Gluon fragmentation to $^{3D_J}$ quarkonia

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Abstract

We present a calculation of the leading order QCD fragmentation functions for gluons to split into spin-triplet D-wave quarkonia. We apply them to evaluate the gluon fragmentation contributions to inclusive $^{3D_J}$ quarkonium production at large transverse momentum processes like the Tevatron and find that the D-wave quarkonia, especially the charmonium $2^{--}$ state, could be observed through color-octet mechanism with present luminosity. Since there are distinctively large gaps between the contributions of two different (i.e., color-singlet and color-octet) quarkonium production mechanisms, our results may stand as a unique test to NRQCD color-octet quarkonium production mechanism.

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I. INTRODUCTION

The production and decays of quarkonium bound states have been under active studies in experiment ever since the first charmonium 1−− state, the $J/\psi$, was found twenty years before. The study of properties of the bound states of heavy quarks has provided a wealth of information on both the properties of heavy quarks ($c$, $b$) themselves and quantum chromodynamics since it stands on the very border between perturbative and nonperturbative domains. Recently, the observations of the Collider Detector at Fermilab (CDF) on the prompt charmonium production [1] has greatly stimulated progress in theoretical studies of quarkonium physics.

The conventional wisdom was that the dominant contributions to quarkonium production cross section at large transverse momentum ($P_T$) in $p\bar{p}$ collisions come from the QCD leading order diagrams, i.e. the so called parton fusion processes. However, these calculations for prompt $J/\psi$ ($\psi'$) production did not reproduce all the aspects of the available data [2]. It was pointed out by Braaten and Yuan [3] in 1993 that the dominant production mechanism at sufficiently large $P_T$ is the fragmentation of a parton produced with large transverse momentum, while formally this is of higher order in the strong coupling constant $\alpha_s$. Unfortunately, even after including the fragmentation contributions, the predictions for the $\psi'$ production rate still falloff far below the data [4]. This large discrepency between theory and experiment has called in question the simple color-singlet model description for quarkonium [5] and suggests that a new paradigm for treating heavy quark-antiquark bound systems that go beyond the color-singlet model might play an important role in the production of quarkonium at large $P_T$.

To this end, a factorization formalism has recently been performed by Bodwin, Braaten, and Lepage [6] in the context of nonrelativistic quantum chromodynamics (NRQCD), which provides a new framework to calculate the inclusive production and decay rates of quarkonia. In this approach, the calculations are organized in powers of $v$, the average velocity of the heavy quark (antiquark) in the meson rest frame, and in $\alpha_s$, the strong coupling constant. In NRQCD, quarkonium is not solely regarded as simply a quark-antiquark pair but rather a superposition of Fock states. The general Fock state expansion starts as

$$|H(nJ^{PC})> = O(1)|Q\bar{Q}(2S+1L_J,1)>$$
$$+ O(v)|Q\bar{Q}(2S+1(L \pm 1)J_J,8)g>$$
$$+ O(v^2)|Q\bar{Q}(2S+1L_J,8)gg> + \cdots$$
$$+ \cdots$$

1
where the angular momentum quantum numbers of the $Q\bar{Q}$ pairs within various Fock components are indicated in spectroscopic notation inside the brackets with color configuration of either 1 or 8.

The breakdown of color singlet model stems from its overlook of high Fock components contributions to quarkonium production cross sections. The color-octet term in the gluon fragmentation to $J/\psi(\psi')$ has been considered by Braaten and Fleming [4] to explain the $J/\psi(\psi')$ surplus problems discovered by CDF. Taking $<\mathcal{O}^{J/\psi} (^3S_1)>$ and $<\mathcal{O}^{\psi'} (^3S_1)>$ as input parameters, the CDF surplus problems for $J/\psi$ and $\psi'$ can be explained as the contributions of color-octet terms due to gluon fragmentation.

Even though the color-octet mechanism has gained some successes in describing the production and decays of heavy quark bound systems [6] [7] [8], it still needs more effort to go before finally setting its position and role in heavy quarkonium physics. Therefore the most urgent task among others needs to do now is to confirm and identify the color-octet quarkonium signals.

While the first charmonium state, the $J/\psi$, has been found over twenty years, $D$-wave states, given the limited experimental data, have received less attention. However, this situation may be changed in both experimental and theoretical investigations. Experimentally, there are hopes of observing charmonium $D$-wave states in addition to the $\psi(3770)$, in a high-statistic exclusive charmonium production experiment [9], and $b\bar{b}$ $D$-wave states in $\Upsilon$ radiative decays [10].

Recently, there is some clue for the $D$-wave $2^{--}$ charmonium state in $E705$ 300 GeV $\pi^\pm$- and proton-Li interaction experiment [11]. In this experiment there is an abnormal phenomenon that in the $J/\psi\pi^+\pi^-$ mass spectrum, two peaks at $\psi(3686)$ and at 3.836 GeV (given to be the $2^{--}$ state) are observed and they have almost the same height. Obviously, this situation is difficult to explain based upon the color-singlet model. However, it might be explained with the NRQCD analysis. Of course, at energies in fixed target experiments like $E705$, the color-octet gluon fragmentation dominance may or may not be the case. Moreover, the strong signal of $J/\psi\pi^+\pi^-$ at 3.836 GeV observed by $E705$ is now questioned by other experiments [12]. Nevertheless, if the $E705$ result is confirmed (even with a smaller rate, say, by a factor of 3, for the signal at 3.836 GeV), the color-octet gluon fragmentation will perhaps provide a quite unique explanation for the $D$-wave charmonium production. It does remind us that in the NRQCD approach, as discussed in Ref. [13], the production rates of $D$-wave heavy quarkonium states may be as large as that of $S$-wave states as long as the color-octet gluon production
mechanism dominantes.

