ and $\bar{\xi} = \xi^1 - i \xi^2$, so that the components of the field strength and of the metric become respectively:

$$F_{\xi \bar{\xi}} = - F_{\bar{\xi} \xi} = - \frac{1}{2i} F_{12}$$

and

$$g_{\xi \bar{\xi}} = g_{\bar{\xi} \xi} = \frac{1}{2} \sqrt{g} \quad g_{\xi \xi} = g_{\bar{\xi} \bar{\xi}} = 0$$

Moreover, the volume form in complex coordinates is given by: $d^2 \xi g_{\xi \bar{\xi}} = d\xi^1 \wedge d\xi^2 \sqrt{g}$, where $d^2 \xi \equiv id\bar{\xi} \wedge d\xi$. Accordingly, the classical equations of motion of the Maxwell field theory take the following form:

$$\partial_\xi \left[ g^{\xi \bar{\xi}} \left( \partial_\xi A_\xi - \partial_\bar{\xi} A_{\bar{\xi}} \right) \right] = 0 \quad (1)$$

$$\partial_{\bar{\xi}} \left[ g^{\xi \bar{\xi}} \left( \partial_\xi A_\xi - \partial_{\bar{\xi}} A_{\bar{\xi}} \right) \right] = 0 \quad (2)$$

with $g^{\xi \bar{\xi}}$ being the inverse metric: $g^{\xi \bar{\xi}} g_{\xi \bar{\xi}} = 1$.  

4
Let us decompose the gauge fields using the Hodge decomposition:

\[ A_\xi = \partial_\xi \varphi + \partial_\xi \rho + A^\text{har}_\xi \]
\[ A^{\bar{\xi}} = -\partial^{\bar{\xi}} \varphi + \partial^{\bar{\xi}} \rho + A^\text{har}_{\bar{\xi}} \]  \hspace{1cm} (3)

The coexact and exact components are expressed using the two scalar fields \( \varphi \) and \( \rho \) respectively. Here \( \varphi \) is purely complex, while \( \rho \) is real. \( A^\text{har}_\xi \) and \( A^\text{har}_{\bar{\xi}} \) take into account the holomorphic differentials. The decomposition 3 is not invertible unless

\[ \int_\Sigma d^2 \xi g_{\xi \bar{\xi}} \varphi(\xi, \bar{\xi}) = \int_\Sigma d^2 \xi g_{\xi \bar{\xi}} \rho(\xi, \bar{\xi}) = 0 \]  \hspace{1cm} (4)

The only nontrivial solutions of the equations 1–2 are provided by the \( h \) harmonic differentials \( A^\text{har}_{i, \xi} \), \( A^\text{har}_{i, \bar{\xi}} \), \( i = 1, \ldots, h \) and by the vector fields \( A^I_\xi \), \( A^I_{\bar{\xi}} \) corresponding to nonvanishing values of the first Chern class. The former satisfy the relations:

\[ \partial_\xi A^\text{har}_{i, \xi} = \partial^{\bar{\xi}} A^\text{har}_{i, \bar{\xi}} = 0 \]

while the latter are built in such a way that

\[ F_{\xi \bar{\xi}} d\xi \wedge d\bar{\xi} = \frac{i \Phi}{A} g_{\xi \bar{\xi}} d\bar{\xi} \wedge d\xi \]

where \( \Phi \) is a constant representing the total magnetic flux associated with the fields \( A^I_\xi \), \( A^I_{\bar{\xi}} \) and \( A = i \int_\Sigma d^2 \xi g_{\xi \bar{\xi}} \) denotes the area of the Riemann surface.

At this point we represent \( \Sigma \) in an explicit way as an algebraic curve associated with Weierstrass polynomials of the kind:

\[ y^n = \prod_{i=1}^{nm} (z - a_i) \]  \hspace{1cm} (5)

In 5, \( z \) and \( \bar{z} \) denote a set of complex variables describing the sphere \( \mathbb{C} \mathbb{P}^1 \) and \( n, m \) are integers. For our purposes, it will be convenient to regard \( \mathbb{C} \mathbb{P}^1 \) as the compactified complex plane, i.e. \( \mathbb{C} \mathbb{P}^1 \equiv \mathbb{C} \cup \{\infty\} \). As usual, we cover \( \mathbb{C} \mathbb{P}^1 \) with two open sets \( U_1 \) and \( U_2 \) containing the points \( z = 0 \) and \( z = \infty \) respectively. The local coordinate \( z' \) on \( U_2 \) is related to \( z \) by the conformal transformation: \( z' = \frac{1}{z} \). The \( a_i \) are complex parameters denoting the branch points of the curve. It is easy to check that the point at infinity \( z = \infty \) is not a branch point. Solving eq. 5 with respect to \( y \), we obtain a multivalued function \( y(z) \), whose branches will be denoted with the symbol
\( y^{(l)}(z), \ l = 0, \cdots, n - 1. \) A generic tensor on the algebraic curve can be multivalued on the complex sphere due to its dependence on \( y^{(l)}(z) \). To indicate the branches of any such tensor \( T(z, \bar{z}) \), we will use the following convenient notation:

\[
T^{(l)}(z, \bar{z}) \equiv T(z, \bar{z}; y^{(l)}(z), \bar{y}^{(l)}(\bar{z}))
\]

