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COMPUTER GRAPHICS AND THE EIGENFUNCTIONS FOR THE KOCH SNOWFLAKE DRUM

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Abstract

Using techniques from numerical analysis, we have approximated – jointly with J. W. Neuberger and R. J. Renka – the smallest 50 eigenvalues, along with their associated eigenfunctions, of the Dirichlet Laplacian on the Koch snowflake domain. Physically, these correspond to the frequencies, and the associated normal modes, of the Koch snowflake drum. Several graphical representations of these eigenfunctions, as well as the magnitudes of their gradients, have been produced using Cray supercomputers and Silicon Graphics machines. We briefly describe here the numerical methods used in the computations and present pictures of a selected set of the eigenfunctions. We also compare the graphical images of the boundary behavior (of the magnitude) of the gradient with the known mathematical results (of Lapidus and Pang). This work is partly motivated by the physical experiments of Sapoval *et al.* related to the formation of fractal structures.

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1 Introduction

The following is a brief description of joint work performed by Michel L. Lapidus, J. W. Neuberger, Robert J. Renka, and Cheryl A. Griffith, to be published in [LaNeRG].

Objects with fractal boundary make poor resonators. Consider, as evidence of this, the following examples: ocean wave barriers are made up of randomly placed boulders of various sizes, high quality factor wave guides and microwave cavities are polished ([Sa1] and [SaGoMa]), and fractal objects in nature, such as trees and coastlines, appear to be very stable. Fractal geometry provides a framework within which to understand such phenomena.

Physical experiments on the mechanical vibrations of fractal drums have been carried out by the physicist B. Sapoval and his collaborators, Th. Gobron and A. Margolina [Sa1-2, SaGo, SaGoMa]. (The “fractal drum” in their experiments consisted of a stainless steel sheet with the boundary laser etched in the pattern of a square snowflake.) Their results have suggested, in particular, that wave motion is “localized” and severely damped in objects with extremely irregular (i.e, fractal) boundary. (Related experiments dealing instead with microwave cavities are described in [KuSr, SrHoKu].)

Most of the mathematical work on this subject—motivated in part by the study of porous media as well as the scattering of electromagnetic waves off fractal surfaces [Be1-2]—has focused on the asymptotic distribution of the eigenvalues (of Dirichlet or Neumann) Laplacians on regions with fractal boundary. See, e.g., [BrCa, La1, EvHa, FlV, HeLa, LaMa, LaPo1-2] as well as the survey article [La2] and the references therein. This was the emphasis of the research course, entitled “Lectures on spectral and fractal geometry”, given by the second author (M. L. L.) in Stockholm. (The latter part of these lectures dealt with analysis and spectral geometry *on* fractals themselves, rather than regions with fractal boundary. See, e.g., [BaPe, Ki, KiLa, La3] and the references therein.) In contrast, the work of M. L. Lapidus and M. Pang [LaPa] studied rigorously the boundary behavior of the Dirichlet Laplacian eigenfunctions and their gradients on a class of planar domains with fractal boundary, including the (triadic and square) Koch snowflake domains, as well as their polygonal approximations. In particular, one of their main results, specialized to the Koch snowflake domain, implies that the gradient of the first eigenfunction “blows up” as one approaches uncountably many boundary points, namely, those with an obtuse angle corner (in an inner

polygonal approximation) as well as those in a Cantor-type set of boundary points (of Hausdorff dimension $\ln 2/\ln 3$) contained in the true snowflake curve. The membrane of the Koch snowflake drum is said to experience “infinite stress” [Sa1] (and would obviously burst) near these points. [In [LaPa], the gradient was also shown to vanish as one approaches boundary points with an acute angle corner as well as those in another Cantor set contained in $\partial\Omega$. (All angles are measured from the inside.)]

The long term goal of this research is to understand the stability and existence of fractal objects in nature. Examples showing the relevance of this work are provided at the beginning of this introduction as well as in [LaNeRG].

