Abstract: We have studied the kinetics of \( q \)-deformed bosons and fermions, within a semiclassical approach. This investigation is realized by introducing a generalized exclusion-inclusion principle, intrinsically connected with the quantum \( q \)-algebra by means of the creation and annihilation operators matrix elements. In this framework, we have derived a non-linear Fokker-Planck equation for \( q \)-deformed bosons and fermions which can be seen as a time evolution equation, appropriate to consider non-equilibrium or near-equilibrium systems in a semiclassical approximation. The steady state of this equation reproduces in a simple mode the \( q \)-oscillators equilibrium statistics.

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Recently, there has been a great deal of interest in the study of the quantum groups and of their applications to \( q \)-deformed bosons and fermions. Several models [1–3] of \( q \)-deformed free bosons and fermions gas [5,4,6–10] have been applied to Bose gas condensation [11,12], anyon fields [13], phonon spectrum in \(^3\)He [14], asymmetric XXZ Heisenberg chain [15], nuclear and molecular physics [16–19], conformal field theories [20] and quantum Lie superalgebras [21].

The \( q \)-boson and \( q \)-fermion particles are defined by an appropriate deformation of the commutation-anticommutation relations of the creation and annihilation operators, modifying the exchange factor between permutated particles.

For \( q \)-boson particles, we have the \( q \)-deformed commutations [3,4]

\[
[a, a^\dagger] = [a^\dagger, a] = 0, \quad a a^\dagger - q a^\dagger a = q^{-N};
\]

the Hilbert space with basis \(| n >\) is constructed with these features

\[
\begin{align*}
& a \mid 0 >= 0, \quad a^\dagger \mid n >= [n + 1]^{1/2} \mid n + 1 > , \\
& a \mid n >= [n]^{1/2} \mid n - 1 >=, \quad N \mid n >= = n \mid n >=
\end{align*}
\]

implying the following operatorial relations

\[
a^\dagger a = [N], \quad a a^\dagger = [1 + N].
\]

The notation [ ] is defined in terms of the deformation parameter \( q \) as

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

In the limit \( q \to 1 \), \([x] \to x\) and one reproduces the familiar boson algebra.

For \( q \)-fermion operators \( b \) and \( b^\dagger \), analogously to the boson case, we have the anticommutation relations [3,4]

\[
\{b, b\} = \{b^\dagger, b^\dagger\} = 0, \quad b b^\dagger + q^{-1} b^\dagger b = q^{-N}.
\]

The number operator \( N \), for this case, can take only the values \( n = 0, 1 \) and the above anticommutations imply the following relations

\[
b \mid 0 >= = b^\dagger \mid 1 >= = 0, \quad \mid 1 >= = b^\dagger \mid 0 >
\]

and, as a consequence, we have

\[
b b^\dagger = [N], \quad b^\dagger b^\dagger = [1 - N].
\]

In the recent past several authors [4,7,11,22], assuming the relations of Eqs.(3) and (7), have derived the statistical distributions of \( q \)-oscillators (let us recall at this point that the statistical distributions can not be derived from the thermal averages \(< a^\dagger a >\) or \(< b^\dagger b >\) because these quantities are the average numbers of particles only in the undeformed \((q = 1)\) case).

In this letter we want to show that the thermodynamic and statistical properties of \( q \)-deformed bosons and fermions can be derived in the framework of a kinetic approach, recently proposed by us [23,24]. The crucial point of our model is the introduction in the transition probability of an exclusion-inclusion Pauli principle which can be seen as the semiclassical consequence of the partial antisymmetrization-symmetrization of the quantum wave function. Assuming that the diffusional current must satisfy general conditions and that the exclusion-inclusion principle be valid in analogy to the pure fermion-boson case, we deduce a generalized non-linear Fokker-Planck equation appropriate to study the equilibrium and also the non-equilibrium conditions of \( q \)-oscillators.

