Calibration of the Tully-Fisher relation in the field

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Received / Accepted

Abstract. A new technique for calibrating slope and relative zero-point of Tully-Fisher like relations by using a sample of distant field galaxies is proposed. Based on a null-correlation approach (NCA), the technique is insensitive to the presence of selection effects on apparent magnitude \( m \) and on log line-width distance indicator \( p \). This interesting property is used for discarding nearby galaxies of the observed sample. It is shown that such a subsampling allows in effect to attenuate biases on calibration parameters created by the presence of radial peculiar velocities.

Key words: galaxies – distance scale – velocity field

1. Introduction

During this last decade, increasing efforts have been made attempting to extract informations concerning the distribution of mass in the universe from the radial peculiar velocity field of distant galaxies (see for example Aaronson et al. 1986, Lynden-Bell et al. 1988, Bertschinger et al. 1990, Rauzy et al. 1992, Newsam et al. 1995, Rauzy et al. 1995). Quantitative results based on peculiar velocity studies, such as constraints on the value of the density parameter \( \Omega_0 \), can nowadays be found in the literature (see Dekel 1994 for a review). Reliability of such results is however closely related to the various working hypotheses assumed throughout the successive steps of the analysis.

First of these steps is to obtain redshift-independent estimates of galaxies distance. It is performed by using Tully-Fisher (TF) like relations (Tully-Fisher 1977 for spirals and Faber-Jackson 1976 for ellipticals). These observed statistical relations linearly correlate the absolute magnitude \( M \) (or similar quantity) of a galaxy with an observable parameter \( p \) (\( p = \log V_{\text{rot}} \) for spirals and \( p = \log \sigma \) for ellipticals). Assuming that the TF relation has been correctly calibrated, an estimate of the absolute magnitude \( M \) is obtained by measuring the observable \( p \). Measurement of the apparent magnitude \( m \) (or similar quantity) provides then with an estimate of the distance of the galaxy, and comparing it with the redshift \( z \) finally furnishes the deviation from the mean Hubble flow (i.e. the radial peculiar velocity).


Motivations leading to introduce a new calibration technique are twofold. First, bias correction requires generally a full description of the calibration sample (i.e. the specific shapes of the observational selection function on \( m \) and \( p \) and of the luminosity function have to be assumed). Since available samples are often constituted of data inherited from various observational programs, modelization of such characteristics still remains a difficult problem. The philosophy is herein to reduce as far as possible the number of dubious assumptions made on these composite samples when deriving calibration parameters. Second, it is not clear how existing calibration procedures are affected by the presence of radial peculiar velocities. The aim is herein to quantify, and if possible to minimize, influences of peculiar velocity fields on the estimates of the calibration parameters.

In section 2 is summarized the basic statistical model describing Tully-Fisher like relations. The new calibration technique is presented section 3. Cumbersome calculations and proofs can be found appendices A, B and C. Potentials of the method are illustrated appendix D where
2. Distance and velocity estimates

The Tully-Fisher like relations are based on an observed linear correlation between the absolute magnitude \( M \) (or similar quantity such as \(-5 \log_{10} D \) with \( D \) the linear diameter) and the log line-width distance indicator \( p \) of galaxies (\( p \approx \log V_{max} \) for spirals and \( p \approx \log \sigma \) for ellipticals). They allow to estimate distance and radial peculiar velocity of an individual galaxy from its measured apparent magnitude \( m \) (or similar quantity such as \(-5 \log_{10} d \) with \( d \) the apparent diameter), parameter \( p \) and redshift \( z \). In this section, we recall in mind the basic statistical model describing these relations (see Triay et al. 1994 or Rauzy&Triay 1996 for details).

Regardless of the distance of the galaxy, selection effects in observation and measurement errors, the theoretical probability density \((pd)\) in the \( M-p \) plane reads:

\[
dP_1 = F(M, p) \; dM \; dp
\]  

The Direct (i.e. Forward) Tully-Fisher (DTF) relation assumes that it exists a random variable \( \zeta^D \),

\[
\zeta^D = \tilde{M}(p) - M = a^D p + b^D - M
\]  

where \( a^D(p) \) is the distribution function of the variable \( p \) in the \( M-p \) plane. The random variable \( \zeta^D \) of zero mean and dispersion \( \sigma^D_\zeta \) accounts for the intrinsic scatter about the DTF straight line \( \Delta_{DTF} \) of zero-point \( b^D \) and slope \( a^D \). The distance modulus \( \mu \) of an object reads:

\[
\mu = m - M = 5 \log_{10} r + 25
\]  

where \( m \) is the apparent magnitude of the galaxy and \( r \) its distance in Mpc. Regardless of measurement errors, the observed probability density takes the following form:

\[
dP_2 = \frac{1}{A_2} \phi(m, p) f_p(p) \; dp \; g(\zeta^D; \mu) \; d\zeta^D \; d\mu
\]  

where \( \phi(m, p) \) is a selection function accounting for selection effects in observation on \( m \) and \( p \), \( h(\mu) \) is the spatial distribution function of sources (along the line-of-sight) and \( A_2 = \int \phi(m, p) f_p(p) g(\zeta^D; \mu) \; d\mu \; d\zeta^D \) is the normalisation factor warranting \( \int dP_2 = 1 \). Under the three following assumptions:

- \( \mathcal{H}(0) \) No measurement errors on \( m \) and \( p \) are present. Particularly, corrections on galactic extinction and on inclination effects are supposed valid.
- \( \mathcal{H}(1) \) The function \( g(\zeta^D; 0, \sigma^D_\zeta) \) is gaussian.

which implies that, whatever the line-of-sight direction, the distance modulus distribution function reads

\[
h(\mu) = \exp[5\alpha \mu] \quad \text{with} \quad \alpha = \frac{\ln 10}{5}
\]

a statistical estimator \( \tilde{r} \) generally adopted for the distance of a galaxy with measured \( m \) and \( p \) reads as follows (see appendix A for details, or Lynden-Bell et al. 1988 Landy&Szalay 1992, Triay et al. 1994):

\[
\tilde{r} = \exp[\alpha(m - a^D p - b^D - 25)] \; \exp\left[\frac{7}{2} \alpha^2 \sigma^D_\zeta^2 \right]
\]  

where the term \( \exp\left[\frac{7}{2} \alpha^2 \sigma^D_\zeta^2 \right] \) accounts for a volume correction. This unbiased distance estimator does not depend on the selection function \( \phi(m, p) \) in \( m \) and \( p \) and on the specific shape of the distribution function \( f_p(p) \). Its accuracy \( \Delta_{\tilde{r}} \) is proportional to the distance estimate of the galaxy (see appendix A):

\[
\Delta_{\tilde{r}} = \tilde{r} \sqrt{\exp[\alpha^2 \sigma^D_\zeta^2] - 1}
\]

The radial peculiar velocity \( v \) of a galaxy, expressed in km s\(^{-1}\) with respect to the velocity frame in which the redshift \( z \) is measured, reads:

\[
v = z - H_0 \; r
\]

where \( H_0 \) is the Hubble’s constant and \( z \) is expressed in km s\(^{-1}\) units. Assuming that the above hypotheses hold, the estimator \( \hat{v} \) of the radial peculiar velocity of a galaxy with measured \( m \), \( p \) and \( z \) reads thus as follows:

\[
\hat{v} = z - H_0 \; \tilde{r} = z - B \; \exp[\alpha(m - a^D p)] \; \exp\left[\frac{7}{2} \alpha^2 \sigma^D_\zeta^2 \right]
\]

where \( H_0 \) and \( b^D \) have been merged into a single parameter \( B = H_0 \exp[\alpha(-b^D - 25)] \). The accuracy \( \Delta_{\hat{v}} \) of this radial peculiar velocity estimator \( \hat{v} \) is:

\[
\Delta_{\hat{v}} = H_0 \; \Delta_{\tilde{r}} = H_0 \; \tilde{r} \; \sqrt{\exp[\alpha^2 \sigma^D_\zeta^2] - 1}
\]

3. Calibration using null-correlation approach

Presence of the radial peculiar velocities \( v \) are included in the statistical modelization by rewriting the density probability \( dP_2 \) of Eq. (5) as follows:

\[
dP_3 = \frac{1}{A_3} \phi(m, p) f_p(p) \; dp \; g(\zeta^D; \mu) \; d\zeta^D \; d\mu \; f_v(v; x) \; dv
\]

The estimator \( \tilde{r} \) given Eq. (6) is unbiased in the following sense. Suppose a sample of \( N \) galaxies homogeneously distributed in space and with the same measured \( m \) and \( p \). For \( N \) large enough, the distances average on the sample \( \langle r \rangle \) will coincide with the estimator \( \tilde{r} \) (i.e. \( \langle r \rangle \rightarrow E(r \mid m, p) = \tilde{r} \); where \( E(r \mid m, p) \) is the mathematical expectancy of \( r \), given \( m \) and \( p \)).

