Perturbative gravitational couplings and Siegel modular forms in $D = 4, N = 2$ string models

G. L. Cardoso,$^a$*

$^a$Theory Division, CERN, CH-1211 Geneva 23, Switzerland

We consider four-parameter $D = 4, N = 2$ string models with Hodge numbers $(4, 214 - 12n)$ and $(4, 148)$, and we express their perturbative Wilsonian gravitational coupling $F_1$ in terms of Siegel modular forms.

1. Introduction

String-string dualities between heterotic strings on $K3 \times T_2$ and corresponding type II strings on suitably chosen Calabi–Yau threefolds in four space-time dimensions have been successfully tested in [1]–[11].

One interesting aspect in the study of these $N = 2$ string-string dualities in four space-time dimensions is the appearance [6] of certain modular forms in the low-energy effective action of these theories. Here, we will focus on $D = 4, N = 2$ four-parameter string models with Hodge numbers $(4, 214 - 12n)$ and $(4, 148)$, and we will express their perturbative Wilsonian gravitational coupling $F_1$ in terms of Siegel modular forms. For the models with Hodge numbers $(4, 214 - 12n)$, this has been achieved in [10], and the results presented below for these models represent a short summary of the ones contained in [10].

2. $D = 4, N = 2$ four-parameter string models

The four-parameter models we will be considering in the following have a perturbative heterotic description in terms of compactifications of the $E_8^{(1)} \times E_8^{(2)}$ heterotic string on $K3 \times T_2$. They also have a dual type II description in terms of compactifications of the type II string on Calabi–Yau threefolds which are $K3$ fibrations [3,12].

We will consider two classes of models. The first class of models can be obtained, in the type II description, by compactifying on Calabi–Yau threefolds with Hodge numbers $(h^{1,1}, h^{2,1}) = (4, 214 - 12n)$ and, consequently, Euler number $\chi = 24n - 420$. We will restrict the discussion to $n = 0, 1, 2$. These models have a perturbative heterotic description in terms of compactifications of the heterotic $E_8^{(1)} \times E_8^{(2)}$ string on $K3 \times T_2$ with $SU(2)$ gauge bundles on the $K3$ with instanton numbers $(d_1, d_2) = (12 - n, 12 + n)$ [1,13,14]. In the heterotic description, the four moduli comprise the dilaton $S$, the two toroidal moduli $T$ and $U$ as well as a Wilson line modulus $V$. The massless spectrum thus contains $N_V = 5$ vector multiplets (here we also count the graviphoton) as well as $N_H = 215 - 12n$ neutral hyper multiplets. At the transition surface $V = 0$, the $U(1)_V$ associated with the Wilson line modulus becomes enhanced to an $SU(2)$. Let $N_V = 2$ and $N_H = 12n + 32$ denote the additional vector and hyper multiplets becoming massless at $V = 0$. Then, as the $SU(2)$ gauge bosons get swallowed, the transition to an $S$-$T$-$U$ model with $N_V = 4$ vector and $N_H = 244$ neutral hyper multiplets takes place [15].

The second class of models can be obtained, in the type II description, by compactifying on Calabi–Yau threefolds with Hodge numbers $(h^{1,1}, h^{2,1}) = (4, 148)$ and, consequently, Euler number $\chi = -288$. These Calabi–Yau threefolds are actually also elliptic, that is, they are elliptically fibred with base $F_n$ ($n = 0, 1, 2$), and their elliptic fibre is of the $E_7$ type [16,17]. These models also have a perturbative heterotic description, apparently in terms of compactifications of the heterotic $E_8^{(1)} \times E_8^{(2)}$ string on $K3 \times T_2$ with
The condition of the moduli space \( \mathcal{H}_2 \) of complex dimension 3.

The term \( p(T, U, V) \) denotes a cubic polynomial, which is generated at one-loop, and which depends on the particular instanton embedding.