(6)

Any \( Z_n \) symmetric algebraic curve \( \Sigma \) is conformally equivalent to a Riemann surface represented as a branched covering of \( \mathbb{CP}_1 \). The genus of the surface is

\[
h = 1 - n + \frac{nm(n - 1)}{2}
\]

(7)

In practice, the Riemann surface \( \Sigma \) is constructed gluing together along a suitable set of branch lines \( n \) copies of the complex sphere, the so-called sheets. Accordingly, the \( z \) variable can be viewed as a mapping \( z : \xi \in \Sigma \rightarrow \mathbb{CP}_1 \). In this paper it will be always understood that \( z \) is a function of \( \xi \), i.e. \( z = z(\xi) \), unless conversely stated. To construct a system of branch lines which is consistent with the multivaluedness of the function \( y(z) \), we group the branch points in \( m \) different sets \( I_i = \{a_{(i-1)n+1}, \cdots, a_{in}\} \), with \( i = 1, \cdots, m \). The branch points in a given set \( I_i \) are connected together by branch lines as shown in fig. 1. As a convention, going around a branch point clockwise (counterclockwise) on the \( j \)-th sheet along a small circle surrounding the point, one encounters the \((j+1)\)-th \(((j-1)\)-th) sheet when crossing a branch line, where \( j = 0, \cdots, n - 1 \mod n \).

In order to describe the topologically nontrivial solutions of the Maxwell field theory on the algebraic curves 5 explicitly, the following relevant divisors are necessary \(^1\):

\[
[dz] = (n - 1) \sum_{p=1}^{nm} a_p - 2 \sum_{j=0}^{n-1} \infty_j
\]

\[
[y] = \sum_{p=1}^{nm} a_p - m \sum_{j=0}^{n-1} \infty_j
\]

(8)

In 8 the symbol \( \infty_j \) denotes the projection of the point \( z = \infty \) on the \( j \)-th sheet. Exploiting the above divisors, it is easy to see that the holomorphic

---

\(^1\)These divisors can be computed using the methods of refs. [12] and [15].
differentials $A_{i \xi}^{\text{bar}}$ and $A_{i \bar{\xi}}^{\text{bar}}$ correspond to linear combinations of the following holomorphic differentials:

$$\Omega_{k,j} dz = \frac{z^{j-1}}{y^{-k+n-1}} dz$$  \hspace{1cm} (9)

where

$$\begin{cases} \ j = 1, \ldots, (n-1)m - km - 1 \\ \ k = 0, \ldots, n - 2 \end{cases}$$

for $m > 1$ and

$$\begin{cases} \ j = 1, \ldots, n - k - 2 \\ \ k = 0, \ldots, n - 3 \end{cases}$$

for $m = 1$.

The calculation of the topologically nontrivial solutions $A_{I}^{\xi}$ is more complicated, for we need an explicit expression of the metric on $\Sigma$. This will be the subject of the next Section.

3 \hspace{0.5cm} METRIC TENSORS ON ALGEBRAIC CURVES

A simple class of conformally flat metrics is provided by tensors of the kind:

$$g_{z\bar{z}} dz d\bar{z} = \frac{dz d\bar{z}}{(y\bar{y})^{n-1}} \left[ 1 + f(z, y)\bar{f}(z, y) \right]^\alpha$$  \hspace{1cm} (10)

where $f(z, y)$ is a rational function of $z$ and $y$ and $\bar{f}(z, y)$ its complex conjugate. Let us notice that the parameter $\alpha$ can also be a rational number. However, it is clear that the term $\left[ 1 + f(z, y)\bar{f}(z, y) \right]^\alpha$ does not introduce further branches on $\mathbb{CP}_1$. Putting $\alpha = 0$ in equation 10, we obtain the degenerate metric

$$g_{1,z\bar{z}} dz d\bar{z} = \frac{dz d\bar{z}}{(y\bar{y})^{n-1}}$$

which has already been used in perturbative string theory. For $\alpha = \frac{(n-1)m-2}{m}$ and $f(z, y) = y(z)$, we have instead the nondegenerate metric

$$g_{2,z\bar{z}} dz d\bar{z} = \frac{dz d\bar{z}}{(y\bar{y})^{n-1}} \left[ 1 + y\bar{y} \right]^\alpha$$  \hspace{1cm} (11)