Study shows that both LEP and Tevatron, especially the latter, are suitable grounds to find the D-wave quarkonia [13] and to test the color-octet signals [14]. In this paper we find that the divergences of the contributions between color-singlet and color-octet mechanisms in quarkonium $^3D_J$ production is enormous. The rest of the paper is arranged as follows: In Sec. II, we describe the formalism, give out the fragmentation functions of $g \to ^3D_J$ to leading order in $\alpha_s$ and further calculate the fragmentation probabilities of the gluon to D-wave quarkonium states. In Sec. III, we apply the fragmentation functions to evaluate the D-wave quarkonium production rates at Tevatron and close with some thoughts and discussions.

II. FORMALISM

Fragmentation is the formation of a hadron within a jet produced by a parton (quark, antiquark or gluon) with large transverse momentum. It is a useful concept because the probability for the formation of hadron within a jet is independent of the process that produces the parton that initiates the jet. By now, the fragmentation functions of quark and gluon splitting to S- and P-wave heavy quark bound states have been calculated [14] [15] [16] [17]. The calculations of fragmentation functions of quark to D-wave states [18] and gluon to spin-singlet D-wave state $^1D_2$ have also been accomplished [19]. However, the study of gluon fragmentation to color-singlet $^3D_J$ states, for its complexity, is still left behind. Here we realize this goal.

In hard process, as Fig.1(a), the most important kinematic region for a virtual gluon split with large $P_T$ is that the gluon is nearly on its massshell. Therefore, we can estimate the decay widths and the branching ratios by the following way [20].

The decay widths of a virtual quark $Q^*$ to color-singlet quarkonium state $^3D_J$ by gluon fragmentation can be evaluated via

$$
\Gamma(Q^* \to Qg^*; g^* \to ^3D_J gg) = \int_{\mu^2_{\text{min}}}^{\infty} d\mu^2 \frac{1}{\pi\mu^3} \Gamma(Q^* \to Qg^*(\mu)) \cdot P(g^* \to ^3D_J gg),
$$

(2)

where $s$ is the invariant mass squared of $Q^*$; $\mu$ is the virtuality of the gluon, and its minimum value squared $\mu^2_{\text{min}} = 12m_Q^2$ corresponding to the infrared cutoff as discussed below; $P$ is the decay distribution defined as

$$
P(g^* \to AX) \equiv \frac{1}{\pi\mu^3} \Gamma(g^* \to AX).
$$

(3)
The general covariant procedure for calculating the production and decay rates of heavy quark bound states may start from the Bethe-Salpeter (BS) amplitudes in the nonrelativistic limit. At leading order in $\alpha_s$, the amplitudes for $g^* \to 3D_J gg$ processes are

$$A = \int \frac{d^4q}{(2\pi)^4} Tr\{\mathcal{O}(P,q)\chi(P,q)\}. \quad (4)$$

Here $\chi(P,q)$ is the BS wave function of the bound states with relative momentum $q$ between the heavy quarks, while $\mathcal{O}(P,q)$ represents the rest of the matrix elements depicted in Fig.1(a),

$$\mathcal{O}(q) = -\frac{i}{4} \left\{ \varphi \frac{k_2 - k_1 + k_2 + k - 2m_Q}{(k_1 - k) \cdot k_2} \varphi - \frac{k_1 + k_2 - k - 2m_Q}{(k_2 - k) \cdot k_1} \varphi \right\} + \text{five permutations of } k_1, k_2, -k \text{ and } \epsilon_1, \epsilon_2, \epsilon. \quad (5)$$

Here the $k_1, k_2, k$ and $\epsilon_1, \epsilon_2, \epsilon$ stand for the momenta and polarization vectors of the two outgoing final gluons and the splitting gluon. Coupling constants and color matrices have been suppressed and contribute a factor

$$\sum_{a,b,c} \left( \frac{1}{\sqrt{3}} g_3 Tr\{T^a T^b T^c\} \right)^2 = \frac{5}{18} g_s^6$$

(6) to the production rates.

