The Gauge-Fixing Fermion in BRST Quantisation

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Alice Rogers
Department of Mathematics
King’s College
Strand
London WC2R 2LS, Great Britain

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Abstract

Conditions which must be satisfied by the gauge-fixing fermion \( \chi \) used in the BRST quantisation of constrained systems are established. These ensure that the extension of the Hamiltonian by the gauge-fixing term \([\Omega, \chi]\) (where \( \Omega \) is the BRST charge) gives the correct path integral.

In canonical BRST quantisation (particularly the path-integral approach developed by Batalin, Fradkin, Fradkina and Vilkovisky in a series of papers [1, 2, 3, 4, 5] (BFV) and Henneaux [6]) the starting point is the simple 2n-dimensional phase space \( \mathbb{R}^{2n} \) with \( n \) position coordinates \( q^i \) and \( n \) momenta
$p_i$, together with $m$ first class constraints $T_a(p, q) = 0$ and a first class Hamiltonian $H(p, q)$. (Here and later, the integer $n$ is greater than $m$, the index $i$ runs from 1 to $n$ and the index $a$ runs from 1 to $m$.) The presence of the constraints means that the true phase space has the more complicated structure of a quotient space of a non-trivial manifold; the key idea in BRST quantization is that cohomology classes on a simpler extended phase space are used instead of observables on this complicated true phase space.

The extended phase space used is the $(2n + 2m, 4m)$-dimensional superspace $\mathbb{R}^{2n+2m, 4m}$ with coordinates $p_i, q^i, k_a, l^a, \eta^a, \pi_a, \theta_a, \phi_a$, where the $k_a$ and $l^a$ are commuting conjugate pairs, while the $\eta^a$ and $\pi_a$ are anticommuting conjugate pairs, as are the $\theta^a$ and the $\phi_a$. (The anticommuting coordinates correspond to ghosts, antighosts and their momenta. Ghost number is defined so that $\eta^a$ and $\theta^a$ have ghost number 1 while $\pi_a$ and $\phi_a$ have ghost number $-1$.) Collectively these coordinates are denoted $P, Q$. The BRST charge $\Omega$, which has ghost number 1, has the form

$$\Omega = T_a \eta^a + k_a \theta^a + \text{higher order terms}, \quad (1)$$

the terms of higher order in the anticommuting variables being determined by the requirement that $\Omega^2 = 0$. (Quantisation is carried out by the standard canonical procedure, with states being functions $f(q, l, \eta, \phi)$ and the corresponding conjugate momenta $p, k, \pi$ and $\theta$ defined as derivatives in the normal manner.)

The main result given by BFV and Henneaux is that the vacuum expectation value for the theory has the path integral expression

$$\int DPDQ \exp i \left( \int_0^t p_a \dot{q}^a + k_a \dot{l}^a + \pi_a \dot{\eta}^a + \theta^a \dot{\phi}_a + H(P, Q) + K(P, Q) \right) dt \quad (2)$$

where $K = [\Omega, \chi]$ is the commutator of the BRST charge $\Omega$ with a field $\chi$ of ghost number $-1$ (the gauge-fixing fermion), and the path integral is taken over closed paths. (In some cases $H$ must be extended by terms involving ghosts so that the modified Hamiltonian commutes with the BRST charge $\Omega$.)

In the work of Henneaux it is implied that the gauge-fixing fermion $\chi$ is arbitrary; however, it is also clear that $\chi$ cannot, for instance, be zero, and so one sees that some non-singularity condition is called for. In the BFV papers, the gauge-fixing fermion is assumed to take the form

$$\chi = l^a \pi_a + X^a \phi_a + \text{higher order terms} \quad (3)$$
where the $m$ even functions $X^a$ must be such that the matrix formed by the commutators $[T_a, X^b]$ is non-singular. (In fact the proof given by BFV is only valid when the constraints commute, but the more general result has recently been established by Batalin and Marnelius [7].)