7
To compute the Ricci tensor $R_{z\bar{z}}$ corresponding to the above conformally flat metric, it is convenient to put
\[
\frac{1}{2}e^{2\sigma} = \frac{(1 + y\bar{y})^\alpha}{(y\bar{y})^{n-1}}
\]
In terms of $\sigma$, $R_{z\bar{z}}$ is defined as follows:
\[
R_{z\bar{z}} = -8\partial_z\partial_{\bar{z}}\sigma
\]
In the explicit calculation of $R_{z\bar{z}}$, the contributions to the total curvature coming from the term $\partial_z\partial_{\bar{z}}\log|y|$ and its complex conjugate are proportional to Dirac $\delta$–functions concentrated at the branch points and at infinity. However, these $\delta$–functions cancel in the final expression of the Ricci tensor because the metric $g_{z\bar{z}}$ is nondegenerate and so it does not have poles or zeros at those points. Thus, after a straightforward calculation, we obtain:
\[
R_{z\bar{z}} = -4\alpha \frac{\partial_z y \partial_{\bar{z}} \bar{y}}{[1 + y\bar{y}]^2}
\]
(12)
Another example of nondegenerate metric is provided by:
\[
g_{3,z\bar{z}} dzd\bar{z} = \frac{dzd\bar{z}}{(y\bar{y})^{n-1}} [1 + z\bar{z}]^\beta
\]
(13)
for $\beta = (n - 1)m - 2$. The Ricci tensor corresponding to this metric has a very simple form:
\[
R_{3,z\bar{z}} = -\frac{4\beta}{(1 + z\bar{z})^2}
\]
To verify that the above tensors represent real metrics on $\Sigma$, we will compute the Euler characteristic $\chi$ for the $Z_n$ symmetric algebraic curves. On a Riemann surface $M_g$ of genus $g$ equipped with a metric $G_{\mu\nu}$ and global complex coordinates $\xi$ and $\bar{\xi}$, $\chi$ is defined as follows:
\[
\chi = \frac{1}{4\pi} \int_{M_g} d^2\xi \sqrt{G} R
\]
where $R$ is the curvature scalar and $d^2\xi \sqrt{G} = \sqrt{G}d\xi \wedge \bar{\xi}$ is the volume form. As already mentioned, in the representation of Riemann surfaces in terms
of algebraic curves, \( z \) can be viewed as the mapping \( z : \xi \in \Sigma \to \mathbb{CP}_1 \). Therefore, the integral of a density \( L^{(l)}_{z\bar{z}}(z, \bar{z}) \) over \( \Sigma \) becomes:

\[
I = \int_{\Sigma} d^2z(\xi) L^{(l)}_{z\bar{z}}(z(\xi), \bar{z}(\bar{\xi}))
\]

To evaluate integrals of this kind, it is possible to use the Poincaré–Lelong equation [13], which expresses \( I \) as an integral over \( \mathbb{CP}_1 \otimes \mathbb{CP}_1 \):

\[
I = \frac{i}{\pi} \int_{\mathbb{CP}_1} d^2z \int_{\mathbb{CP}_1} d^2y \partial_y \partial_{\bar{y}} L^{(l)}_{z\bar{z}}(z, \bar{z}; y, \bar{y}) \log |F(z, y)|
\]  

(14)

where \( F(z, y) = y^n - \prod_{i=1}^{nm}(z - a_i) \). After an integration by parts in \( \partial_y \) and \( \partial_{\bar{y}} \), the latter formula simplifies to

\[
I = \sum_{l=0}^{n-1} \int_{\mathbb{CP}_1} d^2z L^{(l)}_{z\bar{z}}(z, \bar{z})
\]

(15)

This form of the Poincaré–Lelong equation is particularly convenient for our aims. It can be derived as in [13] exploiting the following Cauchy formula:

\[
\frac{1}{2\pi i} \partial_y \partial_{\bar{y}} \log |y - y^{(l)}(z)| = \delta_{y^{(l)}}(y, y^{(l)}(z))
\]

which is a straightforward consequence of the fact that \( F(z, y) \) can be rewritten as follows: \( F(z, y) = \prod_{l=0}^n(y - y^{(l)}(z)) \). Equivalently, if we consider \( y \) as the independent variable instead of \( z \), so that \( z(y) \) becomes a multivalued function with \( s = 1, \ldots, nm \) branches, we have:

\[
I = \sum_{s=0}^{nm-1} \int_{\mathbb{CP}_1} d^2y L^{(s)}_{y\bar{y}}(y, \bar{y})
\]

(16)

In the case of the Euler characteristic, taking into account for instance the metric \( g_{3,z\bar{z}} \) with Ricci tensor \( R_{3,z\bar{z}} \), the relevant integral to be computed is:

\[
\chi = -\frac{1}{4\pi} \int_{\Sigma} d^2z(\xi) \frac{4\beta}{(1 + z(\xi)z(\bar{\xi}))^2}
\]

(17)

