Singular Hypersurfaces in
Scalar-Tensor Theories of Gravity

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Abstract

We study singular hypersurfaces in tensor multi-scalar theories of gravity. We derive in a distributional and then in an intrinsic way, the general equations of junction valid for all types of hypersurfaces, in particular for lightlike shells and write the general equations of evolution for these objects. We apply this formalism to various examples in static spherically symmetric spacetimes, and to the study of planar domain walls and plane impulsive waves.
I. INTRODUCTION

Most of the various attempts to quantize the gravitational field have led to the conclusion that at Planckian energies the Einstein theory of gravity has to be extended in order to include scalar fields. In the low energy limit, string theory gives back classical general relativity with a scalar field partner (the dilaton) and the effective action shows that the dilaton couples to the scalar curvature and to the other matter fields [1]. Scalar fields (compactons) also arise in the process of dimensional reduction of Kaluza-Klein theories [2], and it has been shown that the presence in the action of high-order terms in the curvature and its derivative amounts to introducing scalar fields with appropriate potentials in the Einstein-Hilbert action of general relativity [3].

The importance of the role that scalar fields could play in a full theory of gravity has been noticed since a long time [4]. The pioneering works of Fierz, Jordan, and Brans and Dicke [5] have opened on the first scalar-tensor theory of gravity (usually referred to as the the Brans-Dicke theory) which includes besides the gravitational field $g_{\mu\nu}$, a massless scalar field $\varphi$ and a free parameter $\omega$. This theory was later generalized [6] by making the parameter field-dependent, i.e., $\omega(\varphi)$, and by introducing a potential term $V(\varphi)$. More recently, using nonlinear $\sigma$-models multi-scalar-tensor theories have been considered and their predictions have been dicussed and compared with general relativity in the weak-field and strong-field regimes [7]. All these alternative theories of gravity belong to the class of scalar-tensor theories in the sense that all the other fields (generically denoted by $\Psi_m$) exhibit a universal metric coupling to the gravitational field with the same metric tensor. On the other hand the scalar fields (dilatons, compactons) which appear in the string and Kaluza-Klein theories have a non-metric coupling with the fields $\Psi_m$, and instead induce a local spacetime dependence of the coupling constants. This entails fundamental differences with general relativity, in
particular with the equivalence principle [8].

Besides these theoretical considerations on the possible role played by scalar fields in a complete theory of gravity, an important motivation for considering scalar fields is their application in cosmology to the scenarios of inflation and the formation of topological defects (monopoles, cosmic strings and domain walls). A basic ingredient of all these studies is the introduction of one (or several) scalar fields (inflatons) which trigger the production of phase transitions in the early history of the universe. Many different models of inflation have been proposed [9], some of them within the framework of scalar-tensor or string theories of gravity. For instance, in extended inflation [10], general relativity is replaced by the Brans-Dicke theory, and in a later version sometimes referred to as hyperextended inflation [11] the parameter $\omega$ varies with the scalar field $\phi$. The properties of scalar-tensor cosmological models have been much studied: methods for obtaining exact solutions of the field equations have been given [12], constraints for successful extended inflation and constraints from inflation on scalar-tensor theories have been formulated [13], and the existence of an attractor mechanism towards general relativity have been discussed [14]. The formation and dynamics of spherical bubbles of true vacuum in the Brans Dicke theory have been studied in a thin-wall formalism [15],[16]. String cosmologies have also aroused much interest [17] and solutions to the peculiar difficulties associated with the dilaton have been proposed [18]. A class of supersymmetric domain walls in $N = 1$ supergravity and within effective string theories have been obtained and their gravitational effects have been described [19].

In this paper we study the junction conditions which have to be satisfied by the various fields at an arbitrary singular hypersurface separating two different spacetimes in scalar-tensor theories of gravity. This work generalizes previous descriptions of thin shells in the Brans Dicke theory [15],[16],[20],[21], and [22]
in the sense that we present a general algorithm which can be applied to a singular hypersurface of any type (timelike, spacelike, or lightlike) and we consider the possibility of having discontinuous gauge fields. The results obtained in the timelike case apply to arbitrary surface layers and in particular to domain walls as considered in the refs [15]-[19], with eventually surface currents as it may be the case for some superconducting domain walls coming from a supersymmetric action [23]. The less considered spacelike case might for instance correspond to a transition layer which suddenly appears and disappears all over space at a given time -examples of this situation can be found in [24]. The lightlike case has interesting properties because it can at the same time describe a lightlike shell with surface energy density and surface stresses, and an impulsive gravitational wave which is accompanied by shock waves when discontinuous gauge fields are present. These waves have been shown to be of some interest in string theory [25] as plane waves are exact classical solutions at all order of the string tension parameter [26].

It is known that in scalar-tensor and dilatonic theories two conformally related metrics can be used, the Jordan-Fierz or string metric and the Einstein metric. The Jordan-Fierz or string metric is usually referred to as the physical metric as the stress-energy tensor for the matter fields is conserved in this metric and not in the other one -see however Cho in ref.[2]. However many of the mathematical properties of these theories (asymptotic behavior, Cauchy problem...) are more conveniently investigated in the Einstein metric. While most of the authors having studied thin shells in the Brans-Dicke theory have worked in the Jordan-Fierz frame we have preferred here to use the Einstein frame. The main reasons for making such a choice is that it offers a simpler set of equations, and it enables an easier comparison with previous results obtained in general relativity [27],[28].

This paper is organized as follows. In section 2 we give a brief survey of
the scalar-tensor theories of gravity and in section 3 we present our general re-
results concerning the junction conditions across an arbitrary hypersurface. These
conditions are described in a distributional formalism and within an intrinsic ap-
proach, and the existence of discontinuous gauge fields is considered. In the
next two sections we consider various examples illustrating the general formalism
presented in section 3. These applications concern spherically symmetric shells
(section 4), and planar shells and plane impulsive waves (section 5). In the last
section we briefly discuss the differences which appear in the description of a
shell when using dilatonic theories instead of scalar-tensor theories of gravity.
Finally some static spherically symmetric solutions of the Brans-Dicke theory are
presented in appendix A.

Conventions: Our metric signature is \((-+++)\) and we use the standard
conventions for the Riemann tensor of Misner, Thorne and Wheeler [29]. Greek
indices run from 0 to 3.

II. SURVEY OF SCALAR-TENSOR THEORIES OF
GRAVITATION

The scalar-tensor theories of gravitation are alternative theories of gravity
which generalize in the most natural way the Brans-Dicke theory by introducing
a finite number of scalar fields, \(\varphi^i, i = 1, 2, ..n\), each characterized by a particular
coupling constant to local matter -see for instance Damour and Esposito-Farèse
[7] for a review of scalar-tensor theories. These theories are covariant tensor
field theories and they coincide with general relativity in the post-newtonian
approximation. They are metrics theories, which means that the matter fields
are minimally coupled to a universal covariant 2-tensor, \(\tilde{g}_{\mu\nu}\), usually referred to
as the physical metric or the Jordan-Fierz metric. They can also be described
within another frame, which is conformally related to the previous one, in such
a way that the Einstein-Hilbert term is recovered in the action. This frame is called the Einstein conformal-frame and the corresponding metric $g_{\mu\nu}$, the Einstein metric. The relation between the two metrics is

$$\bar{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu},$$

where the conformal factor $A^2(\varphi)$ is a smooth function of the n scalar fields $\varphi^i$.

The general form of the action for scalar-tensor theories of gravity is, in the Einstein conformal-frame

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - 2g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j \gamma_{ij} - 4B(\varphi) \right] + S_m[\psi_m, A^2(\varphi)g_{\mu\nu}].$$

The second term $S_m$ is the action for the “matter” fields (fermionic or bosonic), collectively denoted by $\psi_m$ which as indicated in $S_m$ couple to the Jordan-Fierz metric. In the first term, the scalar fields $\varphi^i$ appear in a non-linear $\sigma$-model. This means that the quantities $\varphi^i$ can be viewed as internal local coordinates in a n-dimensional manifold (target space) endowed with a metric $d\sigma^2$ which is written

$$d\sigma^2 = \gamma^{ij}(\varphi)d\varphi_id\varphi_j.$$

in the local coordinate system $\{\varphi^i\}$.