Starting point of our approach is the Pauli master equation for the mean occupation function \( n(t, v)\) in terms of the transition probability \( \pi(t, v \to u)\) from the state \( v \) to the state \( u \). In the case of the one dimensional velocity space and in the limit of nearest-neighbor interaction, with infinitesimal transition from the state \( v \) to the state \( v + dv \), the master equation is

\[
\frac{\partial n(t, v)}{\partial t} = \pi(t, v - dv \to v) + \pi(t, v + dv \to v) - \pi(t, v \to v - dv) - \pi(t, v \to v + dv).
\]
The present formalism can be extended to a general dimension velocity space, for this generalization and a more exhaustive discussion of the above master equation we send to Ref. [23].

To introduce, from a semiclassical point of view, the quantum behavior of a generic \( q \)-oscillator boson or fermion system, we postulate a transition probability with a generalized exclusion-inclusion principle as [23,24]

\[
\pi(t,v \to u) = r(t,v,v-u) \varphi[n(t,v)] |\psi[n(t,u)]| , \quad (9)
\]

where \( r(t,v,v-u) \) is the transition rate from the state \( v \) to the state \( u \), \( \varphi(n) \) and \( \psi(n) \) are real functions that inhibit or enhance the transition probability from a site to another one. In fact, \( \varphi(n) \) is a function depending on the occupational distribution at the initial state \( v \) which must satisfy the condition \( \varphi(0) = 0 \) (if the initial state is empty, the transition probability is equal to zero) and \( \psi(n) \) is a function depending on the arrival state and satisfies the condition \( \psi(1) = 1 \) (if the arrival state is empty, the transition probability is not modified). If we choose \( \varphi(n) = n \) and \( \psi(n) = 1 \) we find the standard classical linear kinetics.

Explicit expressions of the functions \( \varphi(n) \) and \( \psi(n) \) can be used to simulate semiclassically a quantum exclusion-inclusion principle in the transition probability. In Ref. [24,25] we have studied, in this framework, the generalized Haldane exclusion statistics [26] and the quantum extension of the non-extensive Tsallis entropy [27].

Because of the quantum meaning of the functions \( \varphi(n) \) and \( \psi(n) \), it must exist an intrinsic relation between these functions and the quantum algebra of the creation and annihilation operators matrix elements. As \( \varphi(n) \) is proportional to the probability of finding in the state \( v \) the occupational number \( n \) and \( \psi(n) \) is proportional to the probability of introducing an extraparticle into a state with occupational number \( n \), we can postulate the following relations

\[
\varphi(n) \propto |n-1| a_n |n>|^2 , \quad (10)
\]

\[
\psi(n) \propto |n+1| a_n^* |n>|^2 , \quad (11)
\]

implying an important connection between the semiclassical kinetic approach and the quantum algebra.

If we expand up to the second order in power of \( dv \) the r.h.s. of Eq.(8), we obtain the non-linear generalized Fokker-Planck equation [23,24]

\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial v} \left[ \left( J + \frac{\partial D}{\partial v} \right) \varphi(n)\psi(n) \right] + D \left( \psi(n) \frac{\partial \varphi(n)}{\partial n} - \varphi(n) \frac{\partial \psi(n)}{\partial n} \right) \frac{\partial n}{\partial v} , \quad (12)
\]

where the drift \( J \equiv J(t,v) \) and the diffusion \( D \equiv D(t,v) \) are respectively the first and the second order momenta of the transition rate.

Eq.(12) is a continuity equation and the factor into the square bracket is the particle current that can be seen as the sum of two terms. The first is the drift current \( j_{d,eff} \), given by

\[
j_{d,eff}(t,v) = \left( J + \frac{\partial D}{\partial v} \right) \varphi(n)\psi(n) . \quad (13)
\]

This quantity is proportional to \( \varphi(n)\psi(n) \), hence proportional to the probability \( \pi(n) \) that the particle get away from the \( v \) site. This is an anomalous current because it is a non-linear algebraic function of \( n \). Only in the case of the standard Fokker-Planck kinetics \( \varphi(n) = n \) and \( \psi(n) = 1 \) this current is proportional to \( n \).