Since the integrand in the above equation is not multivalued and \( z \) maps the curve into \( n \) copies of \( \mathbb{CP}_1 \), it is easy to realize that the right hand side of 17 is equivalent to the following integral over \( \mathbb{CP}_1 \):

\[
\chi = -\frac{n}{4\pi} \int_{\mathbb{CP}_1} d^2z \frac{4\beta}{(1 + z\bar{z})^2}
\]

(18)
This formula can also be directly verified using eq. 15. At this point it is sufficient to notice that the right hand side of 18 is proportional to the Euler characteristic of the complex sphere:

$$\chi_{\mathbb{CP}^1} = \frac{1}{\pi} \int_{\mathbb{CP}^1} \frac{d^2 z}{(1 + z\overline{z})^2} = 2$$  \hspace{1cm} (19)$$

Therefore, eq. 18 yields $$\chi = -4n\beta = (2n - nm(n - 1))$$. Comparing this value of $$\chi$$ with eq. 7, which gives the genus of $$\Sigma$$ in terms of $$n$$ and $$m$$, it is easy to see that $$\chi = 2 - 2g$$ as expected.

The computation of the Euler characteristic starting from the Ricci tensor $$R_{z\overline{z}}$$ of eq. 12 can be performed in an analogous way. In this case $$\chi$$ is given by

$$\chi = -4\alpha \int_\Sigma d^2 z(\xi) \frac{\partial_z y \partial_{\overline{z}} \overline{y}}{[1 + y\overline{y}]^2}$$ \hspace{1cm} (20)$$

To deal with the above integral it is convenient to consider $$y$$ as the independent variable and to solve eq. 5 with respect to $$z$$. Thus $$y$$ maps $$\Sigma$$ into $$nm$$ copies of $$\mathbb{CP}^1$$. With the help of eq. 16, eq. 20 becomes after a straightforward calculation:

$$\chi = -4\alpha \int_\Sigma d^2 y(\xi) \left[\frac{d^2 y(\xi)}{[1 + y(\xi)\overline{y}(\xi)]^2}\right] = -4\alpha nm \int_{\mathbb{CP}^1} d^2 y(\xi) \left[\frac{d^2 y(\xi)}{[1 + y(\xi)\overline{y}(\xi)]^2}\right]$$

Exploiting again eq. 19, we obtain the correct value of the Euler characteristic on a Riemann surface: $$\chi = 2 - 2g$$.

Let us finally notice that we can construct other metric tensors on an algebraic curve which are not of the form 10. For instance the tensor

$$\bar{g}_{z\overline{z}} dz d\overline{z} = \frac{e^{(y\overline{y})^{n-1}}}{e^{(y\overline{y})^{n-1}} - 1} [1 + z\overline{z}]^{\frac{2}{m}} dz d\overline{z}$$ \hspace{1cm} (21)$$

yields a nondegenerate metric, as it is easy to check exploiting the divisors 8. Moreover, all the metric tensors given above are characterized by the fact that they are not multivalued on $$\mathbb{CP}^1$$. In fact, they depend on $$y$$ only through the modulus of this function, which is branch independent. Nondegenerate metrics which are also multivalued can be for instance obtained multiplying eqs. 11, 13 and 21 by integer powers of the factor $$\frac{y}{\overline{y}} + \frac{\overline{y}}{y}$$ or by considering more complicated forms of the functions $$f(z, y)$$ in eq. 10.
4 SOLITONIC SECTORS OF THE MAXWELL FIELD THEORY

At this point, we are ready to derive the fields $A^I_\xi$ and $A^I_{\bar{\xi}}$ explicitly. On the algebraic curve, this is equivalent to solve the equation:

$$F_{z\bar{z}}dz \wedge d\bar{z} = \frac{i\Phi}{A} g_{z\bar{z}}dz \wedge d\bar{z}$$  \hspace{1cm} (22)

The difficulty of computing $A^I_\xi$ and $A^I_{\bar{\xi}}$ explicitly strongly depends on the choice of the metric $g_{z\bar{z}}$. For instance, in the formalism of theta functions, eq. 22 can be easily solved in the canonical $\theta$–metric (C$\theta$M) of [7]. However, in order to construct the C$\theta$M metric, we would need at least the explicit expression of the period matrix, which is not known on an algebraic curve. For these reasons, we will choose here metrics $g_{z\bar{z}}$ which are singlevalued on $CP_1$. Examples of nondegenerate metric tensors of this kind are provided by eqs. 11, 13 and 21. To solve eq. 22, we first define the following Green function:

$$G(z, w) = -\frac{1}{4\pi n} \log \left[ \frac{|z - w|^2}{(1 + z\bar{z})(1 + w\bar{w})} \right]$$

$G(z, w)$ is proportional to the inverse of the Laplace operator on $CP_1$ and it is a well defined Green function also on $\Sigma$, where it satisfies the equation

$$\partial_z \partial_{\bar{z}} G(z, w) = -\frac{1}{2n} \delta^{(2)}_{z\bar{z}}(z, w) + \frac{\gamma_{z\bar{z}}}{4\pi n A}$$

$\delta^{(2)}_{z\bar{z}}(z, w)$ is formally the usual Dirac $\delta$–function on the complex sphere, but we have to remember that on the algebraic curve $\delta^{(2)}_{z\bar{z}}(z, w)$ is singular at all the projections of the point $z = w$ on $\Sigma$.