Under the instantaneous approximation with the negative energy projectors being neglected, the BS wave function $\chi(P,q)$ may be expressed as

$$\chi(P,q) = \frac{i}{2\pi} \frac{P_0 - E_1 - E_2}{(p_{10} - E_1)(p_{20} - E_2)} \Phi(\vec{P}, \vec{q}). \quad (7)$$

Here $P_0$ is the time component of the four momentum of the bound state; $p_{10}$ and $p_{20}$ are the time components of the momenta of quark and antiquark inside the meson, and $E_1, E_2$ are their kinetic energies. From the standard BS wave functions in the approximation that the negative energy projectors are omitted, the vector meson wave function can be projected out as :

$$\Phi(\vec{P}, \vec{q}) = \frac{1}{M} \sum_{S_zM} (JM|1S_zLm) \Lambda^1_+(\vec{p}_1) \gamma_0 \varphi(M+P) \gamma_0 \Lambda^2_-(\vec{p}_2) \psi_Lm(\vec{P}, \vec{q}), \quad (8)$$

where $e$ is the polarization vector associated with the spin-triplet states. $\Lambda^1_+(\vec{p}_1)$ and $\Lambda^2_-(\vec{p}_2)$ are positive energy projection operators of quark and antiquark .

$$\Lambda^1_+(\vec{p}_1) = \frac{E_1 + \gamma_0 \vec{p}_1 \cdot \vec{p}_1 + m_1 \gamma_0}{2E_1}, \quad \Lambda^2_-(\vec{p}_2) = \frac{E_2 - \gamma_0 \vec{p}_2 \cdot \vec{p}_2 - m_2 \gamma_0}{2E_2}. \quad (9)$$

After taking the nonrelativistic approximation the bound state wave function may be further reduced. For D-wave quarkonium production and decay, the first nonzero term is proportional to the second order of the amplitude expansion in powers of $q/M$:
\[ A(P, q) = A(P, 0) + q_\alpha \frac{\partial A(P, q)}{\partial q_\alpha} |_{q=0} + \frac{1}{2} q_\alpha q_\beta \frac{\partial^2 A(P, q)}{\partial q_\alpha \partial q_\beta} |_{q=0} + \cdots \]  

(10)

After integrating \( q_\alpha q_\beta \) over \( d^4q \), the \( ^3D_J \) polarization tensor is related to its nonrelativistic wavefunction by

\[ \int \frac{d^3q}{(2\pi)^3} q_\alpha q_\beta \psi_{2m}(\vec{P}, \vec{q}) = \epsilon^{(m)}_{\alpha\beta} \sqrt{\frac{15}{8\pi}} R^\mu_\nu(0), \]  

(11)

where the polarization tensor’s label \( m \) ranges over the helicity levels of the \( L = 2 \) meson. For spin-singlet case \( \epsilon^{(m)}_{\alpha\beta} \) is identified with \( \epsilon^{(J_s)}_{\alpha\beta} \), while for the spin-triplet case, using explicit Clebsch-Gorden coefficients, we have the following spin-orbit momentum coupling forms \[21\],

\[ \sum_{S_m} \langle 1J_z | 1S_z 2m \rangle \epsilon^{(m)}_{\alpha\beta} \epsilon^{(s)}_{\mu} = -\frac{3}{20} \left\{ (g_{\alpha\rho} - \frac{p_\alpha P_\rho}{4m_Q^2}) \epsilon^{(J_s)}_{\beta\mu} + (g_{\beta\rho} - \frac{P_\beta p_\rho}{4m_Q^2}) \epsilon^{(J_s)}_{\alpha\mu} \right\}, \]  

(12)

\[ \sum_{S_m} \langle 2J_z | 1S_z 2m \rangle \epsilon^{(m)}_{\alpha\beta} \epsilon^{(s)}_{\mu} = \frac{i}{2\sqrt{6m_Q}} \left( \epsilon^{(J_s)}_{\alpha\sigma} \epsilon_{\tau\beta\rho\sigma} \vec{P} \vec{g} \sigma' \sigma'' + \epsilon^{(J_s)}_{\beta\sigma} \epsilon_{\tau\alpha\rho\sigma} \vec{P} \vec{g} \sigma' \sigma'' \right), \]  

(13)

\[ \sum_{S_m} \langle 3J_z | 1S_z 2m \rangle \epsilon^{(m)}_{\alpha\beta} \epsilon^{(s)}_{\mu} = \epsilon^{(J_s)}_{\alpha\beta}. \]  

(14)

Using Eqs. (12)-(14) listed above, the amplitudes of Eq.(4) may be simplified and the averaged squared amplitudes may be obtained when suming up all polarizations of both the meson and gluons. Because the results are lengthy, it is too tedious to write them all here. For the convenience of reference, we just give the expression for the \( ^3D_1 \) state in the Appendix. Then, we have

\[ \Gamma(g^* \to ^3D_J gg) = \int dx_1dx_2 \sum |A|^2, \]  

(15)

where the kinematic variables are defined \( x_1 = \frac{2k_1}{\mu^2} \) and \( x_2 = \frac{2k_2}{\mu^2} \). Furthermore, from the Eq.(15) we can get the expressions of decay distributions \( P(g^* \to ^3D_J gg) \). With them the fragmentation functions can be calculated straightforward

\[ D_{g^* \to ^3D_J}(z, 2m_Q, s) = \frac{d\Gamma(Q^* \to ^3D_J gg Q)/dz}{\Gamma(Q^* \to Qg)}, \]  

(16)

where \( z \equiv \frac{2P \cdot k}{\mu^2} = 2 - x_1 - x_2 \). At high energy limit, the interaction energy \( s \) goes up to infinity, then the definition of \( z \) here is identical with that in Ref. [17] multiplied by a factor of two and the fragmentation functions decouple from any specific gluon splitting processes, which just reflects the universal spirit of fragmentation. The fragmetation function of Eq.(16) is evaluated
at the renormalization scale $2m_Q$, which corresponds to the minimum value of the invariant mass of the virtual gluon. In Fig.2 we display the variation curves of $D_{g^* 	o 3D_J}(z, 2m_c)$ versus $z$. After integrating over variable $z$, the fragmentation probabilities then read as

$$P_{g^* 	o 3D_J} = \frac{\Gamma(Q^* \to 3D_J gg Q)}{\Gamma(Q^* \to Qg)}.$$  \hfill (17)

Studies show [17] that the above method in extracting the gluon fragmentaiton probabilities are equivalent to the method developed in Ref. [3].