Our present work looks at the vibrations of fractal drums from the point of view of numerical analysis and computer graphics. We consider the following eigenvalue problem on a bounded domain $\Omega \subset \mathbf{R}^2$ with irregular boundary $\partial\Omega$:

$$\begin{cases} -\Delta u &= -(u_{xx} + u_{yy}) = \lambda u, & \text{for } (x, y) \in \Omega, \\ u &= 0, & \text{for } (x, y) \in \partial\Omega. \end{cases} \quad (1)$$

Of course, this boundary value problem must be interpreted in the variational sense. That is, $u \neq 0$ is a solution of (1) if $u \in H_0^1(\Omega)$ (the Sobolev space) and

$$\int_{\Omega} \nabla u \nabla v = \lambda \int_{\Omega} uv,$$

for all smooth functions v with compact support in Ω . (See also, for example, Section 2.1 in [La1].)

We wish to approximate the first N_e eigenvalues of (1), as well as their associated eigenfunctions. We choose, as a prototypical model, the Koch snowflake drum; namely Ω is the bounded domain having the Koch snowflake curve [Ma, Fa] as boundary. (See Figure 1.)

Recall that Equation (1) can be considered as a mathematical model for the (steady-state) vibrations of a drum. The eigenvalues (or rather, their square roots) correspond to the natural frequencies (or fundamental modes) of the drum. The first eigenvalue corresponds to the fundamental tone (with associated eigenfunction called the first harmonic) and the higher eigenvalues correspond to the overtones (with associated eigenfunctions referred to as the higher harmonics) of the drum.

Further, recall that the spectrum of the Dirichlet Laplacian on a bounded (planar) domain is discrete and consists of an infinite sequence of eigenvalues

Figure 1: The Koch Snowflake Drum Ω

$\{\lambda_n\}_{n=1}^\infty$ written in increasing order according to their (finite) multiplicity:

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \text{ with } \lambda_n \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (2)$$

Moreover, let φ_n be the eigenfunction corresponding to λ_n (and normalized by $\|\varphi_n\|_{L^2(\Omega)}=1$). Then φ_n is smooth (i.e., infinitely differentiable) in Ω , but not necessarily up to the boundary $\partial\Omega$. Finally, the first eigenvalue λ_1 is *simple* and the first harmonic can be chosen so that $\varphi_1 > 0$ inside Ω .

In the work described here, and that of Sapoval *et al.*, Ω is not a true fractal, but rather a *finite polygonal approximation* to the limiting fractal shape. The eigenvalues are calculated here using a finite difference approximation to $-\Delta$ on a discretization of our domain Ω . This approximation results in a symmetric positive definite matrix A whose positive real eigenvalues and orthonormal eigenvectors are computed by the inverse power method (with deflation).

We conclude this introduction with a brief outline of the remainder of the paper. Section 2 will concentrate on the numerical methods used to approximate the eigenvalues and their corresponding eigenfunctions and Section 3 will give a brief description of the computer algorithm. Section 4 will focus on the computer graphics for the eigenfunctions as well as give a listing of some of the eigenvalues themselves. We end the paper in Section 5 with a short concluding comment about this work and future areas of interest.

2 The Numerical Method

2.1 Approximating the Koch snowflake domain and the Dirichlet Laplacian

The discretized Koch snowflake domain Ω is created by using equilateral triangles (see Figure 2 below, from [LaNeRG]). Therefore, a triangular grid system, rather than a rectangular grid system, is used to fit its geometry. We refer to [NeNo] for a similar method based on rectangular grids. The first approximation to Ω begins with a triangle having unit side lengths (fractal level 0). We “refine” this domain by partitioning each triangle into nine equilateral subtriangles and adding two nodes (vertices of the equilateral subtriangles) on each edge and one node at the barycenter of each triangle. We increase the fractal level of our domain by adding a boundary triangle (and boundary

node) to each side (sequence of three collinear boundary nodes). As can be seen from Figure 2, each interior node has six neighbors.

Note that the limit of this sequence of triangulations defines the Koch snowflake domain Ω . The first three nodes are taken to be $(0, 0)$, $(1, 0)$, and $(.5, \sqrt{3}/2)$. The distance between the nodes (mesh width) is given by $h = (1/3)^{L+R}$ for fractal level L and refinement level R . The initial solution estimates for the eigenvectors at refinement level $R + 1$ are obtained by interpolation from the previous refinement level R . Henceforth, we will denote the number of nodes and the number of boundary nodes by N and N_b , respectively.