Let us denote with $J(z, \bar{z})$ an external singlevalued scalar current on $CP_1$. Since $g_{z\bar{z}}$ is singlevalued, it is easy to show with the help of the Poincaré–Lelong formula that the following result is valid:

$$\partial_z \partial_{\bar{z}} \int_{\Sigma} d^2 w(\xi') G(w(\xi'), z(\xi')) J(w(\xi'), \bar{w}(\bar{\xi}')) g_{w\bar{w}} = -\frac{J(z, \bar{z})}{2} g_{z\bar{z}} +$$

$$\frac{\gamma_{z\bar{z}}}{4\pi n} \int_{\Sigma} d^2 w J(w, \bar{w}) g_{w\bar{w}}$$  \hspace{1cm} (23)
where $\gamma_{z\bar{z}} = \frac{1}{(1+z\bar{z})^2}$ is the standard metric on the complex sphere. We notice that $\gamma_{z\bar{z}}$ could be taken as a metric on $\Sigma$, but unfortunately it is degenerate because of its zeros at the branch points (see eq. 8). However, this fact will not be disturbing in the following calculations, since $\gamma_{z\bar{z}}$ will only appear as an auxiliary tensor.

Besides the Green function $G(z, w)$, we also introduce a gauge field $A^{sph}_\alpha$, $\alpha = z, \bar{z}$, defined in this way:

\begin{align*}
A^{sph}_z &= -\frac{A}{4\pi n} \partial_z \log(1 + z\bar{z}) \quad (24) \\
A^{sph}_{\bar{z}} &= \frac{A}{4\pi n} \partial_{\bar{z}} \log(1 + z\bar{z}) \quad (25)
\end{align*}

(26)

It is easy to check that the following relation

$$\partial_z A^{sph}_\bar{z} - \partial_{\bar{z}} A^{sph}_z = \frac{A\gamma_{z\bar{z}}}{4\pi n} \quad (27)$$

is satisfied over all the algebraic curve $\Sigma$ apart from the points $\infty_0, \ldots, \infty_{n-1}$. At those points, in fact, a $\delta$–function concentrated in $z = \infty$ appears in the right hand side of 27. To show this, it is just sufficient to perform the change of variables $z' = 1/z$ in eqs. 24–25 and to study the behavior of the right hand side in $z' = 0$. The problem of the appearance of $\delta$ functions can be solved as in the case of the Wu–Yang monopoles on the sphere by splitting the algebraic curve into two sets $\Sigma_S$ and $\Sigma_N$. The former contains all the projections of the point $z = 0$ but not those of the point $z = \infty$, while for $\Sigma_N$ the converse is true. Of course, there is a great arbitrariness in choosing the sets $\Sigma_S$ and $\Sigma_N$. To fix the ideas, we will define them as the two sets obtained by cutting the algebraic curve along the contour $\gamma$ shown in fig. 2. Thus $\Sigma_S$ encloses the projections $0, \ldots, 0_{n-1}$ of the point $z = 0$ and all the branch points apart from $a_{nm}$. Consequently, $\Sigma_N$ includes the points $\infty_0, \ldots, \infty_{n-1}$ and the branch point $a_{nm}$. The contour of fig. 2 is valid also in the case in which $z = 0$ is a branch point. In fact we can always put $a_1 = 0$, without any loss of generality. On a two sphere, the above decomposition corresponds to the usual decomposition in southern and northern hemisphere. The difference in the present case is that $\Sigma_S$ and $\Sigma_N$ are not isomorphic to $\mathbb{C}$. Despite of that, this way of covering the algebraic curve will be sufficient for our purposes as we will show below. Indeed, let us write the solution of
eq. 22 on $\Sigma_S$:

$$A_S^z = \frac{i\Phi}{A} \left[ \int_\Sigma d^2 w \partial_z G(z, w) g_{w\bar{w}} + A_{z}^{sph} \right]$$ (28)

$$A_S^\bar{z} = \frac{i\Phi}{A} \left[ - \int_\Sigma d^2 w \partial_{\bar{z}} G(z, w) g_{w\bar{w}} + A_{\bar{z}}^{sph} \right]$$ (29)