The coefficients \( c(4tkl - b^2) \) appearing in the prepotential are, presumably, the expansion coefficients of a Jacobi form \( f(\tau, z) \) of weight \(-2\) and index \( t \)

\[
f(\tau, z) = \sum_{n \geq -1, \ell \in \mathbb{Z}} c(4tm - l^2)q^m r^l, \quad (2)
\]

where \( q = \exp(2\pi it), r = \exp(2\pi iz) \). For the case \( t = 1 \), the appropriate Jacobi form of weight \(-2\) and index 1 is given by \([9, 10]\)

\[
f(\tau, z) = \frac{1}{\eta^4} \left( \frac{12 - n}{24} E_6 E_{4,1} + \frac{12 + n}{24} E_4 E_{6,1} \right). \quad (3)
\]

It would be interesting to find the appropriate Jacobi form for the \( t = 2 \) case.

It can be checked, by comparison with the corresponding type \( \Pi \) prepotential, that the expansion coefficients \( c(4tkl - b^2) \) appearing in (1) indeed only depend on the combination \( 4tkl - b^2 \). In order to do so, one has to consider the contributions of the world sheet instantons to the type \( \Pi \) prepotential which, for four-parameter models, are generically given by

\[
F_{\text{inst}}^{\Pi} = -\frac{1}{(2\pi)^3} \sum_{d_1, d_2, d_3, d_4} n_{d_1, d_2, d_3, d_4}^t \text{Li}_3(\prod_{i=1}^4 q_i^{d_i}), \quad (4)
\]

where \( q_i = \exp(-2\pi t_i) \). The \( n_{d_1, d_2, d_3, d_4}^t \) denote the rational instanton numbers. In order to match (4) with (1), we will perform the following identification of the type \( \Pi \) \( K\ddot{a}hler \) class moduli \( t_1 \) with the heterotic moduli \( \mathcal{S}, T, U \) and \( V \) \([10]\)

\[
t_1 = U - 2V, \quad t_2 = S - \frac{n}{2} T - (1 - \frac{n}{2}) U, \quad (5)
\]

which is valid in the chamber \( T > U > 2V \). Then, the heterotic weak coupling limit \( S \rightarrow \infty \) corresponds to taking the large \( K\ddot{a}hler \) class limit \( t_2 \rightarrow \infty \). In this limit, only the instanton numbers with \( d_2 = 0 \) contribute in the above sum (4). Using the identification \( kT + lU + bV = d_1 t_1 + d_3 t_3 + d_4 t_4 \),
it follows that (independently of $n$)
\[
k = d_3, \\
l = d_1 - d_3, \\
b = d_4 - 2d_1.
\]

Then, (4) turns into
\[
F^{II}_{\text{inst}} = -\frac{1}{(2\pi)^3} \sum_{k,l,b} n_{k,l,b}^c Li_3(e^{-2\pi(kT+U+bV)}),
\]

Comparison with (1) shows that the rational instanton numbers should be related to the expansion coefficients $c(4tkl-b^2)$ by
\[
n_{k,l,b}^c = -2c(4tkl-b^2),
\]
in which case they would have to satisfy the following non-trivial constraint
\[
n_{k,l,b}^c = n^r(4tkl-b^2).
\]

It can now be checked that the constraint (9) is indeed satisfied for the models discussed here. This, in particular, also holds for the model\footnote{We would like to thank P. Mayr for providing us with the intersection numbers and the rational instanton numbers for the $X^{(4)}$ model in the $K3$ phase.} with Hodge numbers $(4,148)$ based on the elliptically fibred Calabi–Yau threefold with $E_7$ fibre $P^{1,1,2}[4]$ and base $F_1$ (denoted by $X^{(4)}$ in [17]). By inspection of the rational instanton numbers for some of the four-parameter models under consideration, it can also be inferred that
\[
c(N) = 0, \quad N > -4t, \\
c(-4t) = 1, \quad c(-1) = -\frac{1}{2}N_H', \quad c(0) = \frac{1}{2}N.
\]

For the $X^{(4)}$ model ($t = 2$), for instance, one finds that
\[
c(-8) = 1, \quad c(-4) = 0, \\
c(-1) = -48, \quad c(0) = -144.
\]

The truncation to a three-parameter Calabi–Yau model is made by setting $t_4 = 0$. The instanton numbers $n_{k,l}^c$ of the three-parameter ($S-T-U$) model are then given by [15]
\[
n_{k,l}^c = \sum_b n^r(4tkl-b^2),
\]

where the summation range over $b$ is finite. For example, when considering the transition from the four-parameter model $X^{(4)}$ to the three-parameter Calabi–Yau model, $n_{0,1}^c = 96 + 288 = 384$. For $n > 0$, $F^{\text{het}}_{\text{cubic}}$ reduces to the cubic prepotential of the $S-T-U$ model [21,20].