We now explain how to approximate the Dirichlet Laplacian by finite difference operators. As was alluded to above, each interior node (x, y) of our triangulation has six neighbors corresponding to the vertices of a regular hexagon centered at (x, y) . We will obtain a second-order finite difference approximation to $\Delta u(x, y)$ by expanding the value of u at each of these neighbors in a Taylor series about (x, y) . Let $\{u_i\}_{i=1}^6$ denote the counterclockwise-ordered sequence of neighboring values, beginning with $u(x + h, y)$ for mesh width h . (See Figure 3.) Then

Figure 3: Interior Node u and its Six Neighbors u_i .

Figure 2: Approximations to the Koch Snowflake, with Fractal Level L and Refinement Level R .

$$\begin{aligned}
u_1 &= u(x+h, y) = u(x, y) + u_x(x, y)h + \frac{1}{2}u_{xx}(x, y)h^2 + \\
&\quad \frac{1}{6}u_{xxx}(x, y)h^3 + O(h^4), \\
u_2 &= u(x + \frac{h}{2}, y + \frac{\sqrt{3}}{2}h) = u(x, y) + u_x(x, y)\frac{h}{2} + u_y(x, y)\frac{\sqrt{3}}{2}h + \\
&\quad \frac{1}{2}\left[u_{xx}(x, y)\frac{h^2}{4} + 2u_{xy}(x, y)\frac{\sqrt{3}}{4}h^2 + u_{yy}(x, y)\frac{3}{4}h^2\right] + \\
&\quad \frac{1}{6}\left[u_{xxx}(x, y)\frac{h^3}{8} + 3u_{xxy}(x, y)\frac{\sqrt{3}}{8}h^3 + 3u_{xyy}(x, y)\frac{3}{8}h^3 + \right. \\
&\quad \left. u_{yyy}(x, y)\frac{3\sqrt{3}}{8}h^3\right] + O(h^4), \\
&\vdots \\
u_6 &= u(x + \frac{h}{2}, y - \frac{\sqrt{3}}{2}h) = u(x, y) + u_x(x, y)\frac{h}{2} - u_y(x, y)\frac{\sqrt{3}}{2}h + \\
&\quad \frac{1}{2}\left[u_{xx}(x, y)\frac{h^2}{4} - 2u_{xy}(x, y)\frac{\sqrt{3}}{4}h^2 + u_{yy}(x, y)\frac{3}{4}h^2\right] + \\
&\quad \frac{1}{6}\left[u_{xxx}(x, y)\frac{h^3}{8} - 3u_{xxy}(x, y)\frac{\sqrt{3}}{8}h^3 + 3u_{xyy}(x, y)\frac{3}{8}h^3 - \right. \\
&\quad \left. u_{yyy}(x, y)\frac{3\sqrt{3}}{8}h^3\right] + O(h^4).
\end{aligned}$$

Thus,

$$\sum_{i=1}^6 u_i = 6u(x, y) + \frac{3}{2}h^2 u_{xx}(x, y) + \frac{3}{2}h^2 u_{yy}(x, y) + O(h^4),$$

and

$$-\Delta u(x, y) = \left(6u(x, y) - \sum_{i=1}^6 u_i\right) / \left(\frac{3}{2}h^2\right) + O(h^2). \quad (3)$$

Denote by $\mathbf{A}$ the matrix representation of the discretized negative Laplace operator. Clearly, $\mathbf{A}$ will be of order $N - N_b$. Now let u_i be the i^{th} nodal eigenfunction value. Then, from Equation (3), the k^{th} component of the matrix-vector product $\mathbf{A}\mathbf{u}$, for each interior node k , is given by

$$(\mathbf{A}\mathbf{u})_k = \left(6u_k - \sum_{j \in N_k} u_j\right) / \left(\frac{3}{2}h^2\right), \quad (4)$$

where N_k is the set of indexes of the neighbors of node k .

Our matrix $\mathbf{A}$ is sparse. This is due to the fact that $\mathbf{A}$ has at most seven nonzero entries per row. Let $c = (3/2)h^2$. Then,

$$A_{ij} = \begin{cases} -1/c, & \text{for } i \text{ and } j \text{ neighbors} \\ 6/c, & \text{for } i = j \\ 0, & \text{otherwise.} \end{cases}$$

From the above, we see that $\mathbf{A}$ is symmetric. Hence, $\mathbf{A}$ has real eigenvalues and a complete set of orthonormal eigenvectors [GoVa, p. 410]. In addition, since $\mathbf{A}$ is diagonally semi-dominant with positive diagonal entries, we can apply the Gershgorin (circle) theorem [GoVa, p. 341] to deduce that the eigenvalues of $\mathbf{A}$ are nonnegative. In fact, it can be shown that $\mathbf{A}$ is non-singular (by a simple generalization of [At, pp. 557–559]) and therefore has positive eigenvalues.