The calculation of color-octet fragmentation functions in $g^* \to 3D_J(3S_1, 8)$ processes, as shown in Fig.1(b), is trivial. They may be obtained directly from color-octet $g^* \to J/\psi(3S_1, 8)$ process [4],

$$D_{g^* \to 3D_J}(z, 2m_Q) = \frac{\pi \alpha_s(2m_Q)}{24m_Q^2} \delta(1 - z) < O^{3D_J}_{\mathcal{S}}(3S_1) > .$$  \hfill (18)

Therefore, the fragmentation probabilities are expressed as:

$$P_{g^* \to 3D_J} = \frac{\pi \alpha_s(2m_Q)}{24m_Q^2} < O^{3D_J}_{\mathcal{S}}(3S_1) > .$$  \hfill (19)

### III. RESULTS AND DISCUSSIONS

From Eq.(17) and (19) we can estimate the quarkonium $3D_J$ production rates at the Tevatron. The color-singlet sector may be factorized into long distance and short distance terms. The former is, to leading order in $v^2$, proportional to the second derivative of the radial wave function at the origin, which may be determined from potential model calculations [22]. The latter can be calculated from perturbative QCD, and it involves the infrared divergence associated with a soft gluon in the final states. In the numerical computation, we impose a lower cutoff $\Lambda$ on the energies of either gluons in the quarkonium rest frame. As discussed in Ref. [17], we choose $\Lambda = m_Q$ to avoid large logarithms and the cutoff dependence of the color-singlet terms is cancelled by the $\Lambda$ dependence of the nonperturbative matrix elements $< O^{3D_J}_{\mathcal{S}}(3S_1) >$ of the corresponding color-octet terms.

The gluon fragmentation contributions to the production of quarkonium $3D_J$ states at large transverse momentum in any high energy process can be approximately obtained by multiplying the cross section for producing gluons with transverse momentum larger than $2m_c$ by appropriate fragmentaion probabilities [8]. Using [15] [22]

$$m_c = 1.5 GeV, m_b = 4.9 GeV, \alpha_s(2m_c) = 0.26, \alpha_s(2m_b) = 0.19,$$
We obtain

\begin{align}
D_{g^*\rightarrow^3D_1(\bar{c}c)}^{(1)} &= 5.6 \times 10^{-8}, \quad D_{g^*\rightarrow^3D_2(\bar{c}c)}^{(1)} = 3.1 \times 10^{-7}, \\
D_{g^*\rightarrow^3D_1(\bar{c}c)}^{(1)} &= 2.2 \times 10^{-7}, \quad D_{g^*\rightarrow^3D_1(\bar{b}b)}^{(1)} = 2.5 \times 10^{-10}, \\
D_{g^*\rightarrow^3D_2(\bar{b}b)}^{(1)} &= 1.4 \times 10^{-9}, \quad D_{g^*\rightarrow^3D_3(\bar{b}b)}^{(1)} = 9.9 \times 10^{-10}.
\end{align}

(21)

For gluon fragmentation color-octet processes, the fragmentation probabilities are proportional to the nonperturbative matrix elements \( < \mathcal{O}^3_{8}^{D_J}(^3S_1) > \) which have not been extracted out from experimental data, nor from the Lattice QCD calculations. Based upon the NRQCD velocity scaling rules and the experimental clues discussed above, here we tentatively assume [13]

\[ < \mathcal{O}^3_{8}^{D_2(\bar{c}c)}(^3S_1) > \approx < \mathcal{O}^{\psi'}_8(^3S_1) > = 4.6 \times 10^{-3} \text{ GeV}^3 \]

(22)

(see Ref. [7]) and further extend this relation to the \( b\bar{b} \) system [7]

\[ < \mathcal{O}^3_{8}^{D_2(b\bar{b})}(^3S_1) > \approx < \mathcal{O}^{\psi'}_8(^3S_1) > = 4.1 \times 10^{-3} \text{ GeV}^3. \]

(23)

The supposed relations (22) and (23) certainly possess uncertainties to some extent, however from the calculated results below we are confident that it will not destroy the major conclusion of this paper. From the approximate heavy quark spin symmetry relation, we have

\[ < \mathcal{O}^3_{8}^{D_1}(^3S_1) > \approx \frac{3}{5} < \mathcal{O}^3_{8}^{D_2}(^3S_1) > \approx \frac{5}{7} < \mathcal{O}^3_{8}^{D_3}(^3S_1) > \]

(24)

for both \( b\bar{b} \) and \( c\bar{c} \) systems.