Next, consider the models with Hodge numbers $(4,148)$. For the model with base $F_1$, the cubic part of the type II prepotential is, in the $K3$ phase ($c_2(J) = (92,24,36,88)$), given by
\[
F^{II}_{\text{cubic}} = t_2(t_1^2 + t_1t_3 + 4t_1t_4 + 2t_3t_4 + 3t_4^2) \\
+ \frac{4}{3}t_1^3 + 8t_1^2t_4 + \frac{n}{2}t_1t_2 + (1 + \frac{n}{2})t_1^2t_3 \\
+ 2(n + 2)t_1t_3t_4 + nt_3^2t_4 + (14 - n)t_1t_2^2 \\
+ (4 + n)t_3t_4^2 + (8 - n)t_3^2.
\]

Then, inserting (5) into (13) yields
\[
F^{II}_{\text{cubic}} = -F^{\text{het}}_{\text{cubic}} = S(TU - V^2) + \frac{1}{3}U^3 \\
+ (\frac{4}{3} + n)V^3 - (1 + \frac{n}{2})UV^2 - \frac{n}{2}TV^2.
\]

For $V = 0$, $F^{\text{het}}_{\text{cubic}}$ precisely reduces to the cubic prepotential of the $S-T-U$ models [21,20].

It can be checked that, for $t_2 = 0$, the above turns into the prepotential of the three-parameter model with base $F_1$ [20]. Inserting (5) (with $n = 1$) into (15) yields
\[
F^{II}_{\text{cubic}} = -F^{\text{het}}_{\text{cubic}} = S(TU - V^2) + \frac{1}{3}U^3 \\
+ 8V^3 - 4UV^2 - 2TV^2.
\]
4. BPS orbits

An important role in the computation of the Wilsonian gravitational coupling $F_1$ is played by BPS states [22,23,6,7],

$$F_1 \propto \log \mathcal{M},$$

where $\mathcal{M}$ denotes the moduli-dependent holomorphic mass of an $N = 2$ BPS state. For the heterotic $S$-$T$-$U$-$V$ models under consideration, the tree-level mass $\mathcal{M}$ is given by [24]–[26]

$$\mathcal{M} = m_2 - im_1 U + im_1 T + n_2(UT + tV^2) + ibV.$$  \hspace{1cm} (18)

Here, $l = (n_1, m_1, n_2, m_2, b)$ denotes the set of integral quantum numbers carried by the BPS state. The level matching condition for a BPS state reads

$$p_L^2 - p_R^2 = 2n^T m + \frac{1}{2t}b^2,$$

where

$$p_R = \frac{|\mathcal{M}|^2}{2Y}, \quad Y = \text{Re} T \text{Re} U - t(\text{Re} V)^2 > 0.$$  \hspace{1cm} (20)

Let us consider states with $n_i = m_j = 0$. For $t = 1$, there are two states with $p_L^2 = 2, p_R^2 = 0$, carrying $b = \pm 2$. These two states are the additional vector multiplets which enhance the $U(1)_V$ to $SU(2)$ at $V = 0$. On the other hand, for $t = 2$, there are no such states with $p_L^2 = 2, p_R^2 = 0$, reflecting the fact that in this case there is no gauge symmetry enhancement of the $U(1)_V$ at $V = 0$.