The potential term $B$ is a smooth function (at least $C^2$) of the scalar-fields $\varphi^i$ and may include a cosmological term. The field equations for the metric tensor $g_{\mu\nu}$ and the scalar fields $\varphi^i$ which follow from the above action are respectively

$$G_{\mu\nu} = 8\pi G (T_{\mu\nu} + T_{\mu\nu}(\varphi)) \equiv 8\pi G T_{\mu\nu},$$

and

$$\Box \varphi^i + \gamma^{ij}_k g^{\mu\nu} \partial_\mu \varphi^j \partial_\nu \varphi^k - \beta^i(\varphi) = -4\pi G \alpha^i(\varphi) T,$$

where $\Box$ is the d’Alembertian with respect to the metric $g$. (Note that eq. (4) is not covariant with respect to the indices of the target space). $T_{\mu\nu}$ is the
total stress-energy tensor, $T_{\mu\nu}$ is the stress-energy tensor of the matter fields (see below) and $T = T_{\mu}^{\mu}$ its trace, and

$$T_{\mu\nu}(\varphi) \equiv \frac{\gamma_{ij}}{4\pi G} \left[ \partial_{\mu} \varphi^{i} \partial_{\nu} \varphi^{j} - \frac{1}{2} g_{\mu\nu} \left( g^{\alpha\beta} \partial_{\alpha} \varphi^{i} \partial_{\beta} \varphi^{j} \right) \right] - \frac{B(\varphi)}{4\pi G} g_{\mu\nu} \quad (5)$$

is the stress-energy tensor of the scalar-fields. The $\gamma_{jk}^{i}$'s are the Christoffel symbols associated with the $\sigma$-model metric $\gamma_{ij}$ and the scalar-fields indices are raised or lowered with this metric or its inverse. An other convenient form of (3) is

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) + \gamma_{ij} \partial_{\mu} \varphi^{i} \partial_{\nu} \varphi^{j}, \quad (6)$$

We have introduced in (4) the following space-time scalars

$$\beta_{i}(\varphi) = \frac{\partial B(\varphi)}{\partial \varphi^{i}}, \quad \alpha_{i}(\varphi) = \frac{\partial \ln A(\varphi)}{\partial \varphi^{i}}, \quad (7)$$

and it is evident from (4) that the $\alpha_{i}(\varphi)$'s represent coupling factors of the scalar fields to matter.

The stress-energy tensor of the matter fields is defined by

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \delta S_{m} \frac{\delta}{\delta g_{\mu\nu}} \quad (8)$$

In the Einstein frame it is not conserved but instead satisfies

$$\nabla_{\nu} T_{\mu\nu} = \alpha_{i}(\varphi) T^{\mu}_{\nu} \partial^{\nu} \varphi^{i} \quad (9)$$

Although this work is performed in the Einstein frame, let us briefly recall for the sake of completeness some properties of the Jordan-Fierz description. In the Jordan-Fierz frame, only one scalar field $\bar{\varphi}$ can be considered and the action takes the general form

$$S = \frac{1}{16\pi G} \int d^{4}x \sqrt{-\bar{g}} \left[ \bar{\varphi} \bar{R} - \frac{\omega(\bar{\varphi})}{\bar{\varphi}} \bar{g}^{\mu\nu} \partial_{\mu} \bar{\varphi} \partial_{\nu} \bar{\varphi} + 2 \bar{\varphi} \Lambda(\bar{\varphi}) \right] + S_{m}[\psi_{m}, \bar{g}^{\mu\nu}] \quad (10)$$
The field equations for $\bar{g}_{\mu\nu}$ and $\bar{\varphi}$ are respectively

\[
\bar{R}_{\mu\nu} - \frac{\bar{R}}{2} \bar{g}_{\mu\nu} = \frac{8\pi G}{\bar{\varphi}} \bar{T}_{\mu\nu} + \Lambda(\bar{\varphi}) \bar{g}_{\mu\nu}
\]

\[
+ \frac{\omega(\bar{\varphi})}{\bar{\varphi}^2} \left[ \partial_\mu \bar{\varphi} \partial_\nu \bar{\varphi} - \frac{1}{2} \bar{g}_{\mu\nu} (\bar{g}^{\alpha\beta} \partial_\alpha \bar{\varphi} \partial_\beta \bar{\varphi}) \right]
\]

\[
+ \frac{1}{\bar{\varphi}} \left( \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\varphi} - \bar{g}_{\mu\nu} \square \bar{\varphi} \right)
\],

(11)

and

\[
\square \bar{\varphi} + \frac{1}{2} \bar{g}^{\alpha\beta} \partial_\alpha \bar{\varphi} \partial_\beta \bar{\varphi} \frac{d}{d\bar{\varphi}} \ln(\frac{\omega(\bar{\varphi})}{\bar{\varphi}}) + \frac{\bar{\varphi}}{\omega(\bar{\varphi})} \left[ \frac{\bar{R}}{2} + \frac{d}{d\bar{\varphi}}(\bar{\varphi}\Lambda(\bar{\varphi})) \right] = 0
\],

(12)

where $\bar{T}_{\mu\nu}$ is the stress-energy tensor of the matter fields

\[
\bar{T}^{\mu\nu} \equiv \frac{2}{\sqrt{-\bar{g}}} \frac{\delta S_m}{\delta \bar{g}_{\mu\nu}}
\],

(13)

and where $\bar{\nabla}$ and $\square$ stand respectively for the covariant derivative and the d’Alembertian associated with the metric $\bar{g}_{\mu\nu}$. As the matter fields couple directly to the Jordan-Fierz metric their stress-energy tensor is conserved, i.e. $\bar{\nabla}_\nu \bar{T}^{\mu\nu} = 0$.

We have the following relations between the two descriptions

\[
\begin{cases}
\bar{\varphi}^{-1} = G A^2(\varphi) \\
2 \omega(\bar{\varphi}) + 3 = \alpha^{-2}(\varphi) \\
2 \Lambda(\bar{\varphi}) = -B(\varphi) A^{-2}(\varphi)
\end{cases}
\]

(14)

and between the matter stress-energy tensors (8) and (13)

\[
T^{\mu\nu} = A^6(\varphi) \bar{T}^{\mu\nu}
\]

(15)

The Brans-Dicke theory corresponds to the particular case where $\omega(\bar{\varphi}) = const.$ and $\Lambda = 0$ in the Jordan-Fierz description, or $\alpha = const.$ and $B = 0$ in the Einstein description. In this case, one obtains from (7)

\[
A(\varphi) = e^{\alpha \varphi}
\]

(16)
III. SINGULAR HYPERSURFACES IN SCALAR-THEORIES OF GRAVITATION

A thin shell corresponds to the situation where there exists a surface in the vicinity of which the distribution of stress and energy is so strongly concentrated that the thin limit approximation can be used. In spacetime this yields a three-dimensional hypersurface (timelike, spacelike or lightlike) along which the metric tensor is only $C^0$, but $C^3$ elsewhere. The description of thin shells in general relativity and in the timelike and spacelike cases is well known since the pioneering works of Lanczos [30] and Israel [31] and an extension to the null or lightlike case has recently been done [27]. It is the purpose of this section to present a general algorithm, similar to the one described in [27], and adapted to scalar-tensor theories of gravitation (the case of theories with a dilaton will be studied in section 4).

Let $\Sigma$ be the singular hypersurface in space-time corresponding to the thin shell. As the metric is only $C^0$ on $\Sigma$ the total stress-energy tensor

$$T_{\mu\nu} = T_{\mu\nu} + \frac{\gamma_{ij}}{4\pi G} (\partial_\mu \varphi^i \partial_\nu \varphi^j - \frac{g_{\mu\nu}}{2} g^{\alpha\beta} \partial_\alpha \varphi^i \partial_\beta \varphi^j) - \frac{B(\varphi)}{4\pi G} g_{\mu\nu}$$

(17)

which appears in the right-hand side of (3) necessarily contains a $\delta$-term with support on $\Sigma$. Comparing the eqs (17) and (4) one easily sees that the singular $\delta$-term can only come from the matter part $T_{\mu\nu}$ of the total stress-energy tensor, and that because of the presence of the trace $T$ in the r.h.s. of (4) the scalar fields $\varphi^i$ are $C^0$ on $\Sigma$ and $C^3$ elsewhere. Therefore the metric and the scalar fields have the same smoothness properties all over spacetime, as expected from the metric coupling of the scalar fields to gravity. For instance, in the scenario of extended inflation, there exists besides the Brans-Dicke scalar field another scalar field (the inflaton) which is responsible for the formation of true-vacuum bubbles.
and yields the $\delta$-term in $T_{\mu\nu}$, while the role of the Brans-Dicke scalar field is to slow down the expansion of the universe.

