Notes on Axion, Inflation and Graceful Exit in Stringy Cosmology

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We study the classical equations of motion and the corresponding Wheeler-De Witt equations for tree level string effective action with the dilaton and axion. The graceful exit problem in certain cases is then analysed.
Recently cosmological implications of string theory has attracted considerable attention [1-2]. It is expected that string theory will provide answers related to the evolution of the Universe in early epochs and eventually shed light on the creation of the Universe. A very attractive proposal has been put forward in order to provide the mechanism for inflation and it stems from the scenario of the pre-big-bang cosmology [3] in the string theoretic framework. We recall that the cosmological solution of the tree level string effective action admits two separate branches labelled as (+) and (−). There is a solution in the (+)-branch which corresponds to weakly coupled dilaton for a cold and flat Universe in the $t \rightarrow -\infty$ limit. As time increases to zero, this background configuration evolves towards a strong coupling regime and the solution admits positive Hubble parameter, $H$, and rapidly accelerating expansion towards strong curvature, progressing to a singularity in its future. On the other hand, the (−)-branch proceeds to a large spatially flat Universe such that the Hubble parameter is positive (expanding Universe), but with deceleration. The singularity is in the past for this particular solution. Furthermore, it can be smoothly joined to a FRW cosmological solution in the late epoch. Naturally, in view of the existence of such cosmological solutions, one is tempted to identify the former solution with the inflationary phase of the Universe whereas the latter endowed with several attractive features of the late time cosmology.

Indeed, these two solutions are related to one another by the stringy symmetry known as scale factor duality (SFD) which is a part of the larger T-duality symmetry group, $O(d,d)$. If we accept that these two solutions correspond, in reality, to a single solution in the temporal evolution of the Universe, we can envision the following scenario. There is a pole-driven superinflationary phase for $t < 0$. It is driven by the dilaton kinetic energy term and the dilaton potential does not affect the growth of the scale factor in this regime. Subsequently, for $t > 0$, we go over to the expanding, decelerating Universe. However, one encounters the problem of graceful exit in string cosmology as is the case in any theory which incorporates inflations. Recently, it has been shown that the graceful exit problem persists[4] for the string theoretic approach to cosmology when one considers the mechanism for branch changes in the classical framework. The no-go theorems have emphasised that the branch changes are not possible even if we include effects of the axion and perfect fluids in the string evolution equations in addition to the dilaton potential. Therefore, so long as one considers tree level string effective action, the graceful exit problem awaits its resolution at the classical level.
A minisuperspace model for spatially homogeneous Bianchi I Universe has been considered by Gasperini, Veneziano and one of us (JM) [5]. These authors have shown that the wave function of the Universe, derived from the Wheeler-De Witt equation [6], can be expanded in terms of plane waves in the minisuperspace. It was found that configurations associated with the pre-big-bang correspond to "right moving" waves, whereas those corresponding to the post-big-bang backgrounds are identified with the "left moving" waves. Thus the two waves move in the opposite directions of the effective spatial coordinates. Moreover, the two branches are related by SFD and a time reversal symmetry. Furthermore, it was demonstrated by explicit examples [5] that transitions between the pre- and post-big-bang domains are possible if the wave function undergoes spatial reflection in the minisuperspace.

The purpose of this note is to explore further the resolution of the graceful exit problem and construct explicit solutions to the WDW equation for a more general class of dilaton potential or to take into account the contributions of the axionic field. First, we shall discuss classical solutions to the string equations of motion and subsequently look for solutions of the Wheeler-De Witt equation.

We consider the low energy string effective action [7] of the form

\[ S = -\frac{1}{2\lambda^2} \int d^4x \sqrt{-g} e^{-\phi} (R + \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} H^2 + V(\phi)), \]  

(1)

where \( \phi \) is the dilaton field (with \( e^{\frac{\phi}{2}} \) being the string coupling constant), \( H_{\mu\nu\rho} \) is the field strength of the anti-symmetric tensor filed \( B_{\mu\nu} \). There can also be cosmological constant term (\( \Lambda \)) in the action, the precise form of which depends on the detail of the compactification scheme used to bring the 10 dimensional action to 4 dimension. Moreover, non-preturbatively there can be dilaton potential contribution to the action. The contribution of cosmological term and the dilaton potential have been grouped together in \( V(\phi) \) of the above action.