2.2 Gauss-Seidel iterative method

The inverse power method requires the solutions to the sequence of linear systems involving the discretized negative Laplacian $\mathbf{A}$. We use an iterative method (known as the Gauss-Seidel method) that eliminates the need to store the matrix $\mathbf{A}$. This method yields a solution for $\mathbf{u}$ in the system $\mathbf{A}\mathbf{u} = \mathbf{v}$ and consists of repeatedly executing the following operation for each k ($k = 1, \dots, N$):

If k is an interior node, then calculate the following:

$$u_k = (cv_k + \sum_{j \in N_k} u_j) / 6.$$

Since $\mathbf{A}$ is positive definite, the Gauss-Seidel method is guaranteed to converge [GoVa, p. 509]. Each step k consists of solving the equation, $(\mathbf{A}\mathbf{u})_k = v_k$, for a new value of u_k and updating the approximate solution vector $\mathbf{u}$. Entries associated with boundary nodes must be zero for the initial solution estimate. Convergence of the method is determined by a bound on the maximum relative change in a solution component between iterations.

The rate of convergence is improved by employing the successive overrelaxation (SOR) method in which the update step u_k is modified to include a relaxation parameter ω . Please see [LaNeRG, Sec. 3.3] and [IsKe, pp. 463–470] for more details.

2.3 Inverse power method

Let $\mathbf{A}$ be a symmetric positive definite linear operator on a real finite dimensional inner product space. Let $\mathbf{A}$ have eigenvalues $\{\lambda_i\}_{i=1}^n$ for $n = N - N_b$, where

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n,$$

with an associated collection of orthonormal eigenvectors $\{\mathbf{u}_i\}_{i=1}^n$; so that $\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i$ and $\mathbf{u}_i^T \mathbf{u}_j = \delta_{ij}$ for $1 \leq i, j \leq n$.

Since the eigenvectors span $\mathbf{R}^n$, an arbitrary vector $\mathbf{x}_0$ may be written in the form $\mathbf{x}_0 = \sum_{i=1}^n \alpha_i \mathbf{u}_i$. (Note that the present λ_i 's are not the same as in Equation (2) above.) The matrix $\mathbf{A}^{-1}$ will have eigenvalues $\{\lambda_i^{-1}\}_{i=1}^n$ with the same eigenvectors. We apply the power method to $\mathbf{A}^{-1}$. Define

$$x_{m+1} = \frac{\|x_m\|^2}{\langle \mathbf{A}^{-1}x_m, x_m \rangle} \mathbf{A}^{-1}x_m, \text{ for } m = 0, 1, \dots.$$

Then the sequence

$$r_m = \left\{ \frac{\langle \mathbf{A}^{-1}x_m, x_m \rangle}{\|x_m\|^2} \right\}_{m=1}^{\infty} \quad (5)$$

converges to the largest eigenvalue λ_1^{-1} of $\mathbf{A}^{-1}$ (provided $\alpha_1 \neq 0$ and $\lambda_1 < \lambda_2$).

The power method finds the largest eigenvalue of $\mathbf{A}^{-1}$, but, of course, the largest eigenvalue of $\mathbf{A}^{-1}$ is the reciprocal of the smallest eigenvalue of $\mathbf{A}$, which is the one that we are interested in.

Using the orthogonality of the eigenvectors, we deduce from (5), after some computation, that:

$$r_m = \lambda_1^{-1} \left(\frac{\alpha_1^2 + \sum_{i=2}^n \alpha_i^2 (\lambda_1/\lambda_i)^{2m+1}}{\alpha_1^2 + \sum_{i=2}^n \alpha_i^2 (\lambda_1/\lambda_i)^{2m}} \right), \quad (6)$$

which converges to λ_1^{-1} even in the case of a multiple eigenvalue.