There exists two equivalent ways of describing shells which we now present - some of the results being common to those obtained in general relativity we refer the reader to [27] for more details. The first description is based on a four-dimensional distributional approach and requires a common set of coordinates covering both sides of the shell. The second one is purely intrinsic, and allows an independent and arbitrary choice of coordinates in the two sides of the shell. The existence of an impulsive gravitational wave and of discontinuous gauge fields will also be examined within these two descriptions.

III.1 Junction conditions : the distributional description

We consider a general smooth hypersurface $\Sigma$ separating two spacetimes $\mathcal{M}_+$ and $\mathcal{M}_-$, endowed with the metrics $g^+$ and $g^-$ (at least of class $C^3$) and with the scalar fields $\varphi^+_i$ and $\varphi^-_i$ (at least of class $C^2$) and we introduce a common system of coordinates $x^\mu$. The matter stress-energy tensors are $T^{\alpha\beta}_+$ and $T^{\alpha\beta}_-$, and the fields equations (3) and (4) are satisfied in each domain. The hypersurface $\Sigma$ results from an isometric soldering of the boundaries $\Sigma^+$ and $\Sigma^+$ which are respectively imbedded in $\mathcal{M}_+$ and $\mathcal{M}_-$. Denoting by $[F] = F^+ - F^-$ the jump across $\Sigma$ of an arbitrary discontinuous function $F$, we thus have

$$ [g^{\alpha\beta}] = 0 \quad , \quad [\varphi^i] = 0 \quad . $$

(18)

Let $\Phi(x) = 0$ be the equation of $\Sigma$ where $\Phi$ is a smooth function (at least $C^1$) taking positive (resp. negative) values in $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) and let $n$ be the normal vector to $\Sigma$ pointing towards the $+$ side and normalized according to

$$ n \cdot n = \epsilon \quad , $$

(19)
where $\epsilon$ is constant over $\Sigma$ and takes respectively a positive, negative or null value whenever $\Sigma$ is timelike, spacelike or lightlike. There exists a non-vanishing smooth function $\chi$ on $\Sigma$ such that we have the relation

$$n = \chi^{-1} \nabla \Phi.$$  \hspace{1cm} (20)

In order to deal with any type of hypersurface (timelike, spacelike or lightlike) we introduce a vector field $N$, transversal to $\Sigma$ and satisfying

$$N \cdot n = \eta^{-1},$$  \hspace{1cm} (21)

where $\eta$ is a given non-vanishing smooth function on $\Sigma$. As $n$ is normal to $\Sigma$, the vector $N$ is defined up to a tangential displacement which has to be the same on each side in order to make sure that the same transversal vector is considered. For a timelike or spacelike hypersurface, $N$ can be chosen identical to the normal $n$ and in that case one has $\epsilon \eta = 1$ and $\eta$ is constant.

For any functions $F^\pm$ defined in each domain $\mathcal{M}_\pm$, we introduce the hybrid quantity

$$\tilde{F} = F^+ \Theta(\Phi) + F^- \Theta(-\Phi),$$  \hspace{1cm} (22)

where $\Theta$ is the Heaviside step-function (which takes values 1, $\frac{1}{2}$ or 0 when its argument is respectively positive, null or negative) and we write distributionally its derivative as

$$\partial_\mu F = (\partial_\mu F)^\sim + \chi n_\mu [F] \delta(\Phi).$$  \hspace{1cm} (23)

The metric $g$ and the scalar-fields $\varphi^i$ as also their tangential derivatives are continuous across $\Sigma$ but their transverse derivatives are not and their jumps are defined by

$$[\partial_\mu g_{\alpha\beta}] = \eta n_\mu \gamma_{\alpha\beta}, \quad [\partial_\mu \varphi^i] = \eta n_\mu \zeta^i,$$  \hspace{1cm} (24)

or using (21) by

$$\gamma_{\alpha\beta} = N^\mu [\partial_\mu g_{\alpha\beta}], \quad \zeta^i = N^\mu [\partial_\mu \varphi^i].$$  \hspace{1cm} (25)
It can be checked [27] that both the $\gamma_{\alpha\beta}$'s and the $\zeta^i$'s are independent on the choice of the transversal vector $N$. Furthermore as only the projection of $\gamma_{\alpha\beta}$ onto $\Sigma$ has an intrinsic meaning, they are not uniquely defined by the above equations and one may perform the gauge transformation

$$\gamma_{\alpha\beta} \rightarrow \gamma_{\alpha\beta} + 2\lambda(\alpha n_\beta),$$

where $\lambda_\alpha$ are the components of an arbitrary vector-field over $\Sigma$. Using the hybrid notation (22) for the metric $g$ and the scalar fields $\varphi^i$, and the relations (24) for their derivatives, it can be shown that the Einstein tensor and the d’Alembertian of the scalar fields are equal to

$$G_{\mu\nu} = G_{\mu\nu}^{\sim} + \eta \chi \delta(\Phi) \left[ \gamma^{(\mu} n^{\nu)} - \frac{1}{2} \left( \gamma n^{\mu} n^{\nu} + \gamma^{\mu} g_{\mu\nu} + \epsilon [\gamma^{\mu\nu} - \gamma g^{\mu\nu}] \right) \right]$$

$$\Box \varphi^i = \left( \Box \varphi^i \right)^{\sim} + \epsilon \eta \chi \delta(\Phi) \zeta^i,$$

where we have defined

$$\gamma_{\mu} = \gamma_{\mu\nu} n^{\nu}, \gamma^{\dagger} = \gamma_{\mu} n^{\nu}, \gamma = \gamma_{\mu\nu} g^{\mu\nu}.$$

We have used the fact that the product $\Theta(\Phi) \Theta(-\Phi)$ vanishes distributionally.

Recalling that as a $\delta$-term has to appear in the stress-energy tensor of the matter fields, one may write the total stress-energy tensor as

$$T_{\mu\nu} = T^{\sim}_{\mu\nu} + S_{\mu\nu} \chi \delta(\Phi),$$

where $S_{\mu\nu}$ represents the surface stress-energy tensor of the shell and $T^{\sim}_{\mu\nu}$ is of the form (22). Then introducing (27), (28) and (30) into the field equations (3) and (4), and extracting their $\delta$-terms, one gets

$$16\pi G \eta^{-1} S_{\mu\nu} = 2\gamma^{(\mu} n^{\nu)} - \gamma n^{\mu} n^{\nu} - \gamma^{\dagger} g^{\mu\nu} - \epsilon (\gamma^{\mu\nu} - \gamma g^{\mu\nu}),$$

$$-4\pi G \eta^{-1} \alpha^i(\varphi) S = \epsilon \zeta^i.$$
where $S = S^{\mu\nu} g_{\mu\nu}$.  

The surface stress-energy tensor $S^{\mu\nu}$ keeps then the same form as in general relativity, eq.(17) of [27]. In the Jordan-Fierz frame it would have taken a different and more complicated form including the jumps of the scalar fields. It can be checked from (31) that $S^{\mu\nu}$ is a tangential quantity

$$S^{\mu\nu} n_\nu = 0 \quad ,$$

and that it is invariant under the gauge transformation (26). Eliminating the trace $S$ of $S_{\mu\nu}$ between (31) and (32), one obtains a relation between the jumps $\gamma_{\mu\nu}$ and $\zeta_i$

$$\alpha^i (\varphi) (\gamma^i - \epsilon \gamma) = 2 \epsilon \zeta^i \quad .$$

As $\gamma^i - \epsilon \gamma$ is invariant under the gauge transformation (26), this fulfils the intrinsic nature of the junction conditions in spite of the non unicity of the transversal $N$. Moreover, as one can see from (34), the jumps in the first derivatives of the metric and the scalar fields cannot be chosen independently of each other, and this provides an extra boundary condition for the resolution of the field equations.

Using again the hybrid notations (22-23), we obtain from the Bianchi identities the conservation relation of the generalized stress-energy tensor (30)

$$\nabla_\mu T^{\mu\nu} = 0 \quad .$$

Introducing (20) and (30) in (35) one gets

$$\nabla_\mu (\chi S^{\mu\nu}) \delta (\Phi) = - \nabla_\mu T^{\mu\nu} \quad ,$$

and extracting the $\delta$-terms one obtains the equation of conservation for the surface stress-energy tensor of the shell

$$\nabla_\nu (\chi S^{\mu\nu}) = - [T^{\mu\nu} n_\nu] \chi - \chi [T^{\mu\nu}(\varphi)] n_\nu \quad ,$$

13
where $\nabla$ here stands for the covariant derivative operator associated with the hybrid metric $g_{\mu\nu}$, i.e. such that $\nabla_\mu g_{\nu\rho} = 0$. The whole set of equations (31), (32) and (37) describe the dynamics of the shell, however as they are not all independent only part of them need to be used.