Using Eqs.(19), (20), and (22)-(24), we readily have

\begin{align}
D_{g^*\rightarrow^3D_1(\bar{c}c)}^{(8)} &= 4.2 \times 10^{-5}, \quad D_{g^*\rightarrow^3D_2(\bar{c}c)}^{(8)} = 7.0 \times 10^{-5}, \\
D_{g^*\rightarrow^3D_3(\bar{c}c)}^{(8)} &= 9.7 \times 10^{-5}, \quad D_{g^*\rightarrow^3D_1(\bar{b}b)}^{(8)} = 2.5 \times 10^{-6}, \\
D_{g^*\rightarrow^3D_2(\bar{b}b)}^{(8)} &= 4.2 \times 10^{-6}, \quad D_{g^*\rightarrow^3D_3(\bar{b}b)}^{(8)} = 5.9 \times 10^{-6}.
\end{align}

(25)

Comparing the above results (25) with (21), we come to an anticipated conclusion that at the Tevatron the gluon fragmentation probabilities through color-octet intermediates to spin-triplet D-wave charmonium and bottomonium states are over 2 \~ 4 orders of magnitude larger than that of color-singlet processes. As a result, the production rates of \(^3D_J \) states are about
the same amount as $\psi'$ and $\Upsilon(2S)$ production rates. Compared with the $\psi'$ production at the Tevatron, the gluon fragmentation color-octet process plays an even more important role in the $^3D_J$ quarkonium production, and it also gives production probabilities larger than the quark fragmentation process [18].

Among the three triplet states of $D$-wave charmonium, $^3D_2$ is the most promising candidate to discover firstly. Its mass falls in the range of $3.810 \sim 3.840$ GeV in the potential model calculation [23], that is above the $D\bar{D}$ threshold but below the $D\bar{D}^*$ threshold. However the parity conservation forbids it decaying into $D\bar{D}$. It, therefore, is a narrow resonance. Its main decay modes are expected to be,

$$^3D_2 \to J/\psi\pi\pi, \quad ^3D_2 \to ^3P_J\gamma (J = 1, 2), \quad ^3D_2 \to 3g.$$ (26)

We can estimate the hadronic transition rate of $^3D_2 \to J/\psi\pi^+\pi^-$ from the Mark III data for $\psi(3770) \to J/\psi\pi^+\pi^-$ [24] and the QCD multipole expansion theory [25] [26]. The Mark III data give [24] $\Gamma(\psi(3770) \to J/\psi\pi^+\pi^-) = (37 \pm 17 \pm 8) \text{ keV}$ or $(55 \pm 23 \pm 11) \text{ keV}$ (see also Ref. [26]). Because the $S-D$ mixing angle for $\psi(3770)$ and $\psi(3686)$ is expected to be small (say, $-10^\circ$, see Ref. [27] for the reasoning), the observed $\psi(3770) \to J/\psi\pi^+\pi^-$ transition should dominantly come from the $^3D_1 \to J/\psi\pi^+\pi^-$ transition, which is also compatible with the multipole expansion estimate [26]. Then using the relation [25]

$$d\Gamma(^3D_2 \to ^3S_12\pi) = d\Gamma(^3D_1 \to ^3S_12\pi)$$

and taking the average value of the $\Gamma(\psi(3770) \to J/\psi\pi^+\pi^-)$ from the Mark III data, we may have

$$\Gamma(^3D_2 \to J/\psi\pi^+\pi^-) = \Gamma(^3D_1 \to J/\psi\pi^+\pi^-) \approx 46 \text{ keV}. \quad (27)$$

For the $E1$ transition $^3D_2 \to ^3P_J\gamma (J = 1, 2)$, using the potential model with relativistic effects being considered [28], we find

$$\Gamma(^3D_2 \to \chi_{c1}\gamma) = 250 \text{ keV}, \quad \Gamma(^3D_2 \to \chi_{c2}\gamma) = 60 \text{ keV}, \quad (28)$$

where the mass of $^3D_2$ is set to be $3.84 GeV$. As for the $^3D_2 \to 3g$ annihilation decay, an estimate gives [29]

$$\Gamma(^3D_2 \to 3g) = 12 \text{ keV} \quad (29)$$

From (27), (28), and (29), we find
\[
\Gamma_{\text{tot}}(3D_2) \approx \Gamma(3D_2 \rightarrow J/\psi \pi \pi) + \Gamma(3D_2 \rightarrow \chi_{c1}\gamma) + \Gamma(3D_2 \rightarrow \chi_{c2}\gamma) + \Gamma(3D_2 \rightarrow 3g) \\
\approx 390 \text{ keV},
\]  

(30)

and

\[
B(3D_2 \rightarrow J/\psi \pi^+\pi^-) \approx 0.12.
\]  

(31)

Considering all the uncertainties this estimate is expected to hold within 50%. Compared (31) with \(B(\psi' \rightarrow J/\psi \pi^+\pi^-) = 0.324 \pm 0.026\), the branching ratio of \(3D_2 \rightarrow J/\psi \pi^+\pi^-\) is only smaller by a factor of 3, and therefore the decay mode of \(3D_2 \rightarrow J/\psi \pi^+\pi^-\) could be observable at Tevatron.