Naturally, the same method may be used to find the k^{th} eigenvalue by simply projecting the initial estimate $\mathbf{x}_0$ onto the orthogonal complement of the eigenspace spanned by the first $k-1$ eigenvectors. Then our new initial estimate $\widehat{\mathbf{x}}_0 = \sum_{i=k}^n \alpha_i \mathbf{u}_i$ yields the Rayleigh quotients:

$$r_m = \lambda_k^{-1} \left(\frac{\alpha_1^2 + \sum_{i=k+1}^n \alpha_i^2 (\lambda_k/\lambda_i)^{2m+1}}{\alpha_1^2 + \sum_{i=k+1}^n \alpha_i^2 (\lambda_k/\lambda_i)^{2m}} \right), \quad (7)$$

which converge to λ_k^{-1} as $m \rightarrow \infty$. Moreover, the sequence x_m will converge to an eigenvector associated with λ_k^{-1} . (Of course, these eigenvectors in turn approximate the eigenfunctions that we want to compute.) For more detail, we refer the reader to [LaNeRG, Sec. 3.4].

3 The Computer Algorithm

In this section, we outline the computer algorithm used to construct the eigenvalues and their associated eigenvectors. In the following, N_e will denote the number of eigenvalues to be calculated, and L_{max} will denote the maximum fractal level. Two files are used for storage. One file, **infile**, contains (unnormalized) initial approximations to the eigenvectors. The other file, **outfile**, contains the approximations to the orthonormal sequence of eigenvectors $\{\mathbf{u}_i\}_{i=1}^{N_e}$.

Start at fractal level $L = 0$, refinement level $R = 1$, and set initial eigenvector estimate $x_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$.

For fractal level $L = 1$ to $(L_{max} - 1)$ do

 If $(L > 0)$ then

 Refine the domain (and increase the number of nodes from n_0 to n).

 Append $n - n_0$ nodes to x_0 .

 End if

For fractal refinement $R = 1$ to $(L_{max} - L)$ do

 Increase fractal level of boundary.

 If $(L \neq 0 \text{ and } R \neq 1)$ then

 Compute initial eigenvector estimates by interpolation from previous refinement level.

 End if

Inverse Power Method:

For $k = 1$ to $\min(N_e, N_b)$ do

 Read in x_0 from **infile**, set $u = x_0$.

```

Set  $\lambda_k = 0$ .
Do
  Set  $\lambda_0 = \lambda_k$ .
  Set  $\mathbf{v} = \mathbf{u} / \|\mathbf{u}\|_2$ .
  Set  $\mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$  (Gauss-Seidel iterative method).
  Rewind outfile.
  For  $i = 1$  to  $k - 1$ 
    Set  $\mathbf{w} = \mathbf{u}_i$  (read in from outfile).
    Set  $\mathbf{u} = \mathbf{u} - (\mathbf{u}^T \mathbf{w}) \mathbf{w}$  (Orthogonalization).
  End for
   $\lambda_k = 1 / \mathbf{u}^T \mathbf{v}$ 
  while (  $|\lambda_k - \lambda_0| > \text{tolerance}$  ).
  Write  $\mathbf{v}$  to outfile.
End for. (Inverse power method loop)
If ( $N_b < N_e$ ) then
  Append last eigenvector  $N_e - N_b$  times to outfile.
End if
If ( $R = 1$ ) then
  Save eigenvector information.
End if
End for (Refinement level loop)
Write all eigenvalue, eigenvector information to output file for fractal
level  $L$ , and all refinements (1 to  $L_{max} - L$ ).

End for (Fractal level loop).

```

4 The Computer Graphics

In the subsection to follow, we will present computer generated images of a selected set of eigenfunctions, along with some of their gradients, on the Koch snowflake domain.

The eigenfunctions were approximated at the University of California using a Fortran program executed on the San Diego Supercomputer Center's

Cray systems. This was due to the large amount of computations involved in the algorithm, as well as to the large output files generated by the program, and to achieve high numerical accuracy. The program output consists of the eigenvalues at all fractal levels and refinements, as well as a binary file which contains the values of the eigenfunctions at each grid point (X , Y and Z). The binary file also contains the number of nodes (grid points), the number of triangles, and a “map” which shows how to connect these nodes to form the Koch snowflake. This provides enough information to plot the eigenfunctions; thus the data can now be rewritten to a format suitable for several commercial plotting packages. For our purposes, the plotting package *geomview* was used. The visualization of the eigenfunctions themselves was performed on a Silicon Graphics Onyx at the University of California’s Center for Visual Computing in Riverside. The Onyx was used because of its superior graphics quality as well as for its speed and ability to represent and rotate large images. All the images were generated in color and many of them will be published in our joint paper [LaNeRG]. In addition, several images will be presented in this paper for the first time. However, for practical reasons, we include only the greyscale version of our graphics in the present paper. Hence, the color scales used in [LaNeRG] are of lesser use here.