The statistics of Eq. (6) is generally used for inferring the distance of individual galaxies. It amounts to apply the above statistical formalism on a sample containing only one object, which cannot be done without ambiguity.
The sample is described by the density probability of Eq. (9) and redshift \( z \) parent magnitude \( m \) along the same line-of-sight of direction \((l, b)\) for which apparent magnitude \( m \), log line-width distance indicator \( p \) and redshift \( z \) are measured. It is assumed hereafter that the sample is described by the density probability of Eq. (11) and satisfies the following hypotheses:

- \( \mathcal{H}_0 \): After appropriate corrections (galactic extinction, inclination effects, ...), residual measurement errors can be neglected.
- \( \mathcal{H}_1 \): \( g(\zeta^D) \) is gaussian, i.e. \( g(\zeta^D) = g_G(\zeta^D; 0, \sigma^D) \).
- \( \mathcal{H}_2 \): The distance modulus distribution function reads \( h(\mu) = \exp[3\alpha \mu] \) with \( \alpha = \ln 10 \). 
- \( \mathcal{H}_3 \): The distribution of radial peculiar velocities \( v \) is a gaussian of mean \( u \) and dispersion \( \sigma_v \), i.e. \( f_v(v; x) = g_G(v; u, \sigma_v) \).

Since the calibration sample is constituted of galaxies lying in the field, \( \mathcal{H}_2 \) appears as a reasonable assumption. Anyway, if the line-of-sight direction of the sample goes across a physical cluster, nothing prevents us to discard galaxies known to belong to the cluster.

Assumption \( \mathcal{H}_3 \) implies that the radial peculiar velocity field is not correlated with the distance \( r \). \( \mathcal{H}_3 \) is thus ruled out if flows such like the Great Attractor are present along the line-of-sight. On the other hand, \( \mathcal{H}_3 \) is less restrictive than the pure Hubble’s flow hypothesis (i.e. \( v = 0 \) everywhere). First, galaxies may have a Maxwellian agitation of velocity dispersion \( \sigma_v \). Second, the mean radial peculiar velocity along the line-of-sight \( u \) is not forced to zero. It allows to mimic the following situation. Suppose that a whole-sky sample is calibrated in a velocity frame of reference where sampled galaxies are not globally at rest (say that the sample has a bulk flow \( \mathbf{u} = (u_x, u_y, u_z) \) in cartesian galactic coordinates with respect to the velocity frame of reference). The radial peculiar velocity of galaxies belonging to the same line-of-sight of direction \((l, b)\) will be shifted by \( u = u_x \cos l \cos b + u_y \sin l \cos b + u_z \sin b \).

Assumption \( \mathcal{H}_3 \) in fact tolerates this kind of situation.

3.1. NCA calibration of the DTF slope \( a^D \)

Calibration of the DTF slope \( a^D \) using null-correlation approach (NCA) is based on the following remark. For a calibration sample satisfying assumptions \( \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 \) and \( \mathcal{H}_3 \) with \( u = \sigma_v = 0 \) (i.e. pure Hubble’s flow hypothesis), the variable \( X = X(a^D) \) defined as:

\[
X = X(a^D) = \alpha(m - a^D_p) - \ln z
\]

If \( u = \sigma_v = 0 \) then \( \text{Cov}(p, X(a^D)) = 0 \)

where \( \text{Cov}(p, X) = E(pX) - E(p)E(X) \) is the covariance of variables \( p \) and \( X \). On the other hand, a wrong value of the DTF slope (i.e. \( a_F = a^D + \Delta a \)) correlates \( p \) and the random variable \( X(a_F) = \alpha(m - a_F p) - \ln z \):

\[
\text{Cov}(p, X(a_F)) = -\alpha \Delta a \text{Cov}(p, p) \neq 0
\]

where \( \text{Cov}(p, p) = E[(p - E(p))^2] = \sigma^2 \) is the square of the standard deviation of \( p \). The null-correlation approach consists in to adopt the value of the parameter \( a \) verifying \( \text{Cov}(p, X(a)) = 0 \) as the correct estimate of the DTF slope \( a^D \):

NCA estimate of \( a^D \): a such \( \text{Cov}(p, X(a)) = 0 \) which gives in practice :

\[
a^\text{NCA} = \frac{\text{Cov}(p, m)}{\text{Cov}(p, p)} - \frac{1}{\alpha} \frac{\text{Cov}(p, \ln z)}{\text{Cov}(p, p)}
\]

Note that in the case of pure Hubble’s flow, the NCA is a quite robust technique for calibrating the slope of the DTF relation. It furnishes indeed an unbiased estimate of \( a^D \) whatever the selection effects \( \phi(m, p) \) on \( m \) and \( p \) which affect the observed sample, and whatever the specific shape of the theoretical distribution function \( f_p(p) \) of the variable \( p \) (see appendix C).

Unfortunately, the presence of radial peculiar velocities such as the ones assumed in \( \mathcal{H}_3 \) with \( u \neq 0 \) or \( \sigma_v \neq 0 \) biases the NCA estimate of the slope \( a^D \) since Eq. (13) rewrites in this case (see appendix C):

\[
\text{Cov}(p, X(a^D)) = -\text{Cov}(p, \ln[1 + \frac{v}{H_0 \bar{r}}]) \neq 0
\]

However, the magnitude of this bias can be attenuated by selecting only distant galaxies of the observed sample. It is not surprising since the term \( \ln[1 + v/(H_0 \bar{r})] \) in Eq. (17) becomes negligible for distances \( r \) large enough. Such a subsampling can be performed by discarding galaxies of the observed sample which have a distance estimate \( \bar{r} \) smaller than a given \( r^* \). It corresponds to introduce an extra selection function \( \psi(m, p) \) defined as follows :

\[
\psi(m, p) = 1 \text{ if } \bar{r} \geq r^* \quad \psi(m, p) = 0 \text{ otherwise.}
\]

where \( \bar{r} \) is given Eq. (6). Introducing this extra selection effect does not alterate the result obtained Eq. (13) since this property is insensitive to the specific shape adopted for the selection function on \( m \) and \( p \).

In order to evaluate amplitude of the bias appearing Eq. (17) and its variation with respect to the cut-off in distance estimate \( r^* \), calculations have been performed on a synthetic sample characterized as follows:

- \( \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 \) and \( \mathcal{H}_3 \) are satisfied by the sample.
Fig. 1. Variation of the ratio $R(r^*)$ of selected object within the observed sample with respect to the extra cut-off in distance estimate $r^*$.

Fig. 2. Variation of the bias $\Delta C_2$ inferred by the presence of a Maxwellian velocity agitation of dispersion $\sigma_v$ with respect to the extra cut-off in distance estimate $r^*$ ($\sigma_v' = (\sigma_v^2 + u^2)^{1/2}$).
DTF zero-point $b^D = -7$.
- DTF intrinsic scatter $\sigma^D = 0.4$
- Hubble's constant $H_0 = 75 \text{ km s}^{-1} \text{ Mpc}^{-1}$
- The distribution function of $p$ is a gaussian of mean $\mu_p = 1.5$ and dispersion $\sigma_p = 0.2$
- The selection effects in observation restrict to a cut-off in apparent magnitude $m_{lim} = 15.5$

The two dominant terms of the bias created by the presence of radial peculiar velocities have been calculated (see appendix C for details):

$$\text{Cov}(p, \ln(1 + \frac{v}{H_0 r^*})) = \Delta C_1 - \frac{1}{2} \Delta C_2 + o(\Delta C_2)$$

(19)

where the analytical expressions of $\Delta C_1$ and $\Delta C_2$ in function of $u$, $\sigma_u$ and $r^*$ can be found Eq. (C11). The amplitude of these terms has to be compared with $\Delta C_0 = -\alpha \Delta a \text{ Cov}(p, p)$ appearing in Eq. (14) when a wrong value of the DTF slope is adopted. Expression of $\Delta C_0$ in function of $\Delta a$ and $r^*$ is given Eq. (C12).

Figure 1 shows variation of $R(r^*)$, i.e. the ratio of selected objects within the observed sample, with respect to the cut-off in distance estimate $r^*$ (analytical expression of $R(r^*)$ is given Eq. (C13)). On one hand, high values of $r^*$ are required in order to minimize as far as possible amplitude of the bias created by radial peculiar velocities. On the other hand, accuracy $\sigma_u$ of the NCA slope estimate depends on the size of the selected subsample (i.e. $\sigma_u \propto R(r^*)^{-1/2}$ due to the intrinsic statistical fluctuations affecting the sample). Since these two features are indeed competitive (i.e. $R(r^*)$ decreases when $r^*$ increases), compromise on the optimal value of $r^*$ has to be chosen with regard to the specific characteristics of the data sample under consideration.

The variations of term $\Delta C_2$, i.e. the contribution of a Maxwellian agitation of velocity dispersion $\sigma_v$ to the bias of Eq. (19), are illustrated Fig. 2. Influence of $\Delta C_2$ on the NCA estimate of the DTF slope $a^D$ given Eq. (15) is obtained by comparing $\Delta C_2$ with the contribution of $\Delta C_0 = -\alpha \Delta a \text{ Cov}(p, p)$ to the covariance of Eq. (14). For example, if the mean radial peculiar velocity along the line-of-sight $u$ is zero, the magnitude of the bias on the NCA estimate of $a^D$ created by a velocity field of dispersion $\sigma_v = 1000 \text{ km s}^{-1}$ is greater than $\Delta a = -1.5$ for $r^* = 0$, falls to $\Delta a = -1.0$ for $r^* \approx 10$ (i.e. $R(r^*) \approx 0.95$), equalizes $\Delta a = -0.5$ for $r^* \approx 20$ (i.e. $R(r^*) \approx 0.8$) and finally falls below $\Delta a = -0.1$ for $r^* = 60$ (i.e. $R(r^*) \approx 0.2$). Figure 2 reveals two important features:

- If nearby galaxies are not discarded from the observed sample, the presence of a Maxwellian velocity agitation for galaxies contaminates strongly the NCA estimate of the DTF slope $a^D$ (a velocity dispersion of $\sigma_v = 500 \text{ km s}^{-1}$ induces a bias on $a^D$ of magnitude $\Delta a \approx -0.5$).
- This bias can be rerendered arbitrarily small by selecting only distant galaxies by means of the extra cut-off in distance estimate $r^*$.