In the cosmological case, all the background fields only depend on cosmic time, \( t \) and we can write the metric and and the antisymmetric tensor field in the following form

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & G_{ij}(t) \end{pmatrix} \]  

(2)

\[ B_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ij}(t) \end{pmatrix} \]  

(3)
Where $G_{ij}$ and $B_{ij}$ are $3 \times 3$ symmetric and antisymmetric matrices respectively. The action can be rewritten in a manifestly $O(3, 3)$ form when the backgrounds assume time dependence.

$$
S = -\frac{\lambda}{2} \int dt e^{-\bar{\phi}} [(\partial_t \bar{\phi})^2 + \frac{1}{8} \text{Tr}(\partial_t M \eta \partial_t M \eta) + V],
$$

(4)

where

$$
\bar{\phi} = \phi - \frac{1}{2} \ln |G_{ij}|.
$$

(5)

Here a constant spatial volume factor has been absorbed into the definition of $\bar{\phi}$ and $M$ is a symmetric $6 \times 6$ matrix given by

$$
M = \begin{pmatrix}
G^{-1} & -G^{-1}B \\
BG^{-1} & G - BG^{-1}B
\end{pmatrix}
$$

(6)

and we have defined the matrices $G$ and $B$ earlier. The action is invariant under $O(3, 3)$

$$
\bar{\phi} \rightarrow \phi, \quad M = \Omega^T M \Omega,
$$

(7)

where

$$
\Omega^T \eta \Omega = \eta, \quad \eta = \begin{pmatrix} 0 & I \\
I & 0 \end{pmatrix}
$$

(8)

with $M$ satisfying

$$
M \eta M = \eta.
$$

(9)

Note that the potential $V = V(\bar{\phi})$ in order that the action remains invariant under $O(3, 3)$. For diagonal metric and zero torsion, with $\Omega = \eta$, this transformation reduces to the scale factor duality. As mentioned earlier, the duality symmetry relates different branches of the scale factor and dilaton when we envisage their time evolutions. Of special interest to us is the one corresponding to accelerated, expanding background with growing curvature (pre-big bang) which gets related to another with desired late time attributes such as expanding decelerated, universe with decreasing curvature (post-big bang).

In what follows, we study the action (1) and the classical solutions relevant in our context, discuss the Wheeler-De Witt equation and analyze the wave function of the universe when the background is given by spatially flat Robertson-Walker metric with

$$
ds^2 = N(t)dt^2 - e^{2\beta(t)} dx_i dx_j \delta^{ij}.
$$

(10)
It is well known that the field strength of the antisymmetric tensor field, in four di-
mensions, is related to the pseudoscalar axion through the Poincare duality transformation

\[ H^{\mu\nu\rho} = e^{\tilde{\phi}} \epsilon^{\mu\nu\rho\lambda} \partial_\lambda \Theta. \]  

Furthermore, the equations of motion for \( H^{\mu\nu\rho} \) is a conservation law and thus the equation
of motion for \( \Theta \) leads to conservation of the axionic charge. Thus, for the case at hand,
the action can be brought to the following form

\[ S = -\frac{\lambda}{2} \int d\tau \left\{ \frac{1}{N} (\partial_\tau \tilde{\phi})^2 - \frac{1}{N} (\partial_\tau \beta)^2 + \frac{q^2 N}{2} e^{-2(\sqrt{3} \beta + \tilde{\phi})} + NV(\tilde{\phi}) e^{-2\tilde{\phi}} \right\}. \]  

where \( dt = e^{-\tilde{\phi}} d\tau \) and \( \tilde{\phi} \) has been defined before. Here the charge \( q \) is related to the
canonical momentum of \( \Theta \), \( p_\Theta = \frac{\partial L}{\partial \partial_\tau \Theta} = q \).

At this stage the following comments are in order. We use the dilaton time variable,
\( \tau \), instead of the cosmic time variable, \( t \), in the definition of the action integral above. This
choice is made for later convenience. If we had used the integral over cosmic time to define
the action integral and derived the the equations of motion, then the classical solutions
would have taken the known form [4] and they would have satisfied the usual constraint
equations. However, when we choose to work in the \( \tau \)-variable, the constraint equation
(see below) takes a simpler form; nevertheless if we go over to \( t \)-variable we recover the
usual constraint equation. Henceforth, we shall express the equations of motion in terms
of \( \tau \) variable and obtain solutions in this variable.

First, we focus our attention to the case where the axionic charge, \( q \), is taken to be
zero and we consider the dilaton potential which is a generalized form of the one considered
by the authors of [5]. In the second example, we shall consider nonvanishing axionic charge
with vanishing dilatonic potential.

When we adopt the first scenario, the equations of motion for \( \beta \) and the dilaton \( \tilde{\phi} \), in
the gauge \( N = 1 \) are

\[ \partial_\tau^2 \beta = 0, \]  

\[ \partial_\tau^2 \tilde{\phi} - \frac{1}{2} \frac{\partial}{\partial \tilde{\phi}} \left( V e^{-2\tilde{\phi}} \right) = 0. \]  

Furthermore, \( \beta \) and \( \tilde{\phi} \) must satisfy the following constraint

\[ (\partial_\tau \tilde{\phi})^2 - (\partial_\tau \beta)^2 - V e^{-2\tilde{\phi}} = 0. \]  

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Now with the potential $V = V_0 e^{m \bar{\phi}}$, $V_0 > 0$ and for $m \neq 2$ the eqn (13), (14) and (15) can be solved exactly with the answers

$$\beta = \frac{k \tau}{\lambda}, \quad \bar{\phi} = \frac{-2}{(m-2)} \ln \left[ \frac{\lambda \sqrt{V_0}}{k} \sinh \frac{k(m-2)\tau}{2\lambda} \right],$$

(16)

here $k$ is a constant of integration. Note that for $m = 4$ the above potential is same as in [5] where the equations of motion could be solved exactly i.e. $\beta$, thus the scale factor $a$ and $\bar{\phi}$ could be expressed in terms of the cosmic time. Thus the present study is a generalization of [5] for $m \neq 4$.