Note that the Narain lattice $L$ of signature $(3,2)$ associated with (19) is given by $L = \Lambda \oplus U(-1)$, where $U(-1)$ denotes the hyperbolic plane

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and where $\Lambda = U(-1) \oplus V < 2t > = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2t \end{pmatrix}$ in a basis which we will denote by $(f_2, f_{-2}, f_3)$ [19]. In the basis

$$j_1 = f_{-2} + f_2, j_3 = f_{-2}, j_4 = 2f_2 + 2f_{-2} - f_3,$$

$L$ is equivalent to the matrix

$$I = \begin{pmatrix} -2 & -1 & -4 \\ -1 & 0 & -2 \\ -4 & -2 & -8 + 2t \end{pmatrix}.$$  \hspace{1cm} (21)

Inspection of (13) and (15) shows that the matrix $-I$ is nothing but the intersection matrix of the $K3$ fibre of the Calabi–Yau threefolds (13) and (15) for $t = 1$ and $t = 2$, respectively.

Of special relevance to the computation of perturbative corrections to $F_1$ are those BPS states, whose tree-level mass vanishes at certain surfaces in the classical moduli space. These surfaces are surfaces in the Siegel modular threefold $\mathcal{A}_t = \Gamma_t \backslash H_2$. Here, $\Gamma_t$ denotes the target-space duality group, which is a paramodular group, $\Gamma_t \subset Sp_4(Q)$ [19]. For $t = 1$, $\Gamma_1 = Sp_4(Z)$. The condition $\mathcal{M} = 0$ is the condition for a rational quadratic divisor $H_1$ [19] of discriminant

$$D(l) = 2t(p_L^2 - p_R^2) = 4tn_1n_1 + 4tn_2m_2 + b^2.$$  \hspace{1cm} (22)

The divisors $H_l$ with discriminant $D(l)$ determine the Humbert surface $H_D$ in the Siegel modular threefold $\mathcal{A}_t$ [19]. Each Humbert surface $H_D$ can be represented by a linear relation in $T, U$ and $V$ [19]. The Humbert surface $H_1$ corresponds to the surface $V = 0$. States becoming massless at $V = 0$ lay on the orbit $D(l) = 1$, that is, on the orbit $n^T m = 0, b^2 = 1$. The Humbert surface $H_{4t}$, on the other hand, corresponds to the surface $T = U$. The states becoming massless on $H_{4t}$ carry quantum numbers $n^T m = 1, b = 0$. They carry $p_L^2 = 2, p_R^2 = 0$, and hence, they correspond to the additional vector multiplets describing the enhancement of $U(1)_T \times U(1)_U$ [24].

5. The heterotic perturbative Wilsonian gravitational coupling $F_1$

The heterotic perturbative Wilsonian gravitational coupling $F_1$ for an $S$-$T$-$U$-$V$ model is, up to ambiguities linear in $T, U$ and $V$, given by [9,10]

$$F_1 = 24S,$$

$$-\frac{2}{\pi} \sum_{k,l,b \in \mathbb{Z}} d(4tkl - b^2)L_i(e^{-2\pi(kT + lU + bV)}),$$

where the condition $(k, l, b) > 0$ means again that: either $k > 0, l, b \in \mathbb{Z}$ or $k = 0, l > 0, b \in \mathbb{Z}$ or $k = l = 0, b > 0$. The expansion coefficients $d(4TN) \ (N = kl - \frac{b^2}{4\pi})$ are determined in terms of
\[ F(\tau) = \sum_{N \in \mathbb{Z}, z+\frac{4t-1}{12}} c(4tN)q^N, \]
\[ E_2 F(\tau) = \sum_{N \in \mathbb{Z}, z+\frac{4t-1}{12}} d(4tN)q^N. \] (24)

Here, the \(c(4tN)\) denote the expansion coefficients occurring in the prepotential (1). The quantity \(E_2 F(\tau)\) should be related to the gravitational threshold corrections, as follows [27,10],
\[ \tilde{I}_{3,2} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left[ Z_{3,2} F(\tau) E_2 - \frac{3}{\pi \tau_2} - d(0) \right]. \] (25)

The Wilsonian coupling \(F_1\) should, on the other hand, also be expressible in terms of target-space duality modular forms, namely Siegel modular forms on \(\mathcal{A}_4 = \Gamma / \mathcal{H}_2\). Below, we will rewrite \(F_1\) in terms of Siegel modular forms.