The term $\Delta C_1$ decreases in function of $r^*$ less rapidly than $\Delta C_2$. For comparable values of $\Delta C_1$ and $\Delta C_2$ at $r^* = 0$, say for $u = -750 \text{ km s}^{-1}$ and $\sigma_v = 1000 \text{ km s}^{-1}$, $\Delta C_1$ bias falls to $\Delta a \approx -1.5$ at $r^* = 10$ (to be compared with $\Delta a \approx -1$ for $\Delta C_2$ bias), to $\Delta a \approx -1$ at $r^* = 20$ ($\Delta a \approx -0.5$ for $\Delta C_2$) and finally to $\Delta a \approx -0.5$ at $r^* = 60$ ($\Delta a \approx -0.1$ for $\Delta C_2$). It thus turns out that a particular attention has to be paid in priority to the presence of constant velocity fields.

As the existence of constant velocities strongly biases the NCA estimate of $a^D$ (at $r^* = 0$, $\Delta a \approx -1.3$ for $u = -500 \text{ km s}^{-1}$), discarding nearby galaxies by means of distance estimate selection appears as a quite crucial step. Note however that the situation is not so stringent for samples affected by bulk flow. Since the $\Delta C_1$ bias is antisymmetric with respect to $u$ (see Fig. 3), the bias on the NCA estimate of $a^D$ cancels in average if the opposite line-of-sight direction is also considered. By scanning the sky by line-of-sight directions, this interesting symmetry allows indeed to detect bulk flows already at the level of the calibration step.

Finally, some upper bounds on the $a^D$ bias created by large-scale coherent velocity fields can be extracted from analysis of Fig. 3. Suppose that the line-of-sight of the calibration sample points toward the direction of a Great Attractor, located at a distance estimate of $\tilde{r} = 40 \text{ Mpc}$ and creating back-side infall velocities, say of amplitude $u = -500 \text{ km s}^{-1}$ at $\tilde{r} = 50 \text{ Mpc}$ and slowly decreasing at larger distances. The bias on the NCA estimate of $a^D$ will be necessarily smaller than the $\Delta C_1$ bias for $u = -500 \text{ km s}^{-1}$ (i.e. at $r^* = 50$, $|\Delta a| < 0.5$). This property is closely related to the subsampling in distance estimate allowed by the null-correlation approach. The peculiar velocity field, whatever its specific form, becomes negligible compared to the mean Hubble’s flow as long as the cut-off in distance estimate $r^*$ is large enough. In this case the null-correlation approach furnishes unbiased estimate of the DTF slope $a^D$.

### 3.2. NCA calibration of the DTF ”zero-point” B*

Assuming that the DTF slope $a^D$ has been correctly calibrated by means of the technique presented section 3.1 or others calibration procedures, the null-correlation approach is herein used for calibrating the remaining calibration parameters entering into the radial peculiar velocity estimator $\tilde{v}$ of Eq. (9). For this purpose, this equation is rewritten as follows:

$$\tilde{v} = z - H_0 \tilde{r} = z - B^* \exp[\alpha(m - a^D p)]$$

(20)
Fig. 3. Variation of the bias $\Delta C_1$ inferred by the presence of a constant velocity $u$ along the line-of-sight with respect to the extra cut-off in distance estimate $r^*$. 

Characteristics of the sample:

- $a^D = -6$; $b^D = -7$; $\sigma^D = 0.4$
- $\rho = 1.5$; $\sigma = 0.2$; $m_{lim} = 15.5$
- $H_0 = 75 \text{ km/s Mpc}^{-1}$


\[ \text{Cov}(p, \tilde{\nu}) = \text{Cov}(p, \tilde{\nu}(d^p, B^*)) = 0 \quad (21) \]

On the other hand, a wrong value of the \( B^* \) parameter (i.e. \( B_F = B^* + \Delta B^* \)) correlates \( p \) and the random variable \( \tilde{\nu}(d^p, B_F) = z - B_F \exp[\alpha(m - aD^p)] \):

\[ \text{Cov}(p, \tilde{\nu}(d^p, B_F)) = -\frac{\Delta B^*}{B^*} \text{Cov}(p, H_0 \tilde{r}) \neq 0 \quad (22) \]

which does not vanish since selection effects on apparent magnitude \( m \) correlate variables \( p \) and \( \tilde{r} \) (i.e. for selected galaxies, observable \( p \) increases in average with the distance estimate \( \tilde{r} \)). Assuming that the DTF slope \( a_d^D \) has been correctly calibrated, the NCA estimate of the "zero-point" \( B^* \) is then defined as:

\[ B_{\text{NCA}} = \frac{\text{Cov}(p, \exp[\alpha(m - aD^p)])}{\text{Cov}(p, z)} \quad (24) \]

NCA estimate of \( B^* \) is clearly robust. It is insensitive to observational selection effects on \( m \) and \( p \), specific shape of the luminosity function \( f(p) \), constant velocity field and Maxwellian agitation of galaxies (see appendix B). Note however that presence of non-constant large scale velocity fields (such GA flow for example) biases NCA estimate of \( B^* \). Unfortunately the subsampling procedure in distance estimate previously proposed is not efficient for dealing with this kind of biases since the contribution of the peculiar velocity \( v \) to \( \text{Cov}(p, z) \) entering Eq. (24) does not decrease with the distance \( r \).

If one want to express the distance estimator \( \tilde{r} \) of Eq. (6) in true distance units (Mpc), the Hubble’s constant \( H_0 \) has to be estimated (i.e. \( H_0 \tilde{r} = B^* \exp[\alpha(m - aD^p)] \)). For this purpose, estimates of the parameters \( B^* \) —by using NCA calibration for example—, DTF zero-point \( b^D \) —using primary distance indicators— and DTF intrinsic dispersion \( \sigma^D_\zeta \) —in galaxies clusters for example—are required (i.e. \( H_0 = B^* \exp[\alpha(b^D + 25)] \left[ \exp\left(-\frac{7}{2}m\alpha^2\sigma^D_\zeta \right) \right] \)). If \( B^* \) is estimated with an error of \( \delta B^* \), \( b^D \) and \( \sigma^D_\zeta \) with errors of \( \delta b^D \) and \( \delta \sigma^D_\zeta \) respectively, the dominant term of the relative error on the \( H_0 \) estimate reads:

\[ \frac{\delta H_0}{H_0} \approx \frac{\delta B^*}{B^*} + \alpha \delta b^D + 7\alpha^2 \delta \sigma^D_\zeta \quad (25) \]

A new technique for calibrating Tully-Fisher like relations was proposed. Based on a null-correlation approach, this calibration procedure is efficient when galaxies constituting the calibration sample are homogeneously distributed in space. The NCA technique is thus adequate for calibrating a sample of field galaxies for example.

In a first step was introduced the random variable \( X(a) = \alpha(m - aD^p) - \ln z \) dependent on a slope parameter \( a \) and on the observed apparent magnitude \( m \), log line-width distance indicator \( p \) and redshift \( z \). In the case of pure Hubble’s flow (i.e. radial peculiar velocities are null everywhere), it was shown that variables \( p \) and \( X(a) \) are not correlated if and only if parameter \( a \) equals the slope \( a_d^D \) of the Direct TF relation. The NCA estimate of the DTF slope \( a_d^D \) was defined as the value of \( a \) such that correlation between \( p \) and \( X(a) \) vanishes. This estimator of \( a_d^D \) was found particularly robust since it does not depend on the selection effects \( \phi(m, p) \) on \( m \) and \( p \) which affect the observed calibration sample and on the specific shape of the luminosity function.

Influences of radial peculiar velocities was investigated in a second step. It was shown that the presence of a peculiar velocity field biases the NCA estimate of \( a_d^D \). A procedure which consists in discarding nearby galaxies of the observed sample was introduced. Such resampling is achieved by adding an extra selection function \( \psi(m, p) \) in distance estimate \( \tilde{r} \). The magnitude of the bias and its variations in function of the cut-off in distance estimate \( r^* \) have been analysed on a synthetic sample and on the Mathewson spirals field galaxies sample (see appendix D). The proposed resampling procedure looks fairly efficient for minimizing bias on the NCA estimate of the DTF slope \( a_d^D \) created by the presence of radial peculiar velocities.

In the third step, calibration of the "zero-point" parameter \( B^* \) entering the definition of the velocity estimator \( \tilde{\nu}(d^p, \beta) \) was investigated. It was shown that the variables \( p \) and \( \tilde{\nu}(d^p, \beta) = z - B \exp[\alpha(m - aD^p)] \) are not correlated if and only if parameter \( \beta \) equals the "zero-point" \( B^* \). The NCA estimate of the second calibration parameter \( B^* \) was thus defined as the value of \( \beta \) such that correlation between \( p \) and \( \tilde{\nu}(d^p, \beta) \) vanishes. This \( B^* \) estimator is robust. It is insensitive to observational selection effects on \( m \) and \( p \) and to the specific shape of the luminosity function. Moreover, NCA estimate of \( B^* \) is not biased by the presence of a Maxwellian velocity agitation nor by the existence of bulk flow.