The purpose of this talk is to describe a criterion which the gauge-fixing fermion must satisfy, together with a simple proof of the main result. This is first to clarify existing work and secondly to assist the study of cases where the Gribov problem might appear to prevent the existence of a gauge-fixing fermion.

The starting point is the knowledge that the true space of states is $H_0(\Omega)$, the $\Omega$-cohomology group of states of ghost number zero. The first observation is that if $L$ is an operator which commutes with $\Omega$ and has ghost number zero, then (as shown by Schwarz [12])

\[
\text{Supertrace } L = \sum_{k=-m}^{k=+m} (-1)^k \text{Trace}_{H_k(\Omega)} L. \tag{4}
\]

The only contribution to the supertrace comes from $\Omega$-closed states which are not exact, since, if $Lf = \lambda f$ then $L\Omega f = \lambda \Omega f$, so that if $\Omega f$ is not zero we have two states with ghost number differing by one and equal $L$ eigenvalues; thus the contribution from non-closed states cancels with that from exact states. (This resembles the argument used by McKean and Singer in the context of index theory [8])

A second observation is that the path integral (2), being over closed paths, gives the supertrace of $\exp i(H + [\Omega, \chi])t$ – provided of course that this operator does have a supertrace – so that, if all cohomology groups other than the zeroth one vanish, the superspace path integral reduces to a trace over the space $H_0(\Omega)$ of physical states. The path integral (2) will then give the required physical result.

The gauge-fixing fermion must thus be chosen so that the commutator $[\Omega, \chi]$ can play a crucial double role, both ensuring the vanishing of the cohomolgy groups other than that at zero ghost number, and making the extended Hamiltonian sufficiently regular for the operator $\exp i(H + [\Omega, \chi])t$ to have a well-defined supertrace. We start with the simple fact that if $[\Omega, \chi]$ were an invertible operator we would have no $\Omega$-cohomology at all. This may be shown by considering a state $f$ such that $\Omega f = 0$; then, if $h = [\Omega, \chi]f$,

\[
f = [\Omega, \chi]^{-1} h = [\Omega, \chi][\Omega, \chi]^{-2} h
\]
\[
= (\Omega \chi + \chi \Omega)[\Omega, \chi]^{-2}h
\]
\[
= \Omega \chi[\Omega, \chi]^{-2}h + \chi[\Omega, \chi]^{-2}\Omega f
\]
\[
= \Omega \chi[\Omega, \chi]^{-2}h,
\]
so that \( f \) must be cohomologically trivial. (Here the fact that \( \Omega \) commutes with \([\Omega, \chi]\), and hence with \([\Omega, \chi]^{-1}\) has been used.)

Now of course we do want some cohomology; if \( \chi \) existed such that \([\Omega, \chi]\) was invertible we would have no physical states; what in fact we require is that \( \chi \) be such that the only states on which \([\Omega, \chi]\) is not invertible are zero ghost number states which are not exact. This then ensures that there is no cohomology except at ghost number zero. This criterion is not quite sufficient to ensure that the path integral (2) gives the correct result, because the operator \( H = [\Omega, \chi] \) may not be sufficiently regular to have a well-defined supertrace. The additional condition which \([\Omega, \chi]\) must satisfy is that \( H + [\Omega, \chi] \) should have positive discrete eigenvalues tending to infinity in each ghost sector, ensuring the necessary absolute convergence of the sums involved in the supertrace.

In the standard simple example it may easily be seen that these conditions are satisfied. In this example, where \( T_a = p_a \) and \( X^a = q^a \), the quartet mechanism of Henneaux and Teitelboim [9] shows that \([\Omega, \chi]\) is a number operator counting ghost and gauge number, and has precisely the required analytic properties.

A longer paper, with more details of this work together with non-trivial examples, is in preparation. Given the group theoretic analysis of BRST cohomology by Kostant and Sternberg [11], it is possible that a development of recent interesting work of Klauder [10] using coherent states can be used to provide a general method for applying the BFV approach to systems affected by the Gribov problem. The relationship of the construction of the gauge-fixing fermion used by McMullan [13] to the analysis in this paper would also be interesting.

References