Exploiting eqs. 23 and 27, it turns out that:

$$\left( \partial_z A_S^\bar{z} - \partial_{\bar{z}} A_S^z \right) dz \wedge d\bar{z} = \frac{i\Phi}{A} g_{z\bar{z}} d\bar{z} \wedge dz$$ (30)

as desired. Analogous expressions of the solutions of equation 22 can be written on $\Sigma_N$. Using the coordinates $z' = 1/z$ and $\bar{z}' = 1/\bar{z}$ we have:

$$A_N^{z'} = \frac{i\Phi}{A} \left[ \int_\Sigma d^2 w' \partial_{z'} G(z', w') g_{w'\bar{w}'} + \tilde{A}_{z'}^{sph} \right]$$ (31)

$$A_N^{\bar{z}'} = \frac{i\Phi}{A} \left[ - \int_\Sigma d^2 w' \partial_{\bar{z}'} G(z', w') g_{w'\bar{w}'} + \tilde{A}_{\bar{z}'}^{sph} \right]$$ (32)

where

$$G(z', w') = -\frac{1}{4\pi n} \log \left[ \frac{|z' - w'|^2}{(1 + z'\bar{z}')(1 + w'\bar{w}')} \right]$$

and

$$\tilde{A}_{z'}^{sph} = -\frac{A}{4\pi n} \partial_{z'} \log(1 + z'\bar{z}')$$ (33)

$$\tilde{A}_{\bar{z}'}^{sph} = \frac{A}{4\pi n} \partial_{\bar{z}'} \log(1 + z'\bar{z}')$$ (34)

As for the fields $A_S^z$ and $A_S^\bar{z}$, it is easy to prove that the following relations are satisfied:

$$\partial_{z'} \tilde{A}_{\bar{z}'}^{sph} - \partial_{\bar{z}'} \tilde{A}_{z'}^{sph} = \frac{A\gamma_{z'\bar{z}'}}{4\pi n}$$ (36)

and

$$\left( \partial_{z'} A_N^{\bar{z}'} - \partial_{\bar{z}'} A_N^{z'} \right) = \frac{i\Phi}{A} g_{z'\bar{z}'} dz' \wedge d\bar{z}'$$ (37)

Equations 28–29 and 31–32 show the reasons for which we only need two sets to cover the Riemann surface $\Sigma$. First of all, the gauge fields $A_N^\alpha$ and $A_S^\alpha$ with $\alpha = z, \bar{z}$, are singlevalued on $\Sigma$. Secondly, they are everywhere well defined, i.e. free of singularities, apart from the projections on the algebraic curve of
the points \( z = 0, \infty \). As a consequence, the behavior of the gauge fields \( A^{N,S}_\alpha \) is not affected by the presence of the branch points and the splitting of \( \Sigma \) into two sets \( \Sigma_N \) and \( \Sigma_S \) is justified.

To be consistent, both fields \( A^N_\alpha \) and \( A^S_\alpha \) should describe the same magnetic field. Indeed, far from the points \( z = 0 \) and \( z = \infty \), where it is possible to use on any sheet of \( \Sigma \) both coordinates \( z(\xi) \) and \( z'(\xi) \), it is easy to see that the fields \( A^{N,S}_\alpha \) are related by a gauge transformation. Introducing the variable \( \Lambda = \frac{i \Phi}{2\pi n} \log \frac{z'}{\bar{z}'} \) we obtain in fact the following relations:

\[
\begin{align*}
A^S_\alpha d'z' &= A^N_\alpha d'z' + \partial_{z'} \Lambda dz' & (38) \\
A^S_\alpha \bar{z}' d\bar{z}' &= A^N_\alpha \bar{z}' d\bar{z}' + \partial_{\bar{z}'} \Lambda d\bar{z}' & (39)
\end{align*}
\]

Using local polar coordinates \( z' = \frac{1}{\rho} e^{-i\theta'} \), we have that

\[ \Lambda = \frac{\Phi \theta'}{2\pi n} \]

It is now easy to see that the transformations 38–39 can be reabsorbed by an \( U(1) \) gauge transformation corresponding to a group element \( U(z', \bar{z}') \) given by:

\[ U = e^{i\Lambda} = e^{i \frac{\Phi \theta'}{2\pi n}} \] (41)

An analogous result can be found working in \( z \) coordinates on \( \Sigma_S \). Since the fields \( A^N_\alpha \) and \( A^S_\alpha \) differ by an exact differential, the corresponding field strength \( F_{zz} \) is globally defined on \( \Sigma \) and we can compute the total magnetic flux associated to the gauge field configurations of eqs. 28–29 and 31–32. Using eqs. 30 and 37 we have:

\[ \int_{\Sigma} d^2 z F^I_{zz} = \int_{\Sigma_S} d^2 z F^S_{zz} + \int_{\Sigma_N} d^2 z' F^N_{zz'} = \Phi \]

as desired, where \( F^I_{zz} = \partial_z A^I_z - \partial_z A^I_z \) and \( I = N, S \) on \( \Sigma_{N,S} \).