*Timelike and spacelike shells ($\epsilon \neq 0$):*

For these hypersurfaces, the gauge freedom (26) enables us to choose $\gamma_{\mu\nu}$ such that its contraction with the normal vector vanishes, hence $\gamma^\mu = 0$ and $\gamma^\dagger = 0$. With that choice the relations (31-32) reduce to

$$-16\pi G \eta^{-1} S_{\mu\nu} = \epsilon \gamma_{\mu\nu} - \gamma (\epsilon g_{\mu\nu} - n_{\mu} n_{\nu})$$ (38)

$$-4\pi G \eta^{-1} \alpha^i(\varphi) S = \epsilon \zeta^i$$ (39)

where the last equation can also be replaced by (34), i.e. by

$$\gamma \alpha^i(\varphi) = -2\zeta^i$$ (40)

The equation (38) is the same as in general relativity but we have here the additional constraints (39) or (40) coming from the presence of the scalar fields $\varphi^i$.

*Lightlike shell ($\epsilon = 0$):*

In that case, it follows from (31) that the stress-energy tensor is equal to

$$-16\pi G \eta^{-1} S^{\mu\nu} = \gamma n^\mu n^\nu - (\gamma^\mu n^\nu + \gamma^\nu n^\mu)$$ (41)

and from (32) that it is tracefree $S = 0$, or equivalently from (34) $\gamma^\dagger = 0$. This last property can be interpreted as a condition for the shell to be pressureless as one can show. At first the normal vector $n$ is tangent to the null generators of the hypersurface $\Sigma$ and satisfies the geodesic equation

$$\nabla_n n = \kappa n$$ (42)
where $\kappa$ vanishes whenever $n$ is associated to an affine parametrization along the null generators. It follows from the definition of $\gamma_{\mu\nu}$ that $\gamma_{\mu\nu} = [L_N g_{\mu\nu}]$, then using $n.n = 0$ and the properties of the Lie derivative $L_N$, one successively gets

$$\gamma^\dagger = n^\mu n^\nu \gamma_{\mu\nu} = -2 \left[ n.L_N n \right] = 2 \left[ n.\nabla_n N \right] = -2 \eta^{-1} [\kappa],$$

where we have also used $[\nabla_n \eta] = 0$. On the other hand, taking the jump across $\Sigma$ of the Raychauduri’s formula the null generators of $\Sigma$, one obtains

$$[\kappa] \Theta = 8\pi G \left[ T_{\mu\nu} n^\mu n^\nu \right], \quad (43)$$

where $\Theta$ is the dilation rate of the null generators. As the normal derivative of the scalar fields $n^\mu \nabla_\mu \varphi^i$ is continuous on $\Sigma$ -recall that the normal is tangent to a null hypersurface- the scalar field part (5) of the total stress energy tensor vanishes in (43) and $T_{\mu\nu}$ can be replaced by the matter stress energy tensor $T_{\mu\nu}$ in this equation. Finally combining these results one gets

$$\gamma^\dagger \Theta = -16\pi \eta \left[ T_{\mu\nu} n^\mu n^\nu \right], \quad (44)$$

which shows that $\gamma^\dagger / 16\pi \eta$ represents the isotropic surface pressure of the null shell. Hence, as $\gamma^\dagger = 0$, a null shell cannot possess any isotropic pressure but only energy density and possibly shears - this restriction does not exist in general relativity where a null shell can have a surface pressure, see examples in [27], [28].

Another consequence is that the vanishing of the right-hand side (44) implies that no energy can be transferred to the shell from the surrounding matter fields.

It has been shown in general relativity [27] that in the lightlike case a shell is generally accompanied by an impulsive wave. This property remains unchanged in scalar-tensor theories and we briefly recall here why such a decomposition into a shell and wave occurs. The Weyl tensor of the space time $\mathcal{M}_+ \cup \mathcal{M}_-$ contains
a $\delta$-term which using (24) and the notations (22-23) can be shown to be equal to
- see the eq.(41) of [27]

$$C_{\mu\nu}^{\alpha\beta} = \{ 2\eta n^{[\alpha} \gamma_{[\mu]} n_{[\nu]} - 16\pi \delta_{[\mu}^{[\alpha} S_{\nu]}^{\beta]} + \frac{8}{3} \pi S_{\lambda}^{\lambda} \delta_{\mu\nu}^{\alpha\beta} \} \chi \delta(\Phi) \quad . \quad (45)$$

It can be seen from (41) and (29) that only $\gamma_{\mu\nu} n^\nu$ and $\gamma$ enters for the expression (41) for $S^{\mu\nu}$, with the additional property $\gamma^\dagger = 0$ in scalar-tensor theories. Therefore there remains a part of $\gamma^{\mu\nu}$ which only contributes to the first term of (45) and can be interpreted as being due to the presence of an impulsive gravitational wave. As a shell and a wave generally co-exist the null hypersurface $\Sigma$ is at the same time the history of a shell and of a wave-front. More on this subject will be said in the next section, and an example of this situation will be described in sect.VI.

### III.2 The intrinsic description

We denote by $\xi^a$ ($a = 1, 2, 3$) a set of intrinsic parameters for the hypersurface $\Sigma$, and $e_{(a)} = \partial / \partial \xi^a$ the corresponding tangent basis vectors. The induced metric $g_{ab}$ on the shell is then given by

$$g_{ab} = e_{(a)} \cdot e_{(b)} \quad . \quad (46)$$

In the case of a timelike or spacelike shell, one needs to introduce the extrinsic curvature $K_{ab} \equiv -n \cdot \nabla_{e_{(b)}} e_{(a)}$ where $n$ is the unit normal, $\epsilon = \pm 1$ and the shell is characterized by the jump of $K_{ab}$ across $\Sigma$. For lightlike shells, this quantity does not carry any extrinsic information because of the tangent nature of the normal vector $n$, and we introduce the transverse extrinsic curvature

$$K_{ab} \equiv -N \cdot \nabla_{e_{(b)}} e_{(a)} = -N \cdot \frac{\delta e_{(a)}}{\delta \xi^a} \quad , \quad (47)$$
where \( N \) is the transversal vector already introduced in the previous section (21). Although \( K_{ab} \) depends on the choice of the vector \( N \) it has been shown in [27] that its jump across \( \Sigma \)
\[
\gamma_{ab} \equiv 2 \left[ K_{ab} \right],
\]
is a well defined quantity which is free of the arbitrariness in the transversal \( N \).

It can be shown [27] that \( \gamma_{ab} \) is the projection onto \( \Sigma \) of the \( \gamma_{\mu
u} \)'s introduced in the eq.(24), i.e. \( \gamma_{ab} = \gamma_{\mu
u} e^{(a)}_{\mu} e^{(b)}_{\nu} \).

The four vectors \((N, e_{(a)})\) form an oblique basis with respect to which the normal vector \( n \) can be decomposed as
\[
n = \epsilon \eta N + l^a e_{(a)},
\]
where \( l^a \) are smooth functions. It follows from this decomposition that
\[
g_{ab} l^b = -\epsilon \eta N_a,
\]
where \( N_a \equiv N.e_{(a)} \).

The induced metric is degenerate whenever the shell is lightlike and in that case its inverse cannot be defined. In order to generalize the notion of an inverse metric as a raising indices operator valid in any case, we introduce as in [27] the symmetric matrix \( g^{ab}_* \) such that
\[
g^{ac}_* g_{cb} = \delta^a_b - \eta l^a N_b.
\]
\( g^{ab}_* \) is not uniquely defined by the above relation because one may perform the following transformation \( g^{ab}_* \rightarrow g^{ab}_* + 2 \lambda l^a l^b \), where \( \lambda \) is an arbitrary function, without changing (51). In the non lightlike case \( (\epsilon \neq 0) \), a convenient choice is \( N = n \), hence \( N_a = 0 \), and \( g^{ab}_* \) is the usual inverse metric \( g^{ab} \). In the lightlike case \( (\epsilon = 0) \), we have \( l^a \neq 0 \) and \( N_a \neq 0 \), and it can be checked that \( g^{ab}_* g_{ab} = 2 \). The completeness relation which in the normal basis \((n, e_{(a)})\) is
\[
g^{\mu\nu} = g^{ab}_* e^{(a)}_{\mu} e^{(b)}_{\nu} + \epsilon^{-1} n^\mu n^\nu,
\]
where \( n^\mu = N.e^{(a)}_{\mu} \).
becomes in the oblique basis \((N, e^{(a)})\)
\[
g^{\mu\nu} = g^{ab} e^{(a)} e^{(b)} + 2\eta l^a e_{(a)} N^{(b)} + \eta^2 N^\mu N^\nu . \tag{53}
\]