Acknowledgements. This work is one of the achievements of a long range program focusing on the statistical modelization of TF like relations, launched five years ago by Roland Triay. He is kindly thanked for constructive discussions. The hospitality of the Centre de Physique Théorique of Luminy is recognized. Stéphane Rauzy is cheerfully thanked for a generous financial support.
The probability density $dP_{m_o,p_o}$ describing a sample of galaxies with the same observed $m_o$ and $p_o$ can be derived from Eq. (5) by using conditional probability:

$$dP_{m_o,p_o} = \frac{1}{A_{m_o,p_o}} \delta(m - m_o) \delta(p - p_o) \phi(m, p) f_p(p) dp \, g(\xi_D; 0, \sigma_D^2) d\xi_D h(\mu) \, d\mu$$

(A1)

where $\delta$ is the Dirac distribution i.e. $\int \delta(x - x_0) \, f(x) \, dx = f(x_0)$ and the normalisation factor $A_{m_o,p_o}$ reads:

$$A_{m_o,p_o} = \int \delta(m - m_o) \delta(p - p_o) \phi(m, p) f_p(p) dp \, g(\xi_D; 0, \sigma_D^2) d\xi_D h(\mu) \, d\mu$$

(A2)

If the two following hypotheses are verified by the sample:

- $H1$) $g(\xi_D; 0, \sigma_D^2)$ is gaussian i.e. $g = g_G$ with $g_G(x; x_0, \sigma) = (\sqrt{2\pi} \sigma)^{-1} \exp[-(x - x_0)^2/(2\sigma^2)]$.
- $H2$) Galaxies are homogeneously distributed along the line-of-sight, i.e. $h(\mu) = \exp[3\alpha \mu]$ with $\alpha = \frac{\ln 10}{5}$. The successive integrations over $m, p$ and $\xi_D$ give for the normalisation factor $A_{m_o,p_o}$:

$$A_{m_o,p_o} = \phi(m_o, p_o) f_p(p_o) \exp[3\alpha (m_o - a^D p_o - b^D)] \exp[\frac{9}{2} \alpha^2 \sigma_D^2]$$

(A3)

where the properties (A4) and (A5a) of gaussian functions have been used:

$$\exp[\lambda x] g_G(x; x_0, \sigma) = \exp[\lambda (x_0 + \lambda \sigma^2/2)] g_G(x; x_0 + \lambda \sigma^2, \sigma)$$

(A4)

$$\int g_G(x; x_0, \sigma) \, dx = 1 \quad (a) \quad ; \quad \int x \, g_G(x; x_0, \sigma) \, dx = x_0 \quad (b) \quad ; \quad \int x^2 \, g_G(x; x_0, \sigma) \, dx = \sigma^2 + x_0^2 \quad (c)$$

(A5)

By using property (A4) and Eq. (A3), the distribution of distance modulus $\mu$ for the sample rewrites thus:

$$dP_{m_o,p_o} = g_G(\mu; m_o - a^D p_o - b^D + 3\alpha \sigma_D^2, \sigma_D^2) \, d\mu$$

(A6)

Properties (A4) and (A5a) imply that the mathematical expectancy $E(r)$ of the distance $r = \exp[\alpha (\mu - 25)]$ reads:

$$E(r) = \int r \, dP_{m_o,p_o} = \exp[\alpha (m_o - a^D p_o - b^D - 25)] \exp[\frac{7}{2} \alpha^2 \sigma_D^2]$$

(A7)

This quantity is generally chosen as distance estimator $\bar{r}$ of individual galaxy with measured $m_o$ and $p_o$. Note however that others estimators could be used, such as the most likely value of the distance $r$ which does not coincide with $E(r)$ since the probability density function (pdf) of the variable $r$ is lognormal, and so not symmetric around its mean. The accuracy $\Delta\bar{r}$ of the distance estimator $\bar{r}$ can be derived from Eqs. (A6,A7) by using properties (A4) and (A5a):

$$\Delta\bar{r} = \sqrt{E((r - \bar{r})^2)} = \bar{r} \sqrt{\exp[\alpha^2 \sigma_D^2]} - 1$$

(A8)

**B. Correlation between $p$ and $\tilde{v}$**

Hereafter, the correlation between $p$ and the velocity estimator $\tilde{v}$ given Eq. (20) is calculated assuming the following hypothesis:

- $H3$) Along a line-of-sight, the distribution of radial peculiar velocities $v$ does not depend on distances $r$ and is a gaussian of mean $u$ and dispersion $\sigma_v$, i.e. $f_v(v; x) = g_G(v; u, \sigma_v)$.

Under assumptions $H1, H2$ and $H3$, the probability density $dP_3$ of Eq. (11) rewrites:

$$dP_3 = \frac{1}{A_3} \phi(m, p) f_p(p) dp \, g_G(\xi_D; 0, \sigma_D^2) d\xi_D \exp[3\alpha \mu] d\mu \, g_G(v; u, \sigma_v) \, dv$$

(B1)

In order to calculate each term involved in the covariance of $p$ and $\tilde{v}$, i.e. $\text{Cov}(p, \tilde{v}) = E(p \, \tilde{v}) - E(p) \, E(\tilde{v})$, the velocity estimator $\tilde{v}$ given Eq. (9) is rewritten in terms of variables $v, \xi_D$ and $\mu$:

$$\tilde{v} = v + H_0 \exp[\alpha (\mu - 25)] \left[ 1 - \exp[-\alpha \xi^D] \exp[\frac{7}{2} \alpha^2 \sigma_D^2] \right]$$

(B2)
for the three quantities $E(p) = \int p \, dp$, $E(\tilde{v} - u) = \int (\tilde{v} - u) \, dP_3$ and $E(p(\tilde{v} - u)) = \int p(\tilde{v} - u) \, dP_3$:

$$E(p) = \frac{1}{A^3} \int \phi(m, p) \, p \, f_p(p) \, dp \, g_C(\zeta^D; 0, \sigma_\zeta^D) \, d\zeta^D \, h(\mu) \, d\mu$$

(B3)

$$E(\tilde{v} - u) = \frac{1}{A^3} \int \phi(m, p) \, f_p(p) \, dp \, g_C(\zeta^D; 0, \sigma_\zeta^D) \, h(\mu) \, H_0 \, \exp[\alpha(\mu - 25)] \left[ 1 - \exp[-\alpha \zeta^D] \exp\left[\frac{7}{2} \alpha^2 \sigma_\zeta^D \right] \right]$$

(B4)

$$E(p(\tilde{v} - u)) = \frac{1}{A^3} \int \phi(m, p) \, p \, f_p(p) \, dp \, g_C(\zeta^D; 0, \sigma_\zeta^D) \, h(\mu) \, H_0 \, \exp[\alpha(\mu - 25)] \left[ 1 - \exp[-\alpha \zeta^D] \exp\left[\frac{7}{2} \alpha^2 \sigma_\zeta^D \right] \right]$$

(B5)

where properties (A5a) and (A5b) have been used. Replacing $h(\mu)$ by $\exp[3\alpha \mu] = \exp[3\alpha(m - a^D p - b^D + \zeta^D)]$ and using definition of $B$ given Eq. (9), Eqs. (B3,B4,B5) expressed in terms of $p$, $m$ and $\zeta^D$ read:

$$E(p) = \frac{1}{A^3} \int \phi(m, p) \, p \, f_p(p) \exp[3\alpha(m - a^D p - b^D)] \, dm \, dp \times \int \exp[3\alpha \zeta^D] \, g_C(\zeta^D; 0, \sigma_\zeta^D) \, d\zeta^D$$

(B6)

$$E(\tilde{v} - u) = \frac{B}{A^3} \int \phi(m, p) \, f_p(p) \exp[4\alpha(m - a^D p - b^D)] \, dm \, dp \times C$$

(B7)

$$E(p(\tilde{v} - u)) = \frac{B}{A^3} \int \phi(m, p) \, p \, f_p(p) \exp[4\alpha(m - a^D p - b^D)] \, dm \, dp \times C$$

(B8)

where the constant $C$ is defined as follows:

$$C = \int \left[ \exp[4\alpha \zeta^D] - \exp[3\alpha \zeta^D] \exp\left[\frac{7}{2} \alpha^2 \sigma_\zeta^D \right] \right] \, g_C(\zeta^D; 0, \sigma_\zeta^D) \, d\zeta^D$$

(B9)

By using properties (A4) and (A5a), it is found that integral of Eq. (B9) vanishes. It thus implies that $C = 0 \Rightarrow E(\tilde{v} - u) = 0, E(p(\tilde{v} - u)) = 0, \text{Cov}(p, \tilde{v} - u) = 0$ and finally $\text{Cov}(p, \tilde{v}) = 0$. This result does not depend on the shape of the selection function $\phi(m, p)$ in $m$ and $p$, on the theoretical distribution function $f_p(p)$ of the variable $p$ and on the mean radial peculiar velocity $u$ along the line-of-sight. For a sample of galaxies satisfying hypotheses $H_1$, $H_2$ and $H_3$, it thus turns out that the observable $p$ is not correlated with the velocity estimator $\tilde{v}$ as long as the model parameters entering into $\tilde{v} = \tilde{v}(a^D, b^D, \sigma_\zeta^D, H_0)$ via Eq. (9) are correct\(^2\). On the other hand, if a wrong value is assumed for one of these calibration parameters, say for example that $a^D$, $b^D$ and $\sigma_\zeta^D$ are correct but not the Hubble’s constant $H_F = H_0 + \Delta H_0$, the covariance between $p$ and $v_F = \tilde{v}(a^D, b_D, \sigma_\zeta^D, H_F)$ gives:

$$\text{Cov}(p, v_F) = \text{Cov}(p, \tilde{v} - \Delta H_0 \tilde{r}) = 0 - \Delta H_0 \text{Cov}(p, \tilde{r}) \neq 0$$

(B10)

which does not vanish since existing selection effects on apparent magnitude $m$ correlate variables $p$ and $\tilde{r}$ (i.e. for selected galaxies, observable $p$ increases in average with the distance estimator $\tilde{r}$). This point motivates the use of a null-correlation approach\(^3\) in order to calibrate Tully-Fisher like relations.