The \(3D_1 c\bar{c}\) state \(\psi(3770)\) could also detected via \(3D_1 \rightarrow D\bar{D}\). The other states, including the \(3D_3(c\bar{c})\) and \(3D_J(b\bar{b})\) are perhaps difficult to detect for reasons of either more decay modes or smaller production rates.

In conclusion, we have calculated the fragmentation functions and fragmentation probabilities of the gluon to \(3D_J\) charmonium and bottomonium states in both color-singlet and color-octet processes with certain numerical assumptions (e.g. Eq.(22)). The results can also be used in other hard gluon fragmentation processes because of the universality of the fragmentation functions. The study shows that, because charmonium \(3D_2\) state may have a production rate as large as that of \(\psi'\) at the Tevatron through color-octet production mechanism, the charmonium \(3D_2\) state as a most promising candidate to discover should be observable at the Tevatron with present luminosity, even the assumption of Eq.(22) with an error of 10 times off the exact case. On the other hand, since the calculated results show that the color-singlet and the color-octet contributions diverge enormously, this will also present a crucial test for the color-octet mechanism. the \(3D_J\) bottomonium states may have less strong signals to be detected comparing with the \(3D_J\) charmonium states because of their small production rates.

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\[ \sum |A|^2 = \frac{25\alpha_s^2 \mu |R_D(0)|^2}{2^{15}\pi m_c^7} \left. \frac{128d}{\sum_{i=0}^{12} f_i d^i} \right| \frac{1}{15(1 - d - x_1)^5(1 - d - x_2)^5(x_1 + x_2)^6}, \]  
(32)

where \(d = M^2/\mu^2\) and the functions \(f_i\) are defined as

\[ f_0 = (x_1 - 1)^4(x_2 - 1)^4[x_1^5(x_2 + 17) + x_1^4(4x_2^2 + 53x_2 - 177) + x_1^3(6x_2^2 + 74x_2 - 420x_2 + 576) + x_1^2(4x_2^4 + 74x_2^3 - 486x_2^2 + 1216x_2 - 928) + x_1(x_2^5 + 53x_2^4 - 420x_2^3 + 1216x_2^2 - 1600x_2 + 768) + 17x_2^5 - 177x_2^4 + 576x_2^3 - 928x_2^2 + 768x_2 - 256], \]

\[ f_1 = (x_1 - 1)^3(x_2 - 1)^3[10x_1^8 + 4x_1^7(15x_2 + 1) + x_1^6(170x_2^2 + 39x_2 - 141) + x_1^5(300x_2^3 - 133x_2^2 - 616x_2 - 271) + x_1^4(360x_2^4 - 566x_2^3 - 803x_2^2 - 587x_2 + 3162) + x_1^3(300x_2^5 - 566x_2^4 + 656x_2^3 + 106x_2^2 + 8336x_2 - 8696) + x_1^2(170x_2^6 - 133x_2^5 - 803x_2^4 + 106x_2^3 + 9772x_2^2 - 21480x_2 + 12720) + x_1(60x_2^7 + 39x_2^6 - 616x_2^5 - 587x_2^4 + 8336x_2^3 - 21480x_2^2 + 24160x_2 - 9856) + 10x_2^8 + 4x_2^7 - 141x_2^6 - 271x_2^5 + 3162x_2^4 - 8696x_2^3 + 12720x_2^2 - 9856x_2 + 3136], \]

\[ f_2 = (x_1 - 1)^2(x_2 - 1)^2[-50x_1^9 + 2x_1^8(-125x_2 + 14) + 2x_1^7(-320x_2^2 + 479x_2 - 97) + x_1^6(1951 + 3593x_2^2 - 5120x_2 - 1120x_2^2 + 1951) + 2x_1^5(-730x_2^4 + 2819x_2^3 - 8038x_2^2 + 7736x_2 - 160) + x_1^4(-1460x_2^5 + 5950x_2^4 - 24146x_2^3 + 44649x_2^2 - 11984x_2 - 17579) + 2x_1^3(-560x_2^6 + 2819x_2^5 - 12073x_2^4 + 31416x_2^3 - 22632x_2^2 - 24258x_2 + 26192) + x_1^2(3593x_2^6 - 640x_2^7 - 16076x_2^5 + 44649x_2^4 - 45264x_2^3 - 57330x_2^2 + 146496x_2 - 75496) + 2x_1(479x_2^7 - 125x_2^8 - 2560x_2^6 + 7736x_2^5 - 5992x_2^4 - 24258x_2^3 - 73248x_2^2 - 76552x_2 + 27952) - 50x_2^9 + 28x_2^8 - 194x_2^7 + 1951x_2^6 - 320x_2^5 - 17579x_2^4 + 52384x_2^3 - 75496x_2^2 + 55904x_2 - 16832], \]