Because of the tangential nature of the surface stress-energy of the shell \(S_{\mu\nu}\), see (33), we can write
\[
S^{\mu\nu} = S^{ab} e^{(a)} e^{(b)} , \tag{54}
\]
where \(S^{ab}\) is now an intrinsic tensor of \(\Sigma\). As it is known from the distributional description that the surface-energy tensor keeps the same form as in general relativity, its intrinsic form is still equal to (see the eq.(31) of ref.[27])
\[
16\pi G \eta^{-1} S^{ab} = \left( g^{ac} l^{b} l^{d} + l^{a} l^{c} g^{bd} - g^{ab} l^{c} l^{d} - l^{a} l^{b} g^{cd} \right) \gamma_{cd} - \epsilon \left( g^{ac} g^{bd} - g^{ab} g^{cd} \right) \gamma_{cd} \tag{55}
\]

Timelike and spacelike shells \(\epsilon \neq 0\):

Making the convenient choice such that \(N = n\), one recovers from (55) the well-known relation
\[
16\pi G S^{ab} = -\gamma^{ab} + \gamma g^{ab} , \tag{56}
\]
where we have used \(l^a = 0\), \(\epsilon \eta = 1\), \(g^{ab} = g^{ab}\), and \(\gamma_{ab} = 2 \left[ K_{ab} \right]\). The equation for the scalar field is again given by (39) or (40) where now the trace \(S\) is taken from (56).

The influence of the scalar fields \(\varphi^i\) can also be seen in the the hamiltonian and momentum constraints which yield here the two following equations
\[
S^b_{a;b} = - \left[ T^{\mu\nu} e^{(a)} n^{\mu} \right] - \frac{\gamma_{ij}}{4\pi G} \zeta^{i} \nabla_{a} \varphi^{j} \tag{57}
\]
\[
S^{ab} \tilde{K}_{ab} = \left[ T^{\mu\nu} n^{\mu} n^{\nu} \right] + \frac{\gamma_{ij}}{8\pi G} \left[ \nabla_{n} \varphi^{i} \nabla_{n} \varphi^{j} \right] , \tag{58}
\]
where \(\nabla\) is the covariant differentiation with respect to \(g_{ab}\), \(\nabla_{n} = n^{\mu} \nabla_{\mu}\) is the normal derivative, and the tilde is the average \(\tilde{F} = (F^{+} + F^{-})/2\).
lightlike shells $\epsilon = 0$:

In that case, we have $g_{ab} g_{*b} = 2$ and from (50) $g_{ab} l^b = 0$. The trace-free (or pressureless) property which was obtained in the distributional description corresponds here to

$$\gamma^l = \gamma_{cd} l^c l^d = 0 \quad ,$$

thus giving for the surface stress-energy tensor (55)

$$16\pi G \eta^{-1} S^{ab} = \left( g_{*c} l^b l^d + l^a l^b g_{*d} - l^a l^b g_{*d} \right) \gamma_{cd} \quad .$$

As already mentioned in the distributional description a shell and a wave generally co-exist in the lightlike case. The part of $\gamma_{ab}$ which only enters the expression (60) of $S^{ab}$ is $\gamma_{ab} l^b$ and $\gamma = g_{*c} \gamma_{cd}$. This leaves two independent components, denoted by $\hat{\gamma}_{ab}$, corresponding to an impulsive gravitational wave and related to the two degrees of freedom of polarization of the wave. The expression of $\hat{\gamma}_{ab}$ is

$$\hat{\gamma}_{ab} = \gamma_{ab} - \frac{\gamma}{2} g_{ab} + \eta (N_a \gamma_{bc} + N_b \gamma_{ac}) l^c \quad ,$$

where we have used (59).

III.3 Existence of discontinuous gauge fields

When gauge fields are present, the stress-energy tensor $T_{\mu\nu}$ which appears in the total stress-energy tensor $T_{\mu\nu}$, see the eq.(17), is the sum of a pure matter part $T_{m\mu\nu}$ and a gauge field part $T_{F\mu\nu}$ i.e. $T_{\mu\nu} = T_{m\mu\nu} + T_{F\mu\nu}$. Therefore, if there exists a thin shell it will in general carry surface charges and currents acting as sources for the gauge fields and producing discontinuities in these fields across the shell. The surface charges and currents will then enter the 4-vector current in the form of a Dirac $\delta$-term which is related to the discontinuities of the gauge fields in the same way as the surface stress-energy tensor $S^{\mu\nu}$ is related to the discontinuity of the first derivatives of the metric tensor.
Let us consider the general case of a non abelian gauge field of the Yang-Mills type
\[ F_{\mu\nu}^a = \nabla_\mu A_{\nu}^a - \nabla_\nu A_{\mu}^a - e(A_\mu \wedge A_\nu)^a \quad . \] (62)

In this section only, the latin indices refer to the gauge group and cannot be confused with the parameters of the hypersurface introduced earlier. We still use the distributional notation of sect. III.1. In order to produce a surface current, the potential vector \( A_{\mu}^a \) must be only \( C^0 \) on the hypersurface \( \Sigma \), i.e. \( [A_{\mu}^a] = 0 \) and one can write the jump of its first derivatives accross \( \Sigma \) as
\[ [\partial_\mu A_{\nu}^a] = \eta n_\mu \lambda_\nu^a \quad , \] (63)
where \( \lambda_\nu^a \) is a vector field defined on \( \Sigma \) only -see below for more on \( \lambda_\nu^a \). The corresponding gauge field is discontinuous accross \( \Sigma \) and using (63) one gets
\[ [F_{\mu\nu}^a] = \eta(n_\mu \lambda_\nu^a - n_\nu \lambda_\mu^a) \quad . \] (64)

Using this result and the hybrid notation of sect. III.1, one gets for the Yang-Mills field equations
\[ \nabla_\nu \tilde{F}^{\mu\nu}_a = 4\pi [\tilde{J}_a^\mu + j_\mu^a \chi \delta(\Phi)] \quad , \] (65)
where \( \tilde{J}_a^\mu = J_a^{\mu+} \Theta(\Phi) + J_a^{\mu-} \Theta(-\Phi) \) represents the 4-current in the domains \( \mathcal{M}_\pm \) and \( j_a^\mu \) is the surface current. Identifying the \( \delta \)-terms of each side of (65) one gets
\[ 4\pi j_\mu^a = \eta(\lambda_\mu^a n^\mu - \epsilon \eta \lambda_\mu^a) \quad \] (66)
It can be checked from this expression that \( j_\mu^a \) is a tangential quantity, \( j_\mu^a n^\mu = 0 \).

The vector field \( \lambda_\mu^a \) which has been introduced in (63) is not uniquely determined by this equation. As it must have a unique projection onto the hypersurface \( \Sigma \), it is in fact only defined up to the transformation
\[ \lambda_\mu^a \rightarrow \lambda_\mu^a + C^a n_\mu \quad , \] (67)
where $C^a$ is an arbitrary spacetime scalar. In the case of a timelike or spacelike shell ($\epsilon \neq 0$) one can use (67) to choose $C^a$ in order that $\lambda^a_\mu$ is purely tangent, i.e. $\lambda_a.n = 0$, and the surface current (66) reduces to

$$4\pi j^\mu_a = -\epsilon \eta \lambda^\mu_a.$$  \hspace{1cm} (68)

For a lightlike shell ($\epsilon = 0$) such a choice cannot be done and one has

$$4\pi j^\mu_a = \eta (\lambda_a.n)n^\mu.$$  \hspace{1cm} (69)

Hence only the part $\lambda_a.n$ of the vector $\lambda_a$ contributes to the surface current. The remaining part $\lambda_a$ can be written as

$$\lambda^\mu_a = \lambda^\mu_a - (\lambda_a.n)N^\mu,$$   \hspace{1cm} (70)

and characterizes the shock wave which is associated with the discontinuity of the gauge field. Finally, let us consider the behaviour of the stress-energy tensor $T_{\mu\nu}$. Its gauge field part $T^\mu_{\nu}$ has the general form

$$T^\mu_{\nu} = F^\mu_{\lambda}F_{\lambda}^{\nu} - \frac{1}{4}F^2 g^{\mu\nu}.$$  \hspace{1cm} (71)

According to (64) it is discontinuous across $\Sigma$, i.e. $[T^\mu_{\nu}] \neq 0$, and therefore the $\delta$-term which is necessary for the existence of a shell can only come from the pure matter part $T_{m}^{\mu\nu}$.