C. Correlation between $p$ and $X$

In this appendix, the reliability of null-correlation approach in order to estimate the slope $a^D$ of the DTF relation is discussed. In a first step, it is shown that the non-observable variable $X_0 = X_0(a^D) = \alpha(m - a^D p) - \ln[H_0 \tilde{r}]$ is not correlated with $p$ as long as hypotheses $H_1$ and $H_2$ are satisfied. Particularly, this property does not depend on the selection function on $m$ and $p$. In a second step, the correlation between $p$ and the observable variable $X$:

$$X = X(a^D) = \alpha(m - a^D p) - \ln z = X_0 - \ln[1 + \frac{u}{H_0 \tilde{r}}]$$

(C1)

is investigated. The presence of radial peculiar velocities such as the ones assumed in $H_3$ indeed correlates $p$ and $X$. However, it is shown that the amplitude of this effect can be weakened by selecting only distant galaxies of the sample.

\(^2\) Note however that the converse does not hold since the Hubble’s constant $H_0$, DTF zero-point $b^D$ and intrinsic DTF dispersion $\sigma_\zeta^D$ degenerate into a single parameter $B^* = H_0 \exp[-\alpha(b^D + 25)] \exp[\frac{7}{2} \alpha^2 \sigma_\zeta^D]$.\(^3\) It can be proven similarly that $\text{Cov}(m, \tilde{v}) = 0$.
a subsampling can be performed by discarding galaxies of the observed sample which have a distance estimate \(\tilde{r}\) less than a given \(r^*\). It corresponds to introduce an extra selection function \(\psi(m, p)\) defined as follows:

\[
\psi(m, p) = \begin{cases} 
1 & \text{if } \tilde{r} = \exp[\alpha(m - a^D p - b^D - 25)] \exp[\frac{7}{2} \alpha^2 \sigma^2_{D^2}] \geq r^* \quad ; \\
0 & \text{otherwise.} 
\end{cases}
\]  
(C2)

Note that introducing this extra selection effect does not alter the result \(\text{Cov}(p, X_0) = 0\). On the other hand, a wrong value of the DTF slope (i.e. \(a_F = a^D + \Delta a\)) correlates \(p\) and \(X(a_F) = \alpha(m - a^D p - 25) - \ln z\):

\[
\text{Cov}(p, X(a_F)) = \text{Cov}(p, X_0(a_F)) + \text{Cov}(p, -\ln[1 + \frac{v}{H_0 r}]) = -\alpha \Delta a \text{ Cov}(p, p) - \text{Cov}(p, \ln[1 + \frac{v}{H_0 r}]) \neq 0 \quad (C3)
\]

where \(\text{Cov}(p, p) = \text{E}(|p - E(p)|^2) = \Sigma(p)^2\) is the square of standard deviation of \(p\). The null-correlation approach thus consists to adopt the value of the parameter \(a\) verifying \(\text{Cov}(p, X(a)) = 0\) as the correct estimate of the slope \(a^D\) of the DTF relation.

It is proven below that \(\text{Cov}(p, X_0) = \alpha(m - a^D p - \ln[H_0 r]) = 0\) as long as assumptions \(\mathcal{H}1\) and \(\mathcal{H}2\) are satisfied by the sample. Rewriting the non-observable variable \(X_0\) in terms of \(m\), \(p\) and \(\mu\) gives \(X_0 = \alpha(m - a^D p - \mu + 25) - \ln H_0\). In terms of \(\zeta^D\), it reads \(X_0 = -\alpha \zeta^D - \ln B\) where \(B = H_0 \exp[\alpha(-b^D - 25)]\) was previously defined Eq. (9). Since \(\alpha\) and \(\ln B\) are constants, \(\text{Cov}(p, X_0) = -\alpha \text{ Cov}(p, \zeta^D)\). The null-correlation between \(p\) and \(\zeta^D\) can be shown by replacing \(h(\mu)\) by \(\exp[3\alpha \mu] = \exp[3\alpha(m - a^D p - b^D + \zeta^D)]\) and using the probability density \(dP_2\) of Eq. (5) expressed in terms of \(p\), \(m\) and \(\zeta^D\):

\[
E(p) = \frac{1}{A_2} Q_1 \times Q_2 = \frac{1}{A_2} \int \phi(m, p) f_p(p) \exp[3\alpha(m - a^D p - b^D)] dm dp \times \int \exp[3\alpha \zeta^D] g_G(\zeta^D; 0, \sigma^2_D) d\zeta^D \quad (C4)
\]

\[
E(\zeta^D) = \frac{1}{A_2} Q_3 \times Q_4 = \frac{1}{A_2} \int \phi(m, p) f_p(p) \exp[3\alpha(m - a^D p - b^D)] dm dp \times \int \zeta^D \exp[3\alpha \zeta^D] g_G(\zeta^D; 0, \sigma^2_D) d\zeta^D \quad (C5)
\]

\[
E(p \zeta^D) = \frac{1}{A_2} Q_1 \times Q_4 = \frac{1}{A_2} \int \phi(m, p) f_p(p) \exp[3\alpha(m - a^D p - b^D)] dm dp \times \int \zeta^D \exp[3\alpha \zeta^D] g_G(\zeta^D; 0, \sigma^2_D) d\zeta^D \quad (C6)
\]

where the normalisation factor \(A_2\) reads:

\[
A_2 = Q_3 \times Q_2 = \int \phi(m, p) f_p(p) \exp[3\alpha(m - a^D p - b^D)] dm dp \times \int \exp[3\alpha \zeta^D] g_G(\zeta^D; 0, \sigma^2_D) d\zeta^D \quad (C7)
\]

It thus turns out that \(\text{Cov}(p, \zeta^D) = E(p \zeta^D) - E(p) E(\zeta^D) = (A_2)^{-2} [A_2 Q_1 Q_4 - Q_1 Q_2 \times Q_3 Q_4] = 0\), which implies \(\text{Cov}(p, X_0) = 0\). This result\(^4\) is insensitive to the shape of the selection function \(\phi(m, p)\) in \(m\) and \(p\) and to the theoretical distribution function \(f_p(p)\) of the variable \(p\).

The term \(\text{Cov}(p, \ln[1 + v/(H_0 r)])\) entering into Eq. (C3) does not vanish since \(p\) is correlated with distance \(r\). However, its amplitude can be rendered arbitrarily small by adding an extra selection effect such as the one mentioned Eq. (C2). This feature is illustrated by the following example. It is assumed that:

- Hypotheses \(\mathcal{H}1\), \(\mathcal{H}2\) and \(\mathcal{H}3\) are satisfied by the sample.
- The distribution function of \(p\) is a gaussian of mean \(p_0\) and dispersion \(\sigma_p\) (i.e. \(f_p(p) = g_G(p; p_0, \sigma_p)\)).
- The selection effects in observation restrict to a cut-off in apparent magnitude (i.e. \(\phi(m, p) \equiv \phi(m, m) = \theta(m_{\text{lim}} - m)\) where \(\theta\) is the Heaveside distribution function, \(\theta(x) = 1\) if \(x \geq 0\) and \(\theta(x) = 0\) otherwise).

The subsampling is performed by using the extra selection function \(\psi(m, p)\) proposed Eq. (C2). By introducing \(m^*\) such that \(r^* = \exp[\alpha(m^* - 25)] \exp[\frac{7}{2} \alpha^2 \sigma^2_{D^2}]\), \(\psi\) rewrites \(\psi(m, p) = \theta([m - a^D p - b^D] - m^*)\). Adding this extra selection function \(\psi(m, p)\), the probability density \(dP_3\) of Eq. (11) reads:

\[
dP_3 = \frac{1}{A_4} \theta(m_{\text{lim}} - m) \theta([m - a^D p - b^D] - m^*) g_G(p; p_0, \sigma_p) dp g_G(\zeta^D; 0, \sigma^2_D) d\zeta^D \exp[3\alpha \mu] d\mu g_G(v; u, \sigma_v) dv \quad (C8)
\]

Since the probability density \(dP_3\) describing the sample is fully specified, calculations of quantities of interest can be performed analytically or by using numerical simulations. Hereafter, cumbersome intermediate calculations have been deliberately omitted and we just furnish the ultimate analytical expressions of these quantities.