The second term in Eq.(12) is a diffusion like current, \( j_{d,eff} \), proportional to \( \partial n/\partial v \) and given by

\[
j_{d,eff}(t,v) = D A(n) \frac{\partial n}{\partial v} = D(t,v,n) \frac{\partial n}{\partial v} , \quad (14)
\]

with

\[
A(n) = \psi(n) \frac{\partial \varphi(n)}{\partial n} - \varphi(n) \frac{\partial \psi(n)}{\partial n} . \quad (15)
\]

We call \( j_{d,eff} \) anomalous diffusional current because the diffusion coefficient \( D(t,v,n) \) depends explicitly on \( v \) and \( n(t,v) \), while the standard diffusional coefficient does not depend on \( n(t,v) \).

Let us now limit ourselves to those physical processes described by Eq.(12) where the diffusional current \( j_{d,eff} \) is a standard current or Fick current with the diffusion coefficient independent of the distribution function \( n(t,v) \).

To realize this condition it is necessary to determine the functions \( \varphi(n) \) and \( \psi(n) \) that satisfy the equation \( A(n) = A \).

In the case of a pure bosonic kinetics [23], one has \( \varphi(n) = n \) and \( \psi(n) = 1 + n \) or \( \psi(n) = \varphi(n) \) with \( n = 1 + n \). For a pure fermionic kinetics [23] one has \( \varphi(n) = n \), \( \psi(n) = 1 - n \) and, in analogy with the bosonic notations, the fermionic condition can be written as \( \psi(n) = \varphi(n) \), where now \( n = 1 - n \).

The condition \( \psi(n) = \varphi(n) \) with \( n = 1 + \sigma n \) holds both for the pure bosonic kinetics (\( \sigma = 1 \)) and for the pure fermionic kinetics (\( \sigma = -1 \)).

Now, we impose that the above conditions are conserved also in the case the kinetics be defined by an arbitrary \( \varphi(n) \) so that we limit ourselves to consider the physical processes which satisfy the two following general conditions

\[
A(n) = A \quad ,
\psi(n) = \varphi(1 + \sigma n) . \quad (16)
\]

After derivation respect to \( n \) of the first condition and taking into account the second condition in Eqs.(16), we easily obtain the following differential equation for the function \( \varphi(n) \)
Introducing the auxiliary function $\lambda(n) = \varphi''(n)/\varphi(n)$, it is easy to verify, by taking into account Eq. (17), that this function is equal to a constant $\alpha^2 \in \mathbb{R}$ and our problem is reduced to the following Cauchy problem

$$
\varphi''(n) - \alpha^2 \varphi(n) = 0 \quad ,
$$

$$
\varphi(0) = 0 \quad ,
$$

$$
\varphi(1) = 1 \quad .
$$

For $\alpha$ real the solution of Eq. (18) is given by

$$
\varphi(n) = \frac{\sinh \alpha n}{\sin \alpha} \quad ,
$$

while for $\alpha$ imaginary: $\alpha = i\theta$

$$
\varphi(n) = \frac{\sin \theta n}{\sinh \theta} \quad .
$$

Eqs. (19) and (20) can be written using the definition of the symbol $[ ]$ of Eq. (4) as

$$
\varphi(n) = [n] \quad ,
$$

where $q = e^\alpha$ (for $q$ real we have the expression (19) and for $q$ complex we have Eq. (20)).

Thus, selecting $\psi(n) = \varphi(1 + \sigma n)$ and imposing diffusion Fick current, we find the function $\varphi(n)$ and $\psi(n)$ for $q$-particles ($q$-boson for $\sigma = 1$ and $q$-fermion for $\sigma = -1$). These functions ($\varphi(n) = [n]$ and $\psi(n) = [1 + \sigma n]$) correspond to the Eqs. (10) and (11), previously postulated, with the quantum $q$-algebra of Eqs. (1)-(7).

Now, if we include the $q$-functions $\varphi(n)$ and $\psi(n)$ in Eq. (13), in the case of Brownian particles ($J = \gamma v$, $D = \gamma/\beta m$, $\beta = 1/kT$ and $\gamma$ is a dimensional constant), we obtain the Fokker-Planck equation for $q$-oscillators

$$
\frac{\partial n}{\partial t} = \frac{\partial}{\partial v} \left[ \gamma v [n] [1 + \sigma n] + \frac{\gamma kT}{m} \alpha \frac{\partial n}{\partial v} \right] \quad .
$$

This equation describes the time evolution of $q$-deformed bosons and fermions in the velocity space.