IV. SPHERICALLY SYMMETRIC SHELLS

The first example of a thin shell in scalar-tensor theories that we consider is a spherical bubble separating two domains $\mathcal{M}_\pm$ where the metrics $g^{\pm}_{\mu\nu}$ and the scalar fields $\varphi^i_{\pm}$ are spherically symmetric. Because of the presence of the scalar fields, the Birkhoff’s theorem no longer apply and the exterior vacuum solution,
i.e. $T_{\mu\nu} = 0$ in (3), is not necessarily static. For simplicity, we shall here restrict ourselves to static solutions and use the following form of the metrics in the two domains

$$ds^2_\pm = -f_\pm(R_\pm) e^{2\psi_\pm(R_\pm)} dt^2 + f^{-1}_\pm(R_\pm) dR_\pm^2 + r_\pm^2(R_\pm) d\Omega^2 ,$$

(72)

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the spherical line element. The scalar fields $\phi^i_\pm$ and the three functions $f$, $\psi$, and $r$ only depend on the coordinate $R$. Some particular solutions to the field equations in static spherically symmetric spacetimes are presented in appendix A.

As the induced metric and the scalar fields are continuous on the hypersurface $\Sigma$ corresponding to the shell we must have on $\Sigma$

$$\begin{cases}
  r_+(R_+) = r_-(R_-) \\
  \phi^i_+(R_+) = \phi^i_-(R_-)
\end{cases}$$

(73)

These matching conditions limit the evolution of the shell and the nature of the junction. Unless they are trivially satisfied, as it may happen if the metrics $g^\pm_{\mu\nu}$ and the scalar fields $\phi^i_\pm$ are identical, the above equations imply that $R_\pm$ take constant values and the shell has to be stationary (examples of these two situations will be given later on). In general relativity only the first equation (73) is present and the shell is not necessarily static but can have a radial motion. It also follows from (73) that in the case of a lightlike shell it can only be located on a common horizon of the two spacetimes $\mathcal{M}_\pm$ as one must have a null and stationary hypersurface. However as it is known that no black hole solution with regular event horizons exist in scalar-tensor theories, no lightlike shell can be introduced - the situation is different with dilaton theories where black hole solutions exist (see for instance [32]). The fact that the junction can only be made on a stationary hypersurface is actually a consequence of our assumption of staticity. Had we removed this condition and considered non static spherically
symmetric spacetimes (for instance analogous to the Vaidya solution), radially expanding or contracting shells might have been introduced.

For simplicity we shall henceforth assume that only one scalar field is present. Let us consider a timelike shell and let it for the moment have an arbitrary radial motion. The induced metric on the timelike surface Σ takes the form

$$ds^2 = -d\tau^2 + r^2(\tau) d\Omega^2,$$

where $\tau$ is the proper time. The normalized velocity $u = d/d\tau$ ($u.u = -1$) and the unit normal $n$ ($n.n = 1$, $u.n = 0$) have components given by (omitting the ± indices)

$$u^a = \left(\frac{\epsilon_1 \sqrt{f + \dot{R}^2}}{f e^\psi}, \dot{R}, 0, 0\right), \quad n^a = \left(\frac{\epsilon_2 \dot{R}}{f e^\psi}, \epsilon_1 \epsilon_2 \sqrt{f + \dot{R}^2}, 0, 0\right)$$

where $\dot{\tau} = d/d\tau$, $\epsilon_1, \epsilon_2 = \pm 1$; note that $\epsilon_1 \epsilon_2 = \text{sign}(n^a \partial_a r)$.

Because of the spherical symmetry, the surface stress-energy tensor has the perfect fluid form

$$S_{ab} = (\sigma + p) u_a u_b + p g_{ab},$$

$\sigma$ being the surface energy density and $p$ the surface pressure.

Using the results of section III.1,2, one obtains, as $N = n$ and $\epsilon = \eta = 1$

$$-4\pi G \sigma = \left[K^{\theta}_\theta\right],$$

$$8\pi G p = \left[K^\tau_\tau\right] + \left[K^\theta_\theta\right]$$

$$4\pi G (\sigma - 2p) \alpha^i = \zeta^i.$$

The non-zero components of the extrinsic curvature $K_{ab}$ are equal to

$$K^{\theta}_\theta = K^{\phi}_\phi = \epsilon_1 \epsilon_2 \frac{\dot{r}'}{r} \sqrt{f + \dot{R}^2},$$

$$K^\tau_\tau = \frac{\epsilon_1 \epsilon_2}{\sqrt{f + \dot{R}^2}} \left[\ddot{R} + \frac{f'}{2} + \frac{\psi'(f + \dot{R}^2)}{2 e^\psi}\right],$$

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with \( \, \frac{d\tau}{\tau} = d/dR \). Furthermore the \( \tau \)-component of the conservation equation (57) gives

\[
\frac{dM}{d\tau} = -p \frac{dA}{d\tau} + A \{ T_{\mu\nu} u^\mu n^\nu \} \; + \; \frac{A C^i}{4\pi G} \phi^j \gamma_{ij} \; ,
\]

(82)

with \( A = 4\pi r^2 \) and \( M = \sigma A \) being resp. the proper area and the inertial mass of the shell. In the particular case of a domain wall, the equation of state is \( \sigma + p = 0 \) and (82) becomes

\[
\dot{\sigma} = \{ T_{\mu\nu} u^\mu n^\nu \} + 3 \sigma \alpha_i \dot{\phi}^i \; .
\]

(83)

The first example that we consider is a static spherical shell carrying an electric charge \( Q \). The interior spacetime \( M_- \) is flat with a metric given by

\[
ds^2_- = -dt^2 + dR^2 + R^2 \, d\Omega^2 \; ,
\]

(84)

and a constant scalar field, \( \varphi_- = \varphi_0 \). The exterior spacetime corresponds to the Reissner-Nordström-Brans-Dicke solution described in appendix A. The shell is static and has a constant radius \( r_0 \) which, according to the matching relations (73), satisfies \( r_0 = R_{-0} = r_+(R_{+0}) \). Using these relations and the solution of appendix A one gets

\[
r_0^2 = e^{2(a-b)\varphi_0/d} \frac{[c h^2 \lambda - e^{2b\varphi_0/d} s h^2 \lambda]}{[1 - e^{2(a-b)\varphi_0/d}]^2} \; .
\]

(85)

where \( \lambda \) is the charge parameter. This equation shows that the constant radius of the shell depends on the parameters of the spacetimes bordering the shell. The case of an uncharged shell is simply obtained by putting \( \lambda \) equal to zero. Finally the surface energy density and pressure are obtained from (77) and (78) with \( \dot{R} = \ddot{R} = 0 \) and the jump for the scalar field is given by (79).

Our next example in spherical symmetry is a spherical domain wall separating two identical vacuum spacetimes. The metric and the scalar fields are still of the form (72). As the spacetimes are identical the matching relations (73) are trivially
satisfied and the shell can radially expand or contract. For a domain wall we have \( \sigma + p = 0 \), and if it is assumed to be embedded in either true or false vacuum we have \( T^\pm_{\mu\nu}u^\mu n^\nu = 0 \). It then follows from (83) that the surface energy density varies according to
\[
\sigma = \sigma_0 e^{3\alpha \varphi}.
\]
where \( \sigma_0 = \text{const} \). It should be noticed that the physical surface energy density is not \( \sigma \) but rather the quantity \( \tilde{\sigma} \) expressed in the Jordan-Fierz frame. If one uses the conformal transformation (15) one obtains the constant value \( \tilde{\sigma} = \sigma_0 \) as expected for a domain wall embedded in true or false vacuum.

The equation of motion of the domain wall is deduced from (77) with \( K^\theta_\theta \) given by (80). As the two sides of the wall are identical, one sees that the shell only exists if the two products \( (\epsilon_1 \epsilon_2)_\pm \) take opposite values, which corresponds to the situation where the shell separates either two interior or two exterior geometries. Squaring (77) one gets the following equation of motion
\[
\dot{R}^2 + V_{\text{eff}}(R) = -1,
\]
where we have introduced the effective potential
\[
V_{\text{eff}}(R) = -1 + f(R) - 4\pi^2 G^2 \sigma^2(R)r^3(R)(dr/dR)^{-2}.
\]
This equation shows that the motion is only possible provided that \( V_{\text{eff}}(R) < -1 \).