\(^4\) It can be proven similarly that \(\text{Cov}(m, X_0) = 0\).
\ln(1 + \frac{v}{H_0 r}) = \frac{v}{H_0 r} - \frac{1}{2} \frac{v^2}{H_0^2 r^2} + o\left(\frac{v^2}{H_0^2 r^2}\right)  \tag{C9}

It implies that the covariance may be expanded as follows:

\[\text{Cov}(p, \ln(1 + \frac{v}{H_0 r})) = \Delta C_1 - \frac{1}{2} \Delta C_2 + o(\Delta C_2) = \text{Cov}(p, \frac{v}{H_0 r}) - \frac{1}{2} \text{Cov}(p, \frac{v^2}{H_0^2 r^2}) + o\left(\text{Cov}(p, \frac{v^2}{H_0^2 r^2})\right) \tag{C10}\]

The calculations give for the terms \(\Delta C_1\) and \(\Delta C_2\):

\[\Delta C_1 = \frac{u}{H_0} \left(\frac{J[\mu^*; 2]}{I[\mu^*; 3]} - \frac{J[\mu^*; 3]}{I[\mu^*; 3]} \right) \quad ; \quad \Delta C_2 = \frac{u^2 + \sigma_v^2}{H_0^2} \left(\frac{J[\mu^*; 1]}{I[\mu^*; 3]} - \frac{J[\mu^*; 3]}{I[\mu^*; 3]} \right)\]  \tag{C11}

where the functions \(I[\mu^*; N]\) and \(J[\mu^*; N]\) are defined Eq. (C14) and Eq. (C15). For a given cut-off in distance estimate \(\mu^*\), the contribution of \(\Delta C_1\) and \(\Delta C_2\) to \(\text{Cov}(p, X)\) has to be compared with \(\Delta C_0 = -\alpha \Delta a \text{Cov}(p, p)\) appearing in Eq. (C3) when a wrong value of the DTF slope is adopted. This term \(\Delta C_0\) reads:

\[\Delta C_0 = -\alpha \Delta a \text{Cov}(p, p) = -\alpha \Delta a \left(\frac{H[\mu^*]}{I[\mu^*; 3]} - \frac{J[\mu^*; 3]}{I[\mu^*; 3]}\right)\]  \tag{C12}

where the function \(H[\mu^*]\) is defined Eq. (C19) and \(I[\mu^*; 3]\) and \(J[\mu^*; 3]\) Eqs. (C14,C15). Finally, the last quantity of interest is the ratio \(R(\mu^*)\) of selected objects within the observed sample. This ratio is equal to the normalisation factor \(A_1(\mu^*) = I(\mu^*; 3)\) divided by \(A_4(\mu^* \to -\infty)\), i.e. the normalisation factor of the sample when no extra selection effect is present \((r^* = 0 \text{ or } \mu^* \to -\infty)\). It reads in function of the extra cut-off in distance estimate \(\mu^*\) introduced Eq. (C8) as follows:

\[R(\mu^*) = M_0(\omega[\mu^*; 3]) - M_0(\omega[\mu^*; 0]) \exp[3\alpha(\mu^* + aDp_0 + bD - m_{\text{lim}} + \frac{3}{2}aD^2\sigma_p^2)] \tag{C13}\]

where \(\omega[\mu^*; N]\) is defined Eq. (C18) and the \(M_0(x)\) function is introduced Eq. (C17).

**NOTATIONS**:  
\(I[\mu^*; N]\) and \(J[\mu^*; N]\), function of the extra cut-off \(\mu^*\) and of an integer \(N\), are defined as follows:

\[I[\mu^*; N] = K[N] \left(\text{M}_0(\omega[\mu^*; N]) - \text{M}_0(\omega[\mu^*; 0]) \exp[N\alpha(\mu^* + aDp_0 + bD - m_{\text{lim}} - \frac{3}{2}\alpha aD^2\sigma_p^2)]\right) \tag{C14}\]

\[J[\mu^*; N] = +K[N] \left(\sigma_p M_1(\omega[\mu^*; N]) + (p_0 - N\alpha aD\sigma_p^2)M_0(\omega[\mu^*; N])\right) \tag{C15}\]

\[\text{M}_0(x), \text{M}_1(x) \text{ and } \text{M}_2(x) \text{ functions reads:} \tag{C17}\]

\[N(t) = g_G(t; 0, 1) ; \quad \text{M}_0(x) = \int_x^{+\infty} N(t) dt ; \quad \text{M}_1(x) = \int_x^{+\infty} t N(t) dt ; \quad \text{M}_2(x) = \int_x^{+\infty} t^2 N(t) dt\]

and \(\omega[\mu^*; N]\), function of the extra cut-off \(\mu^*\) and of an integer \(N\), is:

\[\omega[\mu^*; N] = m_{\text{lim}} - aDp_0 - bD - \mu^* + N\alpha aD^2\sigma_p^2 \tag{C18}\]

The function \(H[\mu^*]\), involved in the calculation of \(\text{Cov}(p, p)\), is defined as follows:

\[\frac{H[\mu^*]}{K[3]} = -(\sigma_p^2 \text{M}_2(\omega[\mu^*; 0]) + 2\sigma_p p_0 \text{M}_1(\omega[\mu^*; 0]) + p_0^2 \text{M}_0(\omega[\mu^*; 0])) \exp[3\alpha(\mu^* + aDp_0 + bD - m_{\text{lim}} + \frac{3}{2}\alpha aD^2\sigma_p^2)] + \sigma_p^2 \text{M}_2(\omega[\mu^*; 3]) + 2\sigma_p (p_0 - N\alpha aD\sigma_p^2)\text{M}_1(\omega[\mu^*; 3]) + (p_0 - N\alpha aD^2\sigma_p^2) \text{M}_0(\omega[\mu^*; 3]) \tag{C19}\]
Fig. 4. Mathewson spiral field galaxies sample versus simulated sample.
The spiral field galaxies sample of Mathewson et al. 1992 is a composite sample of \( N_{\text{gal}} = 1127 \) spiral galaxies lying in the field (galaxies identified as cluster members have been excluded of this catalog). It covers the south hemisphere and extends in redshift up to 10,000 km s\(^{-1}\) with an effective depth of about 4,000 km s\(^{-1}\). Selection effects in observation are not trivial to model since sampled galaxies have been firstly selected in apparent diameter before Mathewson et al. apply their own selection criteria (minimum limits in inclination and velocity rotation). Some galaxies inherited from other observational programs have been also included in the sample. The data used hereafter in the analysis are:

- the apparent magnitude \( m = I_t^\alpha \) where \( I_t^\alpha \) is the total I-band apparent magnitude corrected for internal and external extinction and K-dimming (column (9) in Mathewson et al. 1992).
- the log line-width distance indicator \( p = \log_{10} V_{\text{rot}} \) where \( V_{\text{rot}} \) is the maximum velocity of rotation of the spiral galaxy (column (9) in Mathewson et al. 1992).
- the redshift \( z = V_{\text{CMB}} \) in km s\(^{-1}\) Mpc\(^{-1}\) unit expressed in the CMB velocity frame (column (11), first row in Mathewson et al. 1992).

The application is presented as follows. In section D.1 is explained how numerical simulations, used throughout the analysis for quantifying the amplitude of errors bars and velocity biases, are performed. Section D.2 is devoted to test on the calibration parameters proposed by Mathewson et al. 1992. Finally, NCA calibration of the Mathewson field galaxies sample is performed section D.3.

### D.1. Accuracies, amplitude of biases and simulations

Accuracies of the NCA calibration parameter estimators are herein calculated by using numerical simulations. Reliability of the values of such standard deviations is of course closely related to the way simulated samples succeed in reproducing the characteristics of the real sample under consideration. A preliminar study of the Mathewson field galaxies (MAT) sample characteristics is thus required.

On figure 4 (top left) is shown the decimal logarithm of the cumulative count in function of the apparent magnitude \( m \) for the MAT sample. It turns out that the completeness in magnitude of the MAT sample is violated beyond \( m_{\text{sup}} = 11.3 \), corresponding to about 1/5 of the total number of sampled galaxies. Observational selection effects are then more complex than a mere cut-off \( m_{\text{lim}} \) in apparent magnitude. In order to mimic the real selection effects in observation affecting the MAT sample, simulations are built in two steps.

A virtual sample, complete in apparent magnitude up to \( m_{\text{lim}} = 15.0 \), is firstly generated assuming the following

- Variable \( p \) is generated according to a gaussian distribution function \( f_p(p) = g_G(p; p_0, \sigma_p) \).
- Variable \( \zeta^D \), accounting for the intrinsic scatter of the DTF relation, is generated according to a gaussian distribution function \( g_G(\zeta^D; 0, \sigma^G_\zeta) \).
- The absolute magnitude \( M = a^D p + b^D - \zeta^D \) is then formed, with \( a^D \) and \( b^D \) the slope and the zero-point of the DTF relation.
- Variable \( \mu \) is generated according to an exponential distribution function \( h(\mu) = \exp(3\mu) \) and such that \( \mu < m_{\text{lim}} - M \) (i.e. distances are thus uniformly distributed in space).
- The redshift \( z = H_0 \exp[\alpha(\mu - 25)] \) with \( H_0 \) the Hubble’s constant in km s\(^{-1}\) Mpc\(^{-1}\) is finally formed according to the pure Hubble’s flow hypothesis.

Adopted values for the parameters \( p_0, \sigma_p, \sigma^D, a^D, b^D \) and \( H_0 \), as well as their corresponding ”zero-points” \( B \) and \( B^* \) introduced Eqs. (9,20), are given table 1.