\[ f_3 = 2(x_1 - 1)(x_2 - 1)[67x_1^{10} + x_1^9(277x_2 - 251) + 3x_1^8(328x_2^2 - 618x_2 + 405) + x_1^7(2780x_2^3 - 5699x_2^2 + 3156x_2 - 513) + x_1^6(5153x_2^4 - 12461x_2^3 - 1731x_2^2 + 18247x_2 - 8577) + x_1^5(6294x_2^5 - 19575x_2^4 - 11940x_2^3 + 89313x_2^2 - 82353x_2 + 15559) + x_1^4(5153x_2^6 - 19575x_2^5 - 16760x_2^4 + 178137x_2^3 - 272631x_2^2 + 116247x_2 + 11894) + x_1^3(2780x_2^7 - 12461x_2^6 - 11940x_2^5 + 178137x_2^4 - 401134x_2^3 + 303362x_2^2 + 20372x_2 - 79388) + x_1^2(984x_2^8 - 5699x_7 - 1731x_6^2 + 89313x_5^2 - 272631x_4^2 + 303362x_3^2 + 10876x_2^2) - 249684x_2 + 125056] + x_1(277x_9^2 - 1854x_8^2 + 3156x_7^2 + 18247x_6^2 - 82353x_5^2 + 116247x_4^2 + 20372x_3^2 + 249684x_2^2 + 266656x_2 - 91040) + 67x_2^{10} - 251x_9^2 + 1215x_8^2, \]
\[-513x_2^7 - 8577x_2^6 + 15559x_2^5 + 11894x_2^4 - 79388x_2^3 + 125056x_2^2 - 91040x_2 + 26080,\]
\[f_4 = 98x_1^{11} + 2x_1^{10}(629x_2 - 640) + x_1^9(6804x_2^2 - 13203x_2 + 6503) + x_1^8(23612x_2^3\]
\[-63506x_2^6 + 56515x_2^5 - 17151) + 2x_1^7(26637x_2^4 - 90183x_2^3 + 97672x_2^2 - 32167x_2\]
\[-1868) + 2x_1^6(39621x_2^5 - 167139x_2^4 + 205992x_2^3 - 2891x_2^2 - 135604x_2 + 59500)\]
\[+2x_1^5(39621x_2^6 - 205399x_2^5 + 296235x_2^4 + 142371x_2^3 - 702184x_2^2 + 558303x_2 - 127486)\]
\[+2x_1^4(26637x_2^7 - 167139x_2^6 + 296235x_2^5 + 246477x_2^4 - 1484408x_2^3 + 1835580x_2^2\]
\[-839002x_2 + 84843) + 2x_1^3(11086x_2^8 - 90183x_2^7 + 205992x_2^6 + 142371x_2^5 - 1484408x_2^4\]
\[+2689458x_2^3 - 1978824x_2^2 + 403532x_2 + 100128) + 2x_1^2(3402x_2^9 - 31753x_2^8 + 97672x_2^7\]
\[-2891x_2^6 - 702184x_2^5 + 1835580x_2^4 - 1978824x_2^3 + 640546x_2^2 + 380960x_2 - 242436)\]
\[+x_1(1258x_2^{10} - 13203x_2^9 + 56515x_2^8 - 64334x_2^7 - 271208x_6^6 + 1116606x_5^6\]
\[-1678004x_4^4 + 807064x_3^3 + 761920x_2^2 - 1085712x_2 + 369120) + 98x_1^{11} - 1280x_1^{10}\]
\[+6503x_2^9 - 17151x_2^8 - 3736x_2^7 + 119000x_2^6 - 254972x_2^5 + 169686x_4^4 + 200256x_3^3\]
\[-484872x_2^2 + 369120x_2 - 102720,\]
\[f_5 = 490x_1^{10} + x_1^9(4870x_2 - 4351) + x_1^8(25096x_2^2 - 42443x_2 + 17526) + 2x_1^7(38960x_2^3\]
\[-93761x_2^6 + 63537x_2 - 8870) + 2x_1^6(75695x_2^4 - 240989x_2^3 + 207627x_2^2 + 12406x_2\]
\[-54294) + 2x_1^5(94186x_2^5 - 383133x_2^4 + 407123x_2^3 + 193818x_2^2 - 526995x_2 + 209916)\]
\[+2x_1^4(75695x_2^6 - 383133x_2^5 + 507036x_2^4 + 480502x_2^3 - 1722286x_2^2 + 1369912x_2\]
\[-320530) + 2x_1^3(38960x_2^2 - 240989x_2^5 + 407123x_2^5 + 480502x_2^4 - 2515042x_2^3\]
\[+3194364x_2^2 + 1563232x_2 + 196056) + 2x_1^2(12548x_2^8 - 93761x_2^7 + 207627x_2^6\]
\[+193818x_2^5 - 1722286x_4^4 + 3194364x_2^3 - 2511804x_2^2 + 648200x_2 + 70608) + x_1(4870x_2^9\]
\[-42443x_2^8 + 127074x_2^7 + 24812x_2^6 - 1053990x_2^5 + 2739824x_2^4 - 3126464x_2^3\]
\[+1296400x_2^3 + 360384x_2^2 - 330624) + 490x_1^{10} - 4351x_2^9 + 17526x_2^8 - 17740x_2^7\]
\[-108588x_2^6 + 419832x_2^5 - 641060x_2^4 + 392112x_2^3 + 141216x_2^2 - 330624x_2 + 131712,\]
\[f_6 = 940x_1^9 + x_1^8(9380x_2 - 6619) + 2x_1^7(23754x_2^2 - 33091x_2 + 8148) + 2x_1^6(67062x_2^3\]
\[-142303x_2^2 + 55480x_2 + 22478) + 2x_1^5(111064x_2^4 - 327485x_2^3 + 178620x_2^2 + 209904x_2\]
\[-168145) + 2x_1^4(111064x_2^5 - 429639x_2^4 + 310024x_2^3 + 667150x_2^2 - 1059801x_2 + 386365)\]
\[+2x_1^3(67062x_2^6 - 327485x_2^5 + 310024x_2^4 + 968408x_2^3 - 2432902x_2^2 + 1859180x_2\]
\[-431760) + 2x_1^2(23754x_2^2 - 142303x_2^2 + 178620x_2^2 + 667150x_2^2 - 2432902x_2^2\]
\[+2981054x_2^2 - 1469776x_2 + 192024) + 2x_1(4690x_2^8 - 33091x_2^7 + 55480x_2^6 + 209904x_2^5\]
\[12
\[-1059801x_2^4 + 1859180x_2^3 - 1469776x_2^2 + 388080x_2 + 45024) + 940x_2^6 - 6619x_2^8 + 16296x_2^7 + 44956x_2^6 - 336290x_2^5 + 772730x_2^4 - 863520x_2^3 + 384048x_2^2 + 90048x_2 - 104832,\]