### 5.1. The S-T-U-V models with \((N_V, N_H) = (5, 215 - 12n)\)

For this class of models, the target space duality group is \(\Gamma_1 = S_p l(Z)\). The relevant Humbert surfaces are \(H_1\) and \(H_4\). The Siegel modular forms which vanish on \(H_1\) and \(H_4\) are given as follows [28]. The Siegel modular form which vanishes on the \(T = U\) locus and has modular weight 0 is given by \(C_{30}^{12}\). It can be shown that, as \(V \to 0\),
\[ \frac{C_{30}^{12}}{C_{12}} \to (j(T) - j(U))^2, \] (26)
up to a normalization constant. On the other hand, the Siegel modular form which vanishes on the \(V = 0\) locus and has modular weight 0 is given by \(C_{12}^{12}\). It can be shown that, as \(V \to 0\),
\[ C_5 \to V (\eta^{24}(T)\eta^{24}(U))^\frac{1}{2}, \] (27)
up to a proportionality constant. Here, the Siegel form \(C_{12}\) is a modular form of weight 12 which generalises \(\eta^{24}(T)\eta^{24}(U)\), that is
\[ \mathcal{C}_{12} \to \eta^{24}(T)\eta^{24}(U) \] (28)
as \(V \to 0\).

The Siegel modular forms \(C_5\) and \(C_{30}\) have infinite product expansions, as follows [28],
\[ C_5 = (q^{r-1}s)^{1/2} \prod_{k \neq \mathbb{Z}} (1 - q^{k-r}b_s s^{f(4k \cdot b^2)}), \]
\[ C_{30} = (q^{3r-s}r^{1/2}(q - s) \times \prod_{k \neq \mathbb{Z}} (1 - q^{k-r}b_s s^{f(4k \cdot b^2)}), \] (29)
where \(q = \exp(-2\pi T), s = \exp(-2\pi U)\) and \(r = \exp(-2\pi V)\). The coefficients \(f(4k \cdot b^2)\) and \(f'_2(4k \cdot b^2)\) are defined as follows [28]. Consider the expansion of the weak Jacobi form of weight 0 and index 1
\[ \phi_{0,1} = \frac{\phi_{12,1}}{\eta^{24}} = \sum_{m \geq 0, l \in \mathbb{Z}} f(4m - l^2)q^m r^l \]
\[ = (r + 10 + r^{-1}) \]
\[ + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2) \]
\[ + q^2(\ldots), \] (30)
where the sum over \(l\) is restricted to \(4n - l^2 \geq -1\). Then, \(f(N) = f(4m - l^2)\) if \(N = 4m - l^2 \geq -1\), and \(f(N) = 0\) otherwise. The coefficients \(f'_{2}(N)\) are then given by \(f'_{2}(N) = 8f(4N) + 2(\frac{N}{2} - \frac{N^2}{4} - 3)f(N) + f(\frac{N}{2})\). Here, \(\left(\frac{2}{q}\right) = 1, -1, 0\) depending on whether \(D = 1 \mod 8, 5 \mod 8, 0 \mod 2\).

Then, in analogy to the perturbative Wilsonian gravitational coupling \(F_1\) for the S-T-U model [5,7]
\[ F_1 = 24S_{inv} - \frac{b_{grav}}{24\pi} \log \eta^{24}(T)\eta^{24}(U) \]
\[ + \frac{2}{\pi} \log(j(T) - j(U)), \] (31)
the perturbative Wilsonian gravitational coupling for an S-T-U-V model should now be given by (in the chamber \(T > U > 2V\)) [10]
\[ F_1 = 24S_{inv} - \frac{b_{grav}}{24\pi} \log C_{12} + \frac{1}{\pi} \log \frac{C_{30}^{12}}{C_{12}} \]
\[ - \frac{1}{2\pi} (N'_H - N'_V) \log \left( \frac{C_5}{C_{12}^{12}} \right)^2. \] (32)
Here, \(N'_V = 2\) and \(N'_H = 12n + 32\) denote the vector and the hyper multiplets which become massless at the \(V = 0\) locus.
The invariant dilaton $S_{\text{inv}}$ is given by [29]