The next step is to extract from this virtual sample a subsample of \( N_{\text{gal}} = 1127 \) which has the same distribution in \( m \) and \( p \) than the observed distribution of the MAT sample. In effect this selection is achieved as follows. The \( m-p \) plane is divided in boxes \( \text{box}(i) \) of equal size and the number \( n(i) \) of MAT galaxies belonging to each box \( \text{box}(i) \) is memorized. Boxes \( \text{box}(i) \) are afterwards filled with galaxies belonging to the virtual sample while the observed \( n(i) \) are not reached. This selection procedure ensures that the \( m-p \) distribution of the simulated samples is approximately identical to the observed one. It corresponds to introduce a complex selection function \( \phi_{\text{MAT}}(m, p) \) in \( m \) and \( p \) directly derived from the data.

**Table 1. Adopted values for the parameters of the simulations**

<table>
<thead>
<tr>
<th>( p_0 )</th>
<th>( \sigma_p )</th>
<th>( a^D_\zeta )</th>
<th>( a^D )</th>
<th>( b^D )</th>
<th>( H_0 )</th>
<th>( B )</th>
<th>( B^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.82</td>
<td>0.2</td>
<td>0.35</td>
<td>-7.0</td>
<td>-5.3</td>
<td>85</td>
<td>0.0097</td>
<td>0.0106</td>
</tr>
</tbody>
</table>

Figure 4 (top right) shows the cumulative count in apparent magnitude for the MAT sample and for a simulated sample. The cumulative count expected for a sample complete up to \( m_{\text{lim}} = 15.0 \) is also shown for comparison. Figures 4 (center left) and (center right) show respectively the distribution in the \( m-p \) plane of the MAT sample and a simulated sample. The difference between these two distributions are due to discretization effects which appear when applying the boxes algorithm. Figures 4 (bottom left) and (bottom right) show respectively the \( m-z \) distributions for the MAT sample and for a simulated sample. The fact that simulated sample distribution approximately reproduces the observed one is encourag-
Fig. 5. Accuracies of NCA estimators and amplitudes of velocity biases: (Top left) Ratio $R(r^*)$ of selected galaxies within the simulated samples function of the extra cut-off in distance estimate $r^*(a^D_B^*)$. (Top right) Standard deviation for the correlation coefficient $\rho(p, \tilde{v})$ and velocity biases created by GA, Bulk and Maxwellian flows. (Bottom left) Standard deviation for NCA slope estimator $a^{NCA}$ and velocity biases created by GA, Bulk and Maxwellian flows. (Bottom right) Standard deviation for NCA "zero-point" estimator $B^{NCA}$ and velocity biases created by GA, Bulk and Maxwellian flows.
Figure 5 visualizes results obtained on these simulated samples. At the top left is plotted the ratio $R(r^*)$ of selected galaxies within a simulated sample when the subsampling in distance estimate (i.e. $\tilde{r} > r^*(a^D, B^*)$) is applied. Standard statistical deviation of the correlation coefficient between $p$ and the velocity estimates $\tilde{v}(a^D, B^*)$ in function of the cut-off in distance estimate $r^*$ is shown figure 5 (top right). Standard deviation (i.e. accuracy) of the NCA slope estimator $a^\text{NCA}$ and of the NCA “zero-point” $B^\text{NCA}$ are respectively presented figures 5 (bottom left) and (bottom right). If no peculiar velocity field is present (i.e. pure Hubble flow hypothesis, as it is the case for simulated samples), we see that $a^\text{NCA}$ and $B^\text{NCA}$ estimators are not biased, as it was previously proven appendices B and C (averaged over 1000 simulations, their values coincide with the input slope and "zero-point" of the simulated sample). It illustrates one of the potentialities of the null-correlation approach for calibrating TF like relations, i.e. its insensitivity to observational selection effects in apparent magnitude $m$ and log line-width distance indicator $p$. For these simulated samples supposed to mimic the Mathewson field galaxies sample ($N_{\text{gal}} = 1127$), accuracy $\sigma_a$ of the NCA slope estimator $a^\text{NCA}$ sounds clearly good : $\sigma_a = 0.07$ or 1% of $a^D$ if all the sample is selected, $\sigma_a = 0.1$ if half of the nearby galaxies of the sample are discarded (i.e. $R(r^*) = 0.5$ or $r^* \approx 3000$ km s$^{-1}$) and $\sigma_a = 0.15$ at $r^* \approx 4000$ km s$^{-1}$ (i.e. $R(r^*) = 0.25$)\(^5\). The same remark holds for the NCA “zero-point” $B^\text{NCA}$ accuracy $\sigma_{B^*}$, $\sigma_{B^*} = 0.0003$ or 3% of $B^*$ at $r^* = 0$ and $\sigma_{B^*} = 0.001$ at $r^* \approx 4000$ km s$^{-1}$.

Influence of peculiar velocity field on the NCA estimators is analysed using three examples : "Great Attractor" (GA) flow, constant or bulk flow and gaussian random flow. In order to take into account the peculiarity of the 3D spatial distribution of the Mathewson field galaxies sample, biases created by these flows have been calculated by comparing the estimates of $a^\text{NCA}$, $B^\text{NCA}$ and $\rho(p, \tilde{v})$ when one of these velocity field is added to the observed redshifts of the MAT sample, with these estimates for the real MAT sample. The Maxwellian flow has a velocity agitation of $\sigma_v = 500$ km s$^{-1}$ and the bulk flow has been chosen to point toward direction $l = 310$ and $b = 20$ in galactic coordinates with an amplitude of al. 1988, centered at a distance of 4200 km s$^{-1}$ toward $l = 310$ and $b = 20$ and creating an infall velocity for our Local group of 535 km s$^{-1}$.

Figure 5 (bottom left) illustrates particularly well the discussion of section 3.1 which was based on analytical results. Biases on the NCA slope estimate $a^\text{NCA}$ created by the presence of Maxwellian and Bulk flows become negligible when nearby galaxies of the MAT sample are discarded using the subsampling procedure in distance estimate (say herein for $r^* > 3000$ km s$^{-1}$). Influence of GA flow can be controled likewise. In practice, large values of the cut-off in distance estimate $r^*$ have to be preferred, of course with regards to the accuracy and to the $r^*$-dependent statistical fluctuations affecting the NCA slope estimate at this distance $r^*$.

Influences of velocity fields on $B^\text{NCA}$ and $\rho(p, \tilde{v})$ are shown respectively figures 5 (bottom right) and (top right). As expected, Maxwellian flow does not bias these two quantities. Since the calibration of the simulated sample was not performed line-of-sight by line-of-sight, presence of Bulk flow slightly biases $B^\text{NCA}$ and $\rho(p, \tilde{v})$. On the other hand, we can see that GA flow creates a significative bias on the two quantities (i.e. greater than their standard deviations). The fact that these biases vanish for large values of the cut-off in distance estimate $r^*$ is due to the specific form of the GA flow and to the characteristics of the 3D spatial distribution of the MAT sample. It cannot be interpreted as a general feature since the correlation between $p$ and $\tilde{v}$ is not expected to vanish when the subsampling in distance estimate is applied, as it is the case for the $p$-$X(a^D)$ correlation. Some reasons may however be advocated for favouring large values of the cut-off in distance estimate $r^*$. Since such a subsampling selects preferencially far away galaxies, a slighter coherence of their peculiar velocities is expected, consequently to their mutual distances. Finally, one remarks that amplitude of the bias created by large flows such as the “Great Attractor” is not greater than $\Delta B^* \approx 0.001$, or 10% of the value of $B^*$. Same remark for the bias on the correlation coefficient $\rho(p, \tilde{v})$ which is less than $\Delta \rho = 0.1$ whatever the value of the cut-off in distance estimate $r^*$.

### D.2. Testing on Mathewson et al. calibration parameters

Mathewson et al. 1992 have proposed some values for the DTF calibration parameters. The authors calibrate the DTF slope $a^D$ in the Fornax cluster (14 galaxies). An estimate of $a^{\text{MAT}} = -7.96$ is obtained by performing a linear regression in magnitude. Assuming that the true distance in km s$^{-1}$ units of Fornax cluster is 1340 km s$^{-1}$, authors\(^6\) since the real distance of MAT galaxies are not known, the contribution of the radial peculiar velocity inferred by GA flow model to observed redshift has been added under the approximation that galaxies distances are given by their observed redshifts.

\(^5\) The standard deviation $\sigma_a$ is proportional to $R(r^*)^{-1/2}$, as expected.
derived a value of $B^\text{MAT} = 0.0039$ for DTF relative zero-point $B$. If $H_0 = 85$ km s$^{-1}$ Mpc$^{-1}$, it implies a value of $b^\text{MAT} = -3.31$ for the DTF zero-point $b^D$. Averaging over the seven richest clusters of their catalog, a value of $\sigma^\text{MAT} = 0.32$ is estimated for the intrinsic scatter of the DTF relation, which gives a value of $B^\ast_\text{MAT} = 0.0042$ for the DTF relative "zero-point" $B^\ast$. No error bars are proposed for these estimates.

Mathewson et al. DTF calibration parameters are tested on figure 6. Figure 6 (left) shows the correlation between $p$ and Mathewson et al. peculiar velocity estimate $\tilde{v}^\text{MAT}$ for the $N_{\text{gal}} = 1127$ of the MAT sample. Figure 6 (right) shows variations of $\rho(p, \tilde{v}^\text{MAT})$ with respect to the cut-off in distance estimate $r^*$ and the deviations from 0 expected when a flow such as the "Great Attractor" is present. It looks very unlikely that the strong correlation found between $p$ and $\tilde{v}^\text{MAT}$ can be explained by the presence of large scale coherent peculiar velocity field. The same feature is observed for the $p$-$X(a^\text{MAT})$ correlation (not shown). This test leads to question in the validity of the calibration techniques used by Mathewson et al. when deriving slope and relative zero-point of the DTF relation.