Finally, let us check the conditions for which the group element \( U = e^{i \frac{\Phi \theta'}{2\pi n}} \) of eq. 41 is well defined, i.e. is singlevalued in the intersection between \( \Sigma_S \) and \( \Sigma_N \) along the path \( \gamma \). When \( U \) is transported \( N \) times along \( \gamma \), the angle \( \theta' \) undergoes the following shift:

\[ \theta' \to \theta' + 2\pi n N \]
as the contour $\gamma$ encircles all the projections of the point $z = \infty$ on the $n$ different sheets composing the algebraic curve. In order to ensure single-valuedness, the following condition on the total flux $\Phi$ should be imposed:

$$\Phi = 2\pi k \quad k = 0, \pm 1, \pm 2, \ldots$$

As a consequence, the solutions of eq. 22 provided by the gauge field configurations in 28–29 and 31–32 satisfy the relation:

$$2\pi k = \int_\Sigma d^2z F_{zz}$$

with integer values of $k$. Thus, as expected, we recover exactly the Dirac quantization of the flux. This result does not depend on the form of the contour $\gamma$. A curve which encircles all the projections of the points $z = 0$ and $z = \infty$ has either to cross at least $n$ branch lines as in fig. 2 or to be of the form given in fig. 3. In both cases, the total shift in the angle $\theta'$ has an $n$ factor in front which does not allow for fractional values of $k$ in eq. 42. Also the situation in which the point $z = 0$ is a branch point does not represent a great complication. Indeed, a contour encircling a branch point must be also of the kind given in figs. 2 and 3. In the latter case, one has to suppose that the paths which surround the branch point $z = 0$ on each sheet are builded in such a way to avoid any branch line, including those outgoing from the point $z = 0$. Moreover, if the contour $\gamma$ is very close to the branch point $z = 0$, we can use local uniformization coordinates $z = t^n$ and $\bar{z} = \bar{t}^n$. In these coordinates, assuming polar coordinates $t = \rho e^{i\theta}$, the group element $U$ of eq. 41 becomes:

$$U = e^{i\Phi/2\pi n}$$

When $U$ is transported around the branch point $N$ times, the variable $\theta$ shifts as follows: $\theta \rightarrow \theta + 2\pi N$. This gives again the Dirac quantization condition 42 of the total flux $\Phi$. Since the gauge fields defined in eqs. 28–29 and 31–32 are singlevalued on $\mathbb{C}P_1$, no problem arises when they are transported along the homology cycles.

\footnote{The case of a branch point at infinity can be treated in the same way of the branch point at $z = 0$. We will not discuss it here because branch points at infinity are impossible due to the form of the Weierstrass polynomial.}
5 CONCLUSIONS

In eqs. 9 and 28–32 we have computed all the nontrivial solutions of the Maxwell equations on a $\mathbb{Z}_n$ symmetric algebraic curve. The gauge fields configurations 28–29 and 31–32 have a simple form, as they are expressed through the scalar Green function on the complex sphere and are single-valued. Using these gauge potentials it is possible to write explicitly the Hamiltonian $H$ of an electron of mass $m$ in the presence of a constant magnetic field perpendicular to $\Sigma$. Remembering that $B = \frac{\Phi}{A}$, we have on $\Sigma_{N,S}$:

$$H = \frac{\Phi}{2mA}$$

(43)

with $P_\alpha = -i\partial_\alpha$, $\alpha = z, \bar{z}$. $g^{z\bar{z}}$ is the inverse of one of the singlevalued metrics given in Section 3. The (degenerate) ground state of $H$ is given by

$$\Psi_{N,S} = e^{-\int_{\bar{z}_0}^{\bar{z}} A_{N,S}^{N,S}d\bar{z}}\Psi_0$$

where $\Psi_0$ satisfies the relation: $\partial_\bar{z}\Psi_0 = 0$. Due to the Dirac quantization condition of the magnetic flux and the singlevaluedness of $A_{\alpha}^{N,S}$, $\Psi_{N,S}$ is a well defined quantum mechanical state. In particular, it is not periodic along the homology cycles. In fact, because of the singlevaluedness, the periods of the gauge potentials along any closed curve $C$ defined on $\Sigma$ are zero:

$$\int_{C_N} A_N^N d\bar{z} + \int_{C_S} A_S^S d\bar{z} = 0$$

Here $C_N$ and $C_S$ are the components of $C$ lying on $\Sigma_N$ and $\Sigma_S$ respectively. The way in which the topologically not trivial solutions have been obtained here makes use of the $\mathbb{Z}_n$ symmetry of the curves. As a matter of fact, the singularities in $z = 0$ and $z = \infty$ of the gauge potentials 28–29 and 31–32 appear symmetrically over all sheets of $\Sigma$. Therefore, our procedure is not suitable for general algebraic curves. However, the metrics given in Section 3 can be easily extended to any algebraic curve with Weierstrass polynomial:

$$F(z, y) = \sum_{i=0}^{n} y^i P_i(z)$$

(44)

where the $P_i(z)$ are polynomials in $z$. In this case, the metrics 10 takes the form:

$$g_{z\bar{z}}dzd\bar{z}\left|F_y\right|^2 \left(1 + f(z, y)\overline{f(z, y)}\right)^\alpha$$