Great interest is actually devoted to study the statistical distribution and the dynamical evolution toward equilibrium of several physical systems. However, because of the huge complexity of the kinetic transport theory on a quantum field theoretical basis, it results very difficult to describe the non-equilibrium behavior of physically interesting systems.

Because the quantum dynamics is equivalent to classical dynamics with the inclusion of quantum fluctuations, that in average coincide with the Brownian fluctuations, we conclude that the non-linear kinetic equation (22) can describe very close the quantum dynamics of a system of $q$-oscillators. Consequently, the above kinetic approach can result appropriate to describe non-equilibrium or near-equilibrium $q$-deformed systems with small (average) quantum fluctuations. These phenomena are actually very important in several applications from condensed matter to cosmological and high energy physics [28-30].

In stationary conditions ($t \to \infty$), the particle current in the square bracket vanishes and Eq. (22) becomes a homogeneous first-order differential equation, easily integrated as

$$
\frac{[n]}{[1 + \sigma n]} = e^{-\epsilon} \quad ,
$$

where $\epsilon = \beta(E - \mu)$, $E = \frac{1}{2}mv^2$ is the kinetic energy and $\mu$ is the chemical potential which can be evaluated by fixing the number of particles of the system.

Eq. (23) defines implicitly the statistical distribution $n(v)$ in stationary conditions; it can be explicitly derived as

$$
n = \frac{1}{\log q} \tanh^{-1} \left( \frac{\sinh \log q}{e^\epsilon - \sigma \cosh \log q} \right) \quad .
$$

For $q$ real the quantity $n$ can be written as

$$
n = \frac{1}{2 \log q} \log \left( \frac{e^\epsilon - \sigma q^{-\sigma}}{e^\epsilon - \sigma q^{\sigma}} \right) \quad ,
$$

while for $q = e^{i\theta}$ complex as

$$
n = \frac{1}{\theta} \tan^{-1} \left( \frac{\sin \theta}{e^\epsilon - \sigma \cos \theta} \right) \quad .
$$

We like to stress that to consider complex values of $q$ can be relevant in reproducing the exact interaction between particles in several physical systems. In Ref. [19] it is analyzed the $q$-deformed pairing vibration in nuclei and is shown that the real part of $q$ simulates an attractive residual interaction between the nucleons and the imaginary part of $q$ decreases the binding energy of the pair of nucleons.

If we use as variable the single-particle energy $\epsilon$, it is easy to verify that the second order density fluctuation $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$ can be expressed as [24]

$$
(\Delta n)^2 = \left\langle \left\{ \frac{\partial}{\partial n} \log \left[ \varphi(n) \psi(n) \right] \right\}^{-1} \right\rangle, \quad (27)
$$

or more explicitly for $q$-deformed bosons and fermions

$$
(\Delta n)^2 = \frac{\sinh \alpha}{\alpha} \left[ n \right] [1 + \sigma n] \quad .
$$

The thermodynamic relations for $q$ oscillators can be derived starting from the explicit expression of the $\varphi(n)$ and $\psi(n)$ functions (see Ref. [24] for details). The entropy density can be expressed as $S(n) = \int \log[\psi(n)\psi(n)]dn$ or more explicitly as

$$
S(n) = \int \log \left[ \frac{[1 + \sigma n]}{[n]} \right] \, dn \quad .
$$
In conclusion, we have studied the kinetics of the $q$-oscillators defined in Eqs.(1)-(7) in the framework of a semiclassical non-linear approach. The two families of $q$-bosons and $q$-fermions can be interpreted as Brownian particles ($J \propto v$, $D = \text{const}$). They obey to an exclusion-inclusion principle defined in terms of the transition probability by the functions $\varphi(n) = [n]$ and $\psi(n) = [1 + \sigma n]$ which are implicitly related with the creation-annihilation operator matrix (see of Eqs.(10) and (11)). This relation fixes an important connection between quantum algebra and the semiclassical kinetics. In this framework, we have derived a generalized Fokker Planck equation appropriate to study near-equilibrium and non-equilibrium $q$-oscillator systems. This equation is easily integrated in stationary condition and reproduces the statistical distributions for $q$-deformed bosons and fermions both for real and complex $q$ values.

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