If the spacetimes correspond to the true vacuum with a vanishing mass parameter \( b \)-see the appendix A, it can be shown that the domain wall can only undergo a bouncing motion starting from infinity down to some minimal radius and back to infinity. If \( b \) does not vanish all types of motion are a priori possible (monotonic and bouncing) according to the values taken by the spacetime parameters \( a, b \) and \( d \) and the Brans Dicke parameter \( \alpha \). The situation is similar if we consider false-vacuum instead of true-vacuum.
V. PLANAR SHELLS AND IMPULSIVE WAVES

The gravitational properties of planar shells have been extensively studied in general relativity with particular emphasis on domain walls and application to cosmology. The vacuum reflection-symmetric solution of the Einstein equations for an infinitely thin planar domain wall was obtained by Vilenkin [33] and later generalized to walls with a given equation of state [34] and without reflection symmetry [35]. Some planar solutions for supersymmetric walls including dilaton were also obtained by Cvetic et al [19] and by Schmidt and Wang [21] in the Brans Dicke theory.

In this section, we present two examples with planar symmetry in a scalar-tensor theory where for simplicity only one scalar field $\varphi$ is introduced. We first give the exact solution for a domain wall surrounded by vacuum which is the counterpart in the Brans Dicke theory of the solution given by Vilenkin in general relativity. Then we study, as an illustration of our formalism for the null case, a plane lightlike shell accompanied by a plane impulsive wave.

It is known from Taub [36] that any plane-symmetric metric can be written as

$$ds^2 = e^{2\nu(t,z)} \left(-dt^2 + dz^2\right) + e^{2\mu(t,z)} \left(dx^2 + dy^2\right).$$

(89)

where the plane of symmetry is $z = 0$. Any reflection symmetric solution with respect to the plane $z = 0$, where the domain wall is located, must satisfy $\nu(t, z) = \nu(t, -z)$ and $\mu(t, z) = \mu(t, -z)$, and for the scalar field $\varphi(t, z) = \varphi(t, -z)$. We shall henceforth call $\nu_0(t)$, $\mu_0(t)$ and $\varphi_0(t)$ the values of $\nu$, $\mu$ and $\varphi$ at the hypersurface $\Sigma, z = 0$. In this example the distributional description of section III.1 will be used. The induced metric on $\Sigma$ is

$$ds^2|_{\Sigma} = -e^{2\nu_0(t)} \, dt^2 + e^{2\mu_0(t)} \left(dx^2 + dy^2\right).$$

(90)
The unit normal is \( n^\alpha = (0, e^{-\nu_0}, 0, 0) \) which from (20) implies that \( \chi = e^{\nu_0} \), and \( \epsilon = +1 \) as we are in the timelike case. Choosing \( N = n \) for the transversal we have \( \eta = 1 \) in (21). Then one finds for the jumps \( \gamma_{\alpha\beta} \) and \( \zeta \) of the first derivatives of the metric and the scalar field which were defined in (24-25)

\[
\begin{align*}
\gamma_{00} &= -\gamma_{11} = -2 e^{\nu_0} [\nu_{,z}] \\
\gamma_{22} &= \gamma_{33} = e^{2\mu_0 - \nu_0} [\mu_{,z}]
\end{align*}
\]

and

\[
\zeta = e^{-\nu_0} [\varphi_{,z}].
\]

As the equation of state of the domain wall is \( \sigma + p = 0 \) one derives from the equations of junction (38)

\[
4\pi G \sigma = -e^{-\nu_0} [\mu_{,z}] \quad (93)
\]

\[
[\nu_{,z}] = [\mu_{,z}],
\]

and from the scalar field junction condition (39)

\[
[\varphi_{,z}] = -3\alpha [\mu_{,z}].
\]

The time component of the equation of conservation (37) gives

\[
\frac{\partial \sigma}{\partial t} = -e^{\nu_0} [T^{\mu\nu} u^\mu n^\nu] + 3\alpha \sigma \frac{\partial \varphi}{\partial t}.
\]

and as the domain wall is surrounded by vacuum on each side \( (T^\pm_{\mu\nu} = 0) \), it immediately follows that

\[
\sigma = \sigma_0 e^{3\alpha \varphi_0(t)}.
\]

where \( \sigma_0 \) is a constant. It can be checked that the physical surface energy density \( \tilde{\sigma} \), which is derived from \( \sigma \) by using the conformal transformation (15) is a constant, \( \tilde{\sigma} = \sigma_0 \), as expected for a domain wall in vacuum.
The unknown functions $\mu(t, z), \nu(t, z)$ of the metric and the scalar field $\varphi(t, z)$ are derived from the field equations and the boundary conditions. Gathering all these results it can be shown that the exact solution for the metric and the scalar field is

$$ds^2 = e^{a(9\alpha^2 t-z)}(-dt^2 + dz^2) + e^{a(t-z)}(dx^2 + dy^2)$$

and

$$\varphi(t, z) = -\frac{3\alpha a}{2}(t - |z|),$$

where $a$ is a constant of integration which using (90) and (94), is related to $\sigma_0$ according to, $4\pi G \sigma_0 = a$. The Vilenkin solution is recovered by putting the Brans-Dicke parameter $\alpha$ equal to zero. As in this solution the space-time is still locally flat everywhere except at $z = 0$ and the $(t, z)$ part of the metric can be written in a form which is conformally related to the Rindler metric.

The following example corresponds to the lightlike case. It will be worked out in the intrinsic formalism developed in section 3.2. An appropriate form of the metric is the Szekeres one

$$ds^2 = -2e^{-M}dudv + e^{-U}(e^V dx^2 + e^{-V}dy^2),$$

where the functions $M, U, V$ and the scalar field $\varphi$ only depend on the null coordinates $(u, v)$, and $(x, y)$ are coordinates in the planes $z = \text{const.}$ - we use the ordering $u, v, x, y$ and greek indices range from 0 to 3.

The null hypersurfaces, $u = \text{const.}$, are generated by null geodesics with tangent, $n = \partial / \partial v$ - note that $v$ is not an affine parameter. These null generators have expansion $\rho$ and shear $\sigma$ equal to

$$\rho = -\frac{U_v}{2}, \quad \sigma = \frac{V_v}{2},$$

where the subscript $v$ indicates partial differentiation with respect to $v$ (similar results hold on the hypersurfaces $v = \text{const.}$).
The spacetime is divided at \( \Sigma (u = 0) \) into two halves \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) where \( \mathcal{M}_+ \) (\( \mathcal{M}_- \)) is to the future (past) of \( \Sigma \) and corresponds to \( u > 0 \) (\( u < 0 \)). To save subscripts we shall drop the minus subscripts on any quantity referring to \( \mathcal{M}_- \).

We assume that \( \mathcal{M}_- \) is flat with coordinates \((u, v, x, y)\) and line element of the form (100) with \( M = U = V = 0 \), and the scalar field is constant \( \varphi \equiv \varphi_0 \). In the second half \( \mathcal{M}_+ \) the coordinates are \((u, v_+, x_+, y_+)\), the line element \( ds_+^2 \) is of the form (100), with \( M^+, U^+, V^+ \) and the scalar field \( \varphi^+ \) depending on the null coordinates \((u, v_+)\). We have taken for simplicity \( u_+ = u \). The two spacetimes are glued along \( \Sigma \) by making the identification

\[
(0, v_+, x_+, y_+) = (0, v - F(x, y), x, y)
\]

where \( F \) is an arbitrary smooth function of \( x \) and \( y \) alone and produces a shift in the null coordinate tangent to the hypersurface.