As a matter of fact, it is known that the estimator of the DTF slope $a^D$, obtained in a cluster by a linear regression on $m$, is biased by the presence of observational selection effects on $m$ and $p$ (see Lynden-Bell et al. 1988 for example). This bias can be corrected on in theory but a full description of selection effects in observation is thus required (see for example Willick 1994), and so is difficult to realize in practice. Same kind of remark holds for the relative zero-point estimate $B$. Since existing peculiar velocity field models proposed in the literature are far to be perfect nor accurate, a room of uncertainty remains when attributing a true distance to the calibration cluster (an error of 250 km s$^{-1}$ for the assumed true distance of Fornax cluster will imply a relative error of $\Delta B \approx 20\%$ for the relative zero-point $B$).

D.3. NCA calibration of the Mathewson spiral field galaxies sample

NCA calibration is herein performed by following the discussions of section D.1. The NCA estimate of the DTF slope $a^D$ is obtained from a subsample selected in distance estimate beyond 4 000 km s$^{-1}$, which corresponds to discard about 60% of nearby galaxies of the MAT sample. NCA estimate of the DTF "zero-point" $B^\ast$ was achieved using galaxies beyond 5 000 km s$^{-1}$. Figure 7 shows the results of the NCA calibration of the Mathewson field galaxies catalog$^7$.

The value of NCA estimate of the DTF slope $a^D$ is given figure 7 (bottom left) : $a^\text{NCA} = -7$, with an accuracy at $r^* = 4 000$ km s$^{-1}$ given by the numerical simulations of $\sigma_a = 0.15$ (or in other words $a^\text{NCA} = -7 \pm 0.15$). Since $^7$ Since the subsampling procedure in distance estimate depends on the values of $a^\text{NCA}$ and $B^\ast_{\text{NCA}}$, few iterations were performed in order to achieve the stable situation presented figure 7.
Fig. 7. NCA calibration of the Mathewson spiral field galaxies sample: (Top left) Ratio $R(r^*)$ of selected galaxies within the MAT sample function of the extra cut-off in distance estimate $r^*(a_{\text{NCA}}, b_{\text{NCA}})$. (Top right) Correlation coefficient $\rho(p, \tilde{v}_{\text{NCA}})$ between $p$ and NCA peculiar velocity estimate $\tilde{v}_{\text{NCA}}$. (Bottom left) NCA slope estimate $a_{\text{NCA}}$ function of the extra cut-off in distance estimate $r^*(a_{\text{NCA}}, b_{\text{NCA}})$. (Bottom right) NCA relative "zero-point" estimate $B_{\text{NCA}}$ function of the extra cut-off in distance estimate $r^*(a_{\text{NCA}}, b_{\text{NCA}})$. 

MATHEWSON FIELD GALAXIES: $N_{\text{gal}} = 1127$

NCA estimate of $a^0$ over 4000 km s$^{-1}$:

$a_{\text{NCA}} = -7.$ with $\sigma_a = 0.15$

NCA estimate of $B^0$ over 5000 km s$^{-1}$:

$B_{\text{NCA}} = 0.011$ with $\sigma_{B} = 0.002$

MATHEWSON FIELD GALAXIES: $N_{\text{gal}} = 1127$

NCA calibration: $a_{\text{NCA}} = -7.$; $B_{\text{NCA}} = 0.011$

Plain: $\rho(p, \tilde{v}_{\text{NCA}})$ function of $r^*$
Fig. 8. NCA "zero-point" estimate for $N_{\text{gal}} = 799$ remaining galaxies of the MAT sample when excluding the GA region (defined as a conic area pointing toward $l = 310$ and $b = 20$ with an angular aperture of $45^\circ$): (Left) Standard deviation for NCA "zero-point" estimator $B_{\text{NCA}}^*$ and velocity biases created by GA and Bulk flows. (Right) NCA "zero-point" estimate $B_{\text{NCA}}^*$ function of the extra cut-off in distance estimate $r^*(a_{\text{NCA}}, B_{\text{NCA}}^*)$.

The estimate is obtained with a large cut-off in distance estimate, it is in principle free of biases created by large scale peculiar velocity field. Adding to this point that NCA calibration technique is insensitive to observational selection effects on apparent magnitude $m$ and log line-width distance indicator $p$, NCA estimate of DTF slope $a^D$ appears as fairly secure.

NCA estimate of the relative "zero-point" $B^*$ is shown figure 7 (bottom right). The value of $B_{\text{NCA}}^* = 0.011$ has been arrested accounting for the more or less stable trend of $B_{\text{NCA}}^*$ beyond $r^* = 5,000$ km s$^{-1}$. To give a value of the $B_{\text{NCA}}^*$ accuracy is a little bit more tricky. It was previously mentioned that the subsampling procedure in distance estimate does not remove bias on $B_{\text{NCA}}^*$ estimator created by the presence of large scale peculiar velocity field (excepting the case of bulk flows). It turns out that the value of this bias depends on the specific geometry and amplitude of the real cosmic velocity field, and so cannot be estimated without modeling this velocity field. The amplitude of this bias for the "Great Attractor" flow model is less than $\Delta B = 0.001$ (or about $10\%$ of $B^*$) for the MAT sample. It means that if the GA flow is real, error on $B_{\text{NCA}}^*$ estimator is dominated by velocity bias rather than statistical fluctuations (accounting for the value of the accuracy on $B_{\text{NCA}}^*$ obtained from numerical simulations). At this stage, other criteria for calibrating DTF relative "zero-point" $B^*$ may be preferred, for example in constraining the average $\langle \hat{v} \rangle$ of radial peculiar velocity estimates over the sample to vanish. Unfortunately such property $\langle \hat{v} \rangle = 0$ is theoretically expected for fair samples (i.e. samples large enough to be representative at any scales of the kinematical fluctuations of the Universe), but not expected for catalogs such as the MAT sample.

Amplitude of velocity bias on the $B_{\text{NCA}}^*$ estimator may however be attenuated by removing of the sample areas presumed to have a strong kinematical activity. Such a procedure was achieved by discarding galaxies of the MAT sample belonging to a cone pointing toward $l = 310$ and $b = 20$ in galactic coordinates with an angular aperture of $45^\circ$ (the GA region). This subsample (i.e. MAT sample, GA region excluded) contains $N_{\text{gal}} = 799$ galaxies. Figure 8 (left) shows amplitudes of biases created by bulk and GA flow when the GA region is excluded of the MAT sample. Compared to the biases shown figure 5 (bottom left) for the whole MAT sample, the discarding procedure looks efficient (if of course, the "Great Attractor" model of Bertschinger et al. 1988 succeeds in mimicking the real cosmic peculiar velocity field). Figure 8 (right) shows the NCA estimate $B_{\text{NCA}}^*$ for the MAT sample, GA region excluded. If the situation improves for $r^* < 3,000$ km s$^{-1}$ (compared to figure 6 (bottom left) showing $B_{\text{NCA}}^*$ for the whole MAT sample), a residual bias persists between
**Fig. 9.** Top: Redshifts $z$ versus NCA distance estimates in km s$^{-1}$ for the MAT sample (left) Outside the GA region (right) Inside the GA region. Bottom: Peculiar velocity difference between Mathewson velocity estimates $\tilde{v}_{\text{MAT}}$ and NCA velocity estimates $\tilde{v}_{\text{NCA}}$. 

Mathewson field galaxies
Inside the GA region
$N_{\text{Gal}} = 328$

Mathewson field galaxies
Outside the GA region
$N_{\text{Gal}} = 799$
the NCA relative "zero-point" $B^*_{\text{NCA}} = 0.011$ has then to be read cautiously, keeping in mind that it can be affected by a velocity bias (anyway presumed not to be greater than $\Delta B = 0.001$)$^8$.

Finally, distance estimate $\tilde{r}$ and velocity estimate $\tilde{v}$ given respectively Eq. (6) and Eq. (20) can be inferred using the NCA calibration parameters $a_{\text{NCA}} = -7.$ and $B^*_{\text{NCA}} = 0.011$ previously derived. Figure 9 (left) and (right) show respectively the redshift $z$ versus the NCA predicted distance $H_0 r^\text{NCA} = B^*_{\text{NCA}} \exp[\alpha(m - a_{\text{NCA}} p)]$ for galaxies of the MAT sample outside and inside the GA region. Figure 9 (bottom) illustrates the difference between the peculiar velocity estimates $\tilde{v} = z - B^* \exp[\alpha(m - a_{\text{p}} p)]$ of Mathewson et al. 1992 and the ones derived in this present appendix. The averaged velocity difference by bins of redshift $z$ is plotted function of the redshift. It turns out that an erroneous input value of the calibration parameters can interpret, especially at large redshifts, as fictitious large scale and coherent peculiar velocity flows. The preliminary calibration step of Tully-Fisher like relation is thus of crucial importance for kinematical studies.

References


$^8$ Independently to these velocity bias problems, relative "zero-point" estimator always suffer from an additional bias inherited from the statistical uncertainties on the slope estimate (i.e. $\delta B/B \approx \alpha < p > \delta a_{\text{p}}$)

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