4.1 A selection of the eigenfunctions

We will present a small set of some of the more interesting eigenfunctions. There are four different types of pictures which roughly correspond to varying color scales; namely, the eigenfunctions themselves, the magnitude of their gradients, the nodal lines of the eigenfunctions and the contour plots of the eigenfunctions. The color scales are as follows: a “continuous” color scale with up to 1 million different combinations of red, green and blue, a “discrete” color scale with eight different color values, “nodal line” color scale with just three possible color values, and lastly, no color scale, meaning that the eigenfunction is shaded by colored “lights”. All color scales are based on the eigenfunctions’ numerical values. For the continuous color scale, nonnegative values vary continuously from yellow (the maximum) to red (zero), while the nonpositive values vary from red to blue (the minimum).

We begin our presentation with the first snowflake harmonic, φ_1 , (Plate 1), with colors provided by lighting. As expected, φ_1 is nonnegative and the corresponding eigenvalue λ_1 is simple.

Nodal lines plots are used to visualize the zeros of the eigenfunctions. Regions colored in red correspond to an ϵ -neighborhood of the true nodal lines, and regions colored in yellow and blue represent, respectively, the positive and negative portions of the eigenfunction outside of the ϵ -neighborhood. Plate 2 depicts the nodal lines of the second eigenfunction φ_2 . In contrast to λ_1 , the eigenvalue λ_2 has multiplicity two. (Its value can be found in Table 1 of Section 4.2.)

This multiplicity can better be seen in the contour plots of Plate 3 which depicts φ_2 and φ_3 , in particular. Contour plots were made of the eigenfunctions in orthographic (rather than stereographic) projection to better understand the symmetries. See [LaNeRG, Sec. 2.7] for comments about the symmetries of the eigenfunctions. Plate 4 displays the contour plots of some of the higher eigenfunctions. Note that φ_{19} and φ_{20} are associated to the same eigenvalue of multiplicity two, while φ_{23} and φ_{48} correspond to simple eigenvalues.

The “blow-up” phenomenon is vividly represented in Plates 5 and 6, as well as in Plate 7, which displays a piece of $|\nabla\varphi_{45}|$ for increasing fractal levels. (See also Plate 8 which depicts the bottom view of $|\nabla\varphi_{13}|$.) This suggests that the mathematically derived results of [LaPa] concerning the blow-up of $|\nabla\varphi_1|$ (see Plate 9, upper left) could be extended to the higher eigenfunctions, as is conjectured in [LaNeRG]. The interested reader may find it amusing to compare the pictures of: φ_{45} , the contour plot of φ_{45} and $|\nabla\varphi_{45}|$, as given, respectively, in Plates 10, 9 and 6.

Of course, computers help us to “see” only those boundary points which are vertices of a polygonal approximation to the snowflake curve $\partial\Omega$, but not their limit points (which constitute the remaining points of $\partial\Omega$). Hence, for example, the mathematical results of [LaPa] (alluded to in the introduction) concerning the “explosion” of $|\nabla\varphi_1|$ at uncountably many other boundary points (in a certain Cantor set of $\partial\Omega$) cannot yet be “visualized”. (See especially, [LaPa, Theorem 3.2, p. 375]; also see [LaPa, Theorem 2.11, p. 365] which gives less precise pointwise estimates for $|\nabla\varphi_n|$, $n \geq 1$.) It would be very interesting to bridge this gap between “theory” and “experiment”. (See [LaNeRG, Problem 5.3].)