$$S_{\text{inv}} = \tilde{S} + \frac{1}{10} L,$$

$$\tilde{S} = S - \frac{4}{10} \left( \partial_T \partial_U - \frac{1}{4} \partial_V^2 \right) h ,$$  \hspace{1cm} (33)

where the role of the quantity $L$ is to render $S_{\text{inv}}$ free of singularities. It can be shown [10] that

$$\tilde{S} = S + \frac{1}{5\pi} \log \frac{C_{30}}{C_{12}}$$

$$\quad - \frac{3}{10\pi (2+n)} \log \left( \frac{C_5}{C_{12}^{5/12}} \right)^2 + \text{regular}$$

and, hence,

$$L = -\frac{2}{\pi} \log \frac{C_{30}}{C_{12}} + \frac{3}{\pi} (2+n) \log \left( \frac{C_5}{C_{12}^{5/12}} \right)^2 .$$  \hspace{1cm} (35)

It follows that the Wilsonian gravitational coupling (32) can be rewritten as

$$F_1 = 24\tilde{S}$$

$$\quad - \frac{1}{\pi} \left[ \frac{19}{5} \log C_{30} + \frac{3}{5} \left( 1 - 2n \right) \log C_5 \right] .$$  \hspace{1cm} (36)

Note that the $\log C_{12}$ terms have completely canceled out.

Comparison of (23) and (36)

$$F_1 = 24S - \frac{2}{\pi} \sum_{(k,l,b)>0} d_n(4kl-b^2)L_{i1}$$

$$\quad = 24\tilde{S} - \frac{1}{\pi} \left[ \frac{19}{5} \log C_{30} + \frac{3}{5} \left( 1 - 2n \right) \log C_5 \right]$$

gives a highly non-trivial consistency check on (24) (here, we have ignored the issue of ambiguities linear in $T, U$ and $V$). It yields, using the product expansions for $C_5$ and $C_{30}$ given in (29), that

$$d_n(N) = \frac{6}{5} N c_n(N) - \frac{19}{5} f'_n(N)$$

$$\quad - \frac{3}{5} \left( 1 - 2n \right) f(N) ,$$  \hspace{1cm} (38)

where $N = 4kl-b^2 \in 4\mathbb{Z}, 4\mathbb{Z} + 3$. In order to show that (38) really holds, consider introducing the hatted quantities [9,10]

$$\tilde{Z} = \frac{1}{72} \left( E_4^2 E_{1,1} - E_6 E_{6,1} \right) / \Delta,$$

$$J_C = \frac{2 E_6 E_{6,1}}{\Delta} + 81 \tilde{Z} ,$$  \hspace{1cm} (39)

as well as

$$\tilde{Z}(\tau) = \tilde{Z}(4\tau) = 2 \sum_{N \in 4\mathbb{Z} or 4\mathbb{Z} + 3} f(N) q^N ,$$  \hspace{1cm} (40)

$$J_C(\tau) = J_C(4\tau) = \sum_{N \in 4\mathbb{Z}, 4\mathbb{Z} + 3} c_J(N) q^N .$$  \hspace{1cm} (41)

Then, it can be verified that

$$f'_n(N) = \frac{1}{2} c_J(N) + 6 f(N) .$$  \hspace{1cm} (42)

It can also be shown that, for $m = 4, 6$ [9]

$$\Theta_q E_m = \frac{m}{12} ( E_2 E_m - E_{m+2} ) ,$$

$$\Theta_q E_{m,1} = \frac{2m-1}{24} \left( E_2 \hat{E}_{m,1} - \hat{E}_{m+2,1} \right) ,$$

$$\Theta_q \hat{E}_{m,1} = \frac{2m-1}{6} \left( \hat{E}_2 \hat{E}_{m,1} - \hat{E}_{m+2,1} \right) ,$$  \hspace{1cm} (43)

where

$$\hat{E}_2(\tau) = E_2(4\tau) , \quad \hat{E}_{m,1}(\tau) = \hat{E}_{m,1}(4\tau) ,$$  \hspace{1cm} (44)

and where $\Theta_q = q^d$. Then, by using (42) as well as (43), it can be shown that (38) indeed holds [10].