16
with \( F_y(z, y) = \partial_y F(z, y) \). The values of the parameter \( \alpha \) depend on the form of the polynomial \( f(z, y) \) and of the function \( f(z, y) \). To determine \( \alpha \), one has to derive the divisors of \( dz, y \) and \( F_y \) as in eq. 8. For a large class of algebraic curves, such divisors can be found in refs. [12]. Analogously, the metric 21 becomes on a general algebraic curve:

\[
\tilde{g}_{\bar{z}z} dz d\bar{z} = \frac{e^{|F_y|^2}}{e^{|F_y|^2} - 1} [1 + z\bar{z}]^\beta dz d\bar{z} \tag{45}
\]

for suitable values of \( \beta \). The results of Section 3 allows us to write explicitly the Lagrangians of many field theories on algebraic curves. For instance, let us write the action for the scalar fields \( \varphi(z, \bar{z}; y^l(z), \bar{y}^l(z)) \) with mass \( \mu \):

\[
S = \sum_{l=0}^{n-1} \int_{\mathbb{CP}_1} d^2z \left[ \frac{1}{2} \left( d_z \phi^{(l)} d_{\bar{z}} \phi^{(l)} + \mu^2 g_{z\bar{z}}^{(l)} (\phi^{(l)})^2 + \lambda_1 R_z^{(l)} (\phi^{(l)})^2 + \lambda_2 R_{z\bar{z}}^{(l)} \phi^{(l)} \right) \right] \tag{46}
\]

Here \( g_{z\bar{z}} \) is a general metric on \( \Sigma \) with Ricci tensor \( R_{z\bar{z}} \), \( l = 0, \cdots, n-1 \) is the branch index and \( \lambda_1, \lambda_2 \) represent real parameters. Explicit examples of metrics and curvatures have been given above and in Section 3. Let us notice that the integration in the right hand side of eq. 46 is over \( \mathbb{CP}_1 \) after applying the Poincaré–Lelong equation but the integrand is multivalued. Moreover, \( d_z \) and \( d_{\bar{z}} \) are total derivatives with respect to the variables \( z \) and \( \bar{z} \). Total derivatives are used to remember that the fields \( \varphi \) depend on \( z, \bar{z} \) also through the functions \( y(z), y^l(z) \). Deriving the action 46 with respect to the field \( \varphi \) in a given branch \( l \), we find the equation of motion of the scalar fields:

\[-d_z d_{\bar{z}} \phi^{(l)} + \left( \mu^2 g^{(l)}_{z\bar{z}} + \lambda_1 R^{(l)}_{z\bar{z}} \right) \phi^{(l)} + \lambda_2 R^{(l)}_{z\bar{z}} = 0 \tag{47}\]

where

\[
d_z d_{\bar{z}} \varphi = \partial_z \partial_{\bar{z}} \varphi + \partial_{\bar{z}} \partial_y \varphi \frac{dy}{dz} + \partial_y \partial_z \varphi \frac{dy}{dz} + \partial_y \partial_{\bar{z}} \varphi \left| \frac{dy}{dz} \right|^2 \tag{48}\]

Locally and far from the branch points, it is possible to solve 47 with the standard methods of the theory of partial differential equations on the complex plane. In the case of the \( \mathbb{Z}_n \) symmetric curves we can even choose singlevalued metrics and curvatures, so that the coefficients appearing in the differential equation 47 are singlevalued. Nevertheless, any local solution
derived in this way is in general multivalued and needs to be analytically continued in order to extend it over the whole algebraic curve. Despite of the difficulties that may arise in the analytic continuation, the possibility of transforming differential equations on a Riemann surface in differential equations on the sphere is remarkable. Moreover, numerical calculations are allowed due the explicitness which is intrinsic in the representation of Riemann surfaces in terms $n$–sheeted coverings of the complex sphere.

6 ACKNOWLEDGEMENTS

The author would like to thank J. Sobczyk for participating in the preliminary stages of this work and for many helpful discussions. This work has been in part supported by the European Community, TMR grant ERB4001GT951315.

References


FIGURE CAPTIONS

1) A possible set of branch cuts on the complex sphere for the $Z_n$ symmetric algebra curves. The cuts appear symmetrically on the sheets composing the curve.

2) A possible covering of the curve $\Sigma$ in two sets $\Sigma_N$ and $\Sigma_N$. Only the part of the contour $\gamma$ which lies on the $i$-th sheet is showed.

3) An alternative form of the two sets $\Sigma_N$ and $\Sigma_S$. $\Sigma_S$ is disconnected in $n$ pieces lying on the different sheets. In the figure only the piece belonging to the $i$-th sheet has been given.