EVAL	L,R = 2,1	L,R = 3,1	L,R = 4,1	L,R = 5,1
1	40.120	39.657	39.463	39.390
2	98.887	98.164	97.720	97.537
3	98.887	98.164	97.720	97.537
4	167.072	166.603	165.898	165.622
5	167.037	166.603	165.897	165.622
6	191.894	191.707	190.938	190.571
7	210.212	210.173	209.274	208.837
8	273.059	274.252	273.246	272.755
9	273.059	274.252	273.246	272.755
10	310.760	313.895	313.138	312.645
11	313.792	316.358	315.368	314.784
12	313.792	316.358	315.368	314.784
13	357.472	361.539	362.540	359.879
14	419.978	427.330	426.501	425.726
15	436.636	445.277	444.626	443.830
16	436.636	445.277	444.625	443.830
17	450.877	460.447	459.795	459.099
18	450.877	460.431	459.795	459.099
19	546.000	562.195	561.785	560.913
20	545.998	562.195	561.785	560.913

Table 1: First 20 Eigenvalues at Fractal Levels $L=2$ through 5 and fixed Refinement Level $R=1$

4.2 The computed eigenvalues

Table 1 above contains the approximations to the first twenty eigenvalues for fixed refinement level and varying fractal levels. The best estimates of the eigenvalues on the Koch snowflake are at fractal level 5. Please refer to [LaNeRG, Sec. 4.2] for more extensive eigenvalue tables and comments.

In the case of the equilateral triangle (Level $L=0$), we have compared our computed eigenvalues with theoretically known eigenvalues ([Pi] and, in a more general case, [Bé]). The agreement was excellent; see [LaNeRG, Sec. 4.2].

Note that as the fractal level increases, the domain size increases and, as expected, the eigenvalue decreases ($L \geq 3$). Among the first 50 eigenvalues,

14 are isolated and the other 36 have multiplicity 2.

5 Conclusion

We hope to continue this work by extending it to different geometrical shapes, fractals themselves and regions with varying fractal boundary, exhibiting various localization and blow-up phenomena. (Please see [LaNeRG, Sec. 5] for a more detailed discussion of open problems.) Eventually, this may help us better understand the formation of fractal structures in nature.

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Figures, Tables and Plates — Captions and Placement

Section 1

Figure 1: The Koch Snowflake Drum Ω

Section 2

Subsection 2.1

Figure 2: Approximations to the Koch Snowflake, with Fractal Level L and Refinement Level R

Figure 3: Interior Node u and its Six Neighbors u_i

Section 4

Subsection 4.1

Please place the Plates at the end of Section 4.1 in numerical order

Plate 1: The First Snowflake Harmonic

Plate 2: The Nodal Lines of the Second Eigenfunction

Plate 3: Contour Plots of φ_1 , φ_2 , φ_3 and φ_{13}

Plate 4: Contour Plots of φ_{19} , φ_{20} , φ_{23} and φ_{48}

Plate 5: $|\nabla\varphi_{13}|$ and a 1/6 slice at a higher level

Plate 6: The Magnitude of the Gradient of the 45th Eigenfunction

Plate 7: 1/6 slice of $|\nabla\varphi_{45}|$ at various refinements

Plate 8: The “medusa”, i.e., the Magnitude of the Gradient of the 13th Eigenfunction (bottom view)

Plate 9: A Potpourri of Five Images

Plate 10: Selected Higher Eigenfunctions: φ_{36} , φ_{37} , φ_{38} and φ_{45}

Subsection 4.2

Table 1: The First 20 Eigenvalues at Fractal Levels $L=2$ through 5 and fixed Refinement Level $R=1$

The First Snowflake Harmonic

Level 4 - 1st Eigenfunction

Plate 2: The Nodal Lines of the Second Eigenfunction

Nodal Lines of the 2nd Eigenfunction

Plate 3: Contour Plots of φ_1 , φ_2 , φ_3 and φ_{13}

Plate 4: Contour Plots of φ_{19} , φ_{20} , φ_{23} and φ_{48}

Magnitude of the Gradient for the 13th Eigenfunction

Plate 5: $|\nabla\varphi_{13}|$ and a 1/6 slice at a higher level

Plate 6: The Magnitude of the Gradient of the 45th Eigenfunction

Magnitude of the Gradient for the 45th Eigenfunction
(one-sixth slice)

Plate 8: The "medusa", i.e., the Magnitude of the Gradient of the 13th Eigenfunction (bottom view)

Snowflake Harmonics

The "extraterrestrial", i.e., Magnitude of the Gradient for the 13th Eigenfunction (one-sixth slice)