5.2. The $S-T-U-V$ models with $(N_V, N_H) = (5, 149)$

For this class of models, the target-space duality group is $\Gamma_2$ [19]. Thus, the appropriate Siegel modular forms for this class of models will be different from the ones considered in the previous subsection. Of relevance for the following are going to be the Siegel modular forms with divisors $H_1$ and $H_8$, denoted by $\Delta_2$ and $\Psi^{(2)}_{12}$ in [19], with modular weight 2 and 12, respectively. Both $\Delta_2$ and $\Psi^{(2)}_{12}$ enjoy infinite product expansions, as follows [19],

$$\Delta_2 = q^{1/4} r^{-1/2} s^{1/2} \times \prod_{k,l,b \in \mathbb{Z}} \left( 1 - q^{k^2} r^{-1} s^{1/2} \right) f_2(8kl-b^2) ,$$

$$\Psi^{(2)}_{12} = q \prod_{k,l,b \in \mathbb{Z}} \left( 1 - q^{k^2} r^{-1} s^{1/2} \right) \left. f_2(8kl-b^2) \right|_{r=0} ,$$  \hspace{1cm} (45)

where $q = \exp(-2\pi T), s = \exp(-2\pi U)$ and $r = \exp(-2\pi V)$. The expansion coefficients $f_2(8kl-$
are the expansion coefficients of the following weak Jacobi form of weight 0 and index 2 [19]
\[
\phi_{0,2} = \frac{\phi_{2,2}}{\eta^4} = \sum_{m \geq 0} \sum_{l \in \mathbb{Z}} f_2(8m - l^2)q^m r^l
\]
= \left( r + 4 + r^{-1} \right) + q(r^{\pm 3} - 8r^{\pm 2} - r^{\pm 1} + 16) + q^2(\ldots),
\]
whereas the expansion coefficients \( c_2(8kl - b^2) \) are the expansion coefficients of the nearly holomorphic Jacobi form of weight 0 and index 2 [19]
\[
\Psi_{0,2} = \frac{E_{12,2}}{\eta^{24}} = \sum_{m \geq -1} \sum_{l \in \mathbb{Z}} c_2(8m - l^2)q^m r^l
\]
= \left( r^{-1} + 24 + q(\ldots) \right). \quad (47)
\]
Finally, we will denote the analogue of \( C_{12} \) by \( X_{12} \). The Siegel form \( X_{12} \) should be such that it doesn’t vanish in the interior of moduli space, and that it reduces to \( r^{24}(T)\eta^{24}(U) \) in the limit \( V \to 0 \). Below, we will see that \( X_{12} \) will in the end drop out of the relevant expressions.

Thus, the Siegel modular form which vanishes on the \( T = U \) locus and has modular weight 0 is given by \( \Psi_{0,2}^{(2)} / X_{12} \). As \( V \to 0 \),
\[
\frac{\Psi_{12}^{(2)}}{X_{12}} \to T - U \quad , \quad (48)
\]
up to a normalization constant. On the other hand, the Siegel modular form which vanishes on the \( V = 0 \) locus and has modular weight 0 is given by \( \Delta_{12}^{\delta} / X_{12} \). As \( V \to 0 \),
\[
\frac{\Delta_{12}^{\delta}}{X_{12}} \to V^6 \quad , \quad (49)
\]
up to a proportionality constant.