\[f_7 = 2[410x_1^8 + 3x_1^7(1849x_2 - 804) + 2x_1^6(14361x_2^2 - 15578x_2 - 1348) + x_1^5(71493x_2^3,\]
\[-131020x_2^4 - 7327x_2 + 72719) + x_1^4(95816x_2^4 - 253812x_2^3 - 3044x_2^2 + 426335x_2 - 254736) + x_1^3(71493x_2^5 - 253812x_2^4 + 582x_2^3 + 950322x_2^2 - 1194424x_2 + 406920) + 2x_1^2(14361x_2^6 - 65510x_2^5 - 1522x_2^4 + 475161x_2^3 - 951640x_2^2 + 691660x_2 - 157488) + x_1(5547x_2^7 - 3156x_2^6 - 7327x_2^5 + 426335x_2^4 - 1194424x_2^3 + 1383320x_2^2 - 661824x_2 + 77952) + 410x_2^8 - 2412x_2^7 - 2696x_2^6 + 72719x_2^5 - 254736x_2^4 + 406920x_2^3 - 314976x_2^2 + 77952x_2 + 19392],\]

\[f_8 = 454x_1^7 + 2x_1^6(5231x_2 - 879) + x_1^5(49146x_2^2 - 37439x_2 - 29865) + x_1^4(98210x_2^3 - 138246x_2^2 - 144901x_2 + 193291) + 2x_1^3(49105x_2^4 - 102277x_2^3 - 148785x_2^2 + 43590x_2 - 221376) + 2x_1^2(24573x_2^5 - 69123x_2^4 - 148785x_2^3 + 686625x_2^2 - 744064x_2 + 240264) + x_1(10462x_2^6 - 37439x_2^5 - 144901x_2^4 + 871180x_2^3 - 1488128x_2^2 + 1017888x_2 - 221760) + 454x_2^7 - 1758x_2^6 - 29865x_2^5 + 193291x_2^4 - 442752x_2^3 + 480528x_2^2 - 221760x_2 + 13440,\]

\[f_9 = 550x_1^6 + x_1^5(8230x_2 + 637) + x_1^4(28258x_2^2 - 6151x_2 - 38526) + 2x_1^3(20578x_2^3 + 10371x_2^2 - 81952x_2 + 71372) + 2x_1^2(14129x_2^4 - 10371x_2^3 - 126946x_2^2 + 234980x_2 - 106504) + x_1(8230x_2^5 - 6151x_2^4 - 163904x_2^3 + 469960x_2^2 - 451168x_2 + 140480) + 550x_2^6 + 637x_2^5 - 38526x_2^4 + 142744x_2^3 - 213008x_2^2 + 140480x_2 - 25280,\]

\[f_{10} = 572x_1^5 + x_1^4(4036x_2 + 2537) + 4x_1^3(2288x_2^2 + 2455x_2 - 6348) + 2x_1^2(4576x_2^3 + 7379x_2^2 - 40504x_2 + 27420) + 4x_1(1009x_2^4 + 2455x_2^3 - 20252x_2^2 + 28844x_2 - 12392) + 572x_2^5 + 2537x_2^4 - 25392x_2^3 + 54840x_2^2 - 49568x_2 + 14144,\]

\[f_{11} = 4[51x_1^4 + 2x_1^3(101x_2 + 245) + 2x_1^2(151x_2^2 + 751x_2 - 928) + 2x_1(101x_2^3 + 751x_2^2 - 1920x_2 + 1152) + 51x_2^4 + 490x_2^3 - 1856x_2^2 + 2304x_2 - 976],\]

\[f_{12} = 8[51x_1^4 + 6x_1(17x_2 - 14) + 51x_2^2 - 84x_2 + 56].\]
Fig.1. Virtual gluon fragmentation processes (a) gluon fragments to $^3D_J$ via color-singlet process, (b) gluon fragments to $^3D_J$ via color-octet process.

Fig.2 The variation of charmonium fragmentation functions $D_{(g\to^3D_J)}(z, 2m_c)$ versus $z$. 