We take \( \xi^a = (v, x, y) \) with \( a = 1, 2, 3 \), as intrinsic parameters on \( \Sigma \), and as the normal is \( n = e_1 \) we obtain from (49) \( l^a = \delta^a_1 \). Continuity at \( u = 0 \) of the induced metric and of the scalar field requires that

\[
U^+(0, v_+) = V^+(0, v_+) = 0,
\]

\[
\varphi^+(0, v_+) = \varphi_0.
\]

The induced metric reduces to \( g_{ab} = \text{diag}(0, 1, 1) \) and one may take for its ‘inverse’ (51), \( g^{ab}_* = \text{diag}(0, 1, 1) \). A convenient choice for the transversal \( N \) corresponds to \( N.n = -1, N.e_{(2)} = N.e_{(3)} = 0, N.N = 0 \), thus leading to components equal to \( N^a = (1, 0, 0, 0) \) in \( \mathcal{M}_- \), and

\[
N^a_+ = (-e^{M_0^+}, F_x^2 + F_y^2, -F_x, -F_y)
\]

in \( \mathcal{M}_+ \) where \( M_0^+ \equiv M^+(0, v_+) \). Introducing these results into (47) one obtains the values of the transverse extrinsic curvature \( K_{ab} \) on each side of the shell.
As it vanishes in $\mathcal{M}_-$ one simply gets from (48) for the jumps, $\gamma_{ab} = 2\mathcal{K}_{ab}^+$. Then using the trace-free property (59) one shows that $M_0^+ = \text{const.}$ which must be equal to zero as the other side is flat, and the non-zero components of $\gamma_{ab}$ are equal to

$$
\gamma_{22} = -2F_{xx} - U_u^+(0, v_+) + V_u^+(0, v_+)
$$
$$
\gamma_{33} = -2F_{yy} - U_u^+(0, v_+) - V_u^+(0, v_+)
$$
$$
\gamma_{23} = \gamma_{32} = -2F_{xy}
$$

Therefore the surface stress-energy tensor (60) is of the form, $-S^{ab} = \sigma l^a l^b$, and has only one non-vanishing component equal to

$$
16\pi G S^{11} = \gamma_{22} + \gamma_{33}
$$

These results show that the shell has no shear and is only characterized by its surface energy density which has the following expression in terms of the functions $F$ and $U^+$

$$
8\pi G \sigma = \Delta F + U_u^+(0, v_+).
$$

On the other hand the wave part (61) of $\gamma_{ab}$ has the only non-vanishing components

$$
\hat{\gamma}_{22} = -\hat{\gamma}_{33} = -F_{xx} + F_{yy} + V_u^+(0, v_+)
$$
$$
\hat{\gamma}_{23} = \hat{\gamma}_{32} = -2F_{xy}
$$

According to the form taken by the functions $F(x, y), U^+(u, v_+)$ and $V^+(u, v_+)$ different types of situations can occur: we can have only a lighlike shell, only an impulsive wave or both. A more extended version describing the geometry of the spacetimes $\mathcal{M}_\pm$ and the properties of the shell and the wave will be presented in forthcoming paper [37].
VI. CONCLUDING REMARKS

Besides the scalar-tensor theories of gravity which have here been considered there exist, as mentioned in the introduction, other alternative theories of gravity which also introduce scalar fields such as the dilaton. Although they look quite similar to scalar-tensor theories when no matter field is present, the dilatonic theories of gravity present an important difference which is due to the way the dilaton couples to the other fields. In this last section we would like to briefly discuss the smoothness properties of the dilaton across a singular hypersurface Σ and point out how they differ from those obtained in the scalar-tensor theories.

Let us use the following expression for the action in the presence of a dilaton in the Einstein metric [8]

\[ S = \int d^4x \sqrt{-g} \left[ R - 2 (\nabla \varphi)^2 \right] - \int d^4x \frac{\sqrt{-g}}{4} k_F e^{-2\kappa \varphi} F^2 + S_m [\Psi, \varphi, g] , \]

(110)

where \( \varphi \) is the dilaton, \( \Psi \) a matter field, \( F \) a Maxwell field with \( F = dA \), \( A \) being the potential. Here, \( q \) is the gravitational coupling constant \( (q = 4\pi G) \), \( \kappa \) is the coupling constant to the dilaton, and \( k_F \) is the coupling constant for the gauge field \( F \). The action \( S_m \) for the matter fields \( \Psi \) can for instance be taken as

\[ S_m [\Psi, \varphi, g^{\alpha\beta}] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (D_\alpha \Psi) (D^\alpha \Psi)^* - e^{2\kappa \varphi} V(\Psi) \right] , \]

(111)

where \( D_\alpha = \partial_\alpha + ieA_\alpha \) is the gauge-covariant derivative, \( e \) the associated charge, and \( V \) a potential.

Such an expression for the total action \( S \) shows that the dilaton does not minimally (i.e. metrically) couple to the different fields but that it induces spacetime dependent coupling factors. The field equation for the dilaton which follows from this action is

\[ \Box \varphi = -\frac{q\kappa k_F}{2} e^{-2\kappa \varphi} F^2 + 2q\kappa e^{2\kappa \varphi} V(\Psi) . \]

(112)
In comparison with the analog equation (4) in the scalar-tensor theories, one immediately sees that only the potential $V(\Psi)$ and not the trace $T(\Psi)$ of the stress-energy tensor of the matter field appear in the r.h.s. Therefore as the matter field $\Psi$ is discontinuous at the hypersurface $\Sigma$ and as the kinetic terms no longer appear in the field equation for the dilaton, one concludes that the dilaton $\varphi$ is necessarily $C^1$ on $\Sigma$—recall that the Maxwell field $F$ is at most discontinuous across $\Sigma$. The jump in the second derivatives of $\varphi$ across $\Sigma$ is thus given by

$$\left[ g^{\mu\nu} \partial_{\mu\nu} \varphi \right] = -\frac{q \kappa k_F}{2} e^{-2\kappa \varphi|\Sigma} [F^2] + 2 q \kappa e^{2\kappa \varphi|\Sigma} [V(\Psi)] .$$

(113)

It follows from this rapid investigation that the scalar fields in scalar-tensor theories behave differently than the dilaton across a singular hypersurface: the former are only $C^0$ while the latter is $C^1$ across a singular hypersurface. This is a consequence of the difference of their coupling to the matter field. As the dilaton is $C^1$ it will not contribute to the singular $\delta$-term appearing in the field equations, and the expression for the surface stress-energy tensor of an arbitrary shell is not affected by the presence of the dilaton and remains the same as in general relativity.

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**APPENDIX : STATIC SPHERICALLY SYMMETRIC SOLUTIONS**

We present in this appendix some static spherically symmetric solutions of the Brans-Dicke theory. As they have an analog in general relativity we call them by the same name.

*Analog of the Schwarzschild solution*
As this solution has already been elsewhere—see Damour and Esposito-Farese [7]—we only recall here its main properties. In the Einstein-frame, an exterior spherically symmetric solution satisfying the system of equations (3) and (4) in vacuum (ψ = 0) is given by:

\[
f(R) = \left(1 - \frac{a}{R}\right)^{\frac{b}{a}} \quad (A.1)
\]

\[
r^2(R) = R^2 \left(1 - \frac{a}{R}\right)^{1 - \frac{b}{a}} \quad (A.2)
\]

\[
e^{2\varphi} = \left(1 - \frac{a}{R}\right)^{\frac{d}{a}} \quad , \quad (A.3)
\]

where \(a\), \(b\) et \(d\) are positive constants, the values of which are restricted by the condition

\[
a^2 = b^2 + d^2 \quad . \quad (A.4)
\]

The above form of the metric is valid in the domain \(R > a\). This solution is the analog of the Schwarzschild metric with \(b\) playing the role of the mass parameter. We obtain an analog of the Minkovski spacetime by putting \(b = 0\) in the above solution. Note that the resulting metric is not flat because of the presence of the scalar field.

**Analog of the Reissner-Nordstöm solution**

In general relativity the Harisson transformation allows to generate a charged solution of the Einstein-Maxwell fields equations from a static vacuum solution. In the same way, starting from the static spherically symmetric uncharged solution obtained above (A1-4), one gets, using a similar transformation a family of static spherically symmetric charged solutions indexed by an arbitrary non-zero parameter \(\lambda\) and satisfying the Brans-Dicke-Maxwell equations, which is given by

\[
f(R) = g^{-2}(R) \left(1 - \frac{a}{R}\right)^{\frac{b}{a}} ; \psi = 0 \quad (A.5)
\]
\[ r^2(R) = g^2(R) R^2 \left( 1 - \frac{a}{R} \right)^{1 - \frac{b}{a}} \]  
\[ e^{2\varphi} = \left( 1 - \frac{a}{R} \right)^{\frac{d}{a}} \]  
\[ (A.6) \]

where the function \( g \) is defined by

\[ g(R) = \cosh^2 \lambda - \left( 1 - \frac{a}{R} \right)^{\frac{b}{a}} \sinh^2 \lambda . \]  
\[ (A.8) \]

The only non-zero component of the electromagnetic potential \( A_\mu \) is equal to

\[ A_t = \frac{\left[ \left( 1 - \frac{a}{R} \right)^{\frac{b}{a}} - 1 \right] \sinh(2\lambda)}{2g(R)} \]  
\[ (A.9) \]

which yields an electromagnetic field \( F_{\mu\nu} \) with the only non-zero component

\[ F_{rt} = \frac{Q}{r^2} \]  
\[ (A.10) \]

where \( Q \) is the electric charge which is related to the parameter \( \lambda \) as

\[ Q = \frac{b}{2} \sinh(2\lambda) \]  
\[ (A.11) \]

It can be checked that putting \( \lambda = 0 \) in these results gives back the analog of the Schwarzschild solution (A.1-4).
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