The perturbative Wilsonian gravitational coupling should then, in analogy to (32), be given by (in the chamber \( T > U > 2V \))
\[
F_1 = 24S_{\text{inv}} - \frac{b_{\text{grav}}}{24\pi} \log X_{12} + \frac{2}{\pi} \log \left( \frac{\Psi_{12}^{(2)}}{X_{12}} \right) - \frac{1}{6\pi} \left( N_H' - N_V' \right) \log \left( \frac{\Delta_{12}^{\delta}}{X_{12}} \right) . \quad (50)
\]
Using that \( b_{\text{grav}} = 336, N_V' = 0 \) and \( N_H' = 96 \), it follows that
\[
F_1 = 24S_{\text{inv}} + \frac{2}{\pi} \log \Psi_{12}^{(2)} - \frac{96}{\pi} \log \Delta_2 \quad . \quad (51)
\]
Note that, in contrast to (32), the dependence on \( X_{12} \) already drops out when using \( S_{\text{inv}} \).

The invariant dilaton \( S_{\text{inv}} \) is given by [29]
\[
S_{\text{inv}} = \tilde{S} + \frac{1}{10} L ,
\]
\[
\tilde{S} = S - \frac{4}{10} \left( \partial_T \partial_U - \frac{1}{8} \partial_V^2 \right) h \quad , \quad (52)
\]
where the role of the quantity \( L \) is to render \( S_{\text{inv}} \) free of singularities. Using (1), it follows that, as \( T \to U \),
\[
\partial_T \partial_U h = -\frac{1}{\pi} \log(T - U) = -\frac{1}{\pi} \log \left( \frac{\Psi_{12}^{(2)}}{X_{12}} \right) + \text{regular} \quad (53)
\]
and that, as \( V \to 0 \),
\[
-\partial_V^2 h = -\frac{1}{\pi} c(-1) \log V = -\frac{1}{12\pi} N_H' \log \left( \frac{\Delta_{12}^{6}}{X_{12}} \right) + \text{regular} \quad . \quad (54)
\]
Hence,
\[
\tilde{S} = S - \frac{4}{10\pi} \left( -\log \Psi_{12}^{(2)} + 6 \log \Delta_2 \right) + \text{regular} \quad (55)
\]
and
\[
L = -\frac{4}{\pi} \left( \log \Psi_{12}^{(2)} - 6 \log \Delta_2 \right) \quad . \quad (56)
\]
It follows that
\[
F_1 = 24 \tilde{S} - \frac{1}{\pi} \sum_{k,l,b \in \mathbb{Z}} \sum_{(k,l,b) > 0} d(8kl - b^2) Li_1
\]
\[
= 24 \tilde{S} - \frac{1}{\pi} \left[ \frac{38}{5} \log \Psi_{12}^{(2)} + \frac{192}{5} \log \Delta_2 \right] \quad (57)
\]
Note that \( X_{12} \) has again cancelled out.

Comparison of (23) and (57)
\[
F_1 = 24S - \frac{2}{\pi} \sum_{k,l,b \in \mathbb{Z}} \sum_{(k,l,b) > 0} d(8kl - b^2) Li_1
\]
\[
= 24S - \frac{1}{\pi} \left[ \frac{38}{5} \log \Psi_{12}^{(2)} + \frac{192}{5} \log \Delta_2 \right] \quad (58)
\]
gives again a highly non-trivial consistency check on (24) (here, we have again ignored the issue of ambiguities linear in \( T, U \) and \( V \)). It yields, using the product expansions for \( \Delta_2 \) and \( \Psi_{12}^{(2)} \) given in (45), that
\[
-2d(N) = \frac{6}{5} N c(N) + \frac{38}{5} c_2(N) + \frac{192}{5} f_2(N) , \quad (59)
\]
where $N = 8k - b^2 \in 8\mathbb{Z}, 8\mathbb{Z} + 7$. This relation can be explicitly checked for some of the coefficients. Consider, for instance, the cases where $N = -8, -1, 0$. Then, using that
\[
d(-8) = c(-8) = 1 , \quad d(-1) = c(-1) = -48 , \\
d(0) = c(0) - 24c(-8) = -168 
\]
as well as
\[
f_2(-1) = 1 , \quad f_2(0) = 4 , \\
c_2(-8) = 1 , \quad c_2(0) = 24 , 
\]
it is straightforward to check that (59) indeed holds.

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