However, although we could get $\bar{\phi}$ and $\beta$ explicitly as functions of $\tau$, it is hard to write them in terms of cosmic time $t$ in closed forms. Nevertheless, it is straight forward to notice that we can get desired solutions such as the inflationary one in the pre-big-bang regime and expanding, decelerating one in the positive $\tau$ domain. From (16) we see that for negative $\tau$ (or $t$) as $\tau \to -\infty$ the scalefactor $a \to 0$ and we begin in the weak coupling phase ($\bar{\phi} \to -\infty$) and evolve towards the strong coupling ($\bar{\phi} \to +\infty$) domain as $\tau \to 0$ from the negative side. It is easy to check that in terms of cosmic time, $\tau = 0$ corresponds to finite value of $t$. Furthermore, for $|t|$ less than this value, the fields become complex. Hence there would always be a classically forbidden region. In particular, it is easy to check that close to this region, $\bar{\phi}$ behaves as

$$\bar{\phi} = -\frac{2}{m} \ln \left[ -\frac{\lambda V_0}{k^2} + \frac{mk^2}{4\lambda} t^2 - \left( \frac{m}{4} - 1 \right) \frac{mk^4}{24\lambda^3 V_0} t^4 + .... \right].$$

(17)

By using the definition of $H$ and the defining relation between the cosmic time $t$ and $\tau$ we can check that $H > 0$ and $\dot{H} > 0$ as required by pre big-bang scenario; here dot denotes derivative with respect to the cosmic time. Similarly, it is quite evident that for large positive $\tau$ (or $t$), we proceede towards the scenario of late time cosmology, $H > 0$, but $\dot{H} < 0$.

If we adopt the Hamiltonian analysis of [5], then the the WDW equation in this case can be written down as

$$[\left( \frac{\partial}{\partial \bar{\phi}} \right)^2 - \left( \frac{\partial}{\partial \beta} \right)^2 + \lambda^2 V_0 e^{(m-2)\bar{\phi}}] \Psi(\bar{\phi}, \beta) = 0.$$
equation (18) can be split as $\Psi(\bar{\phi}, \beta) = \psi_k(\bar{\phi}) e^{-ik\beta}$ where $k$ is a constant which belongs to the continuous eigenvalue spectrum of the momentum operator corresponding to the $\beta$-direction and the function $\psi_k(\bar{\phi})$ satisfies

$$(\partial_{\bar{\phi}}^2 + k^2 + \lambda^2 V_0 e^{(m-2)\bar{\phi}})\psi_k(\bar{\phi}) = 0,$$  

whose general solution is a linear combination of first and second kind Hankel functions, $H^{(1)}_{\nu}(z)$ and $H^{(2)}_{\nu}(z)$ where $\nu = \frac{2k}{m-2}$ and $z = \frac{2\lambda}{m-2} \sqrt{V_0} e^{\frac{m-2}{2} \bar{\phi}}$. However, we require the wave-function to be regular in the limit $\bar{\phi} \to \infty$ and this fixes the wave function to be

$$\Psi_k(\bar{\phi}, \beta) = NH\left(\frac{2k}{m-2}\right) \left(\frac{2\lambda}{m-2} \sqrt{V_0} e^{\frac{m-2}{2} \bar{\phi}}\right) e^{-ik\beta},$$

where $N$ is a normalization constant. Thus, asymptotically, in the regime $\bar{\phi} \to -\infty$, we have

$$\lim_{\bar{\phi} \to -\infty} \Psi_k(\bar{\phi}, \beta) = \Psi_k^+( - \Psi_k^-),$$

where

$$\Psi_k^+(\bar{\phi}) = iN \csc\left(\frac{2ik}{m-2} - \frac{2k}{m-2}\right) e^{\frac{2ik}{m-2} \left(\frac{\lambda V_0}{m-2} - \frac{2ik}{m-2}\right)} e^{-ik(\beta + \frac{2\bar{\phi}}{m-2})} \Gamma\left(1 + \frac{2ik}{m-2}\right),$$

and

$$\Psi_k^-(\bar{\phi}) = iN \csc\left(\frac{2ik}{m-2} - \frac{2k}{m-2}\right) e^{\frac{2ik}{m-2} \left(\frac{\lambda V_0}{m-2} - \frac{2ik}{m-2}\right)} e^{-ik(\beta - \frac{2\bar{\phi}}{m-2})} \Gamma\left(1 - \frac{2ik}{m-2}\right),$$

The probability for transitions from the classical trajectory with $\beta = \bar{\phi}$ to the duality related trajectory $\beta = -\bar{\phi}$ is then found to be

$$R_k = \frac{|\Psi_k^-|^2}{|\Psi_k^+|^2} = e^{\frac{-4\pi k}{m-2}}.$$  

Thus, we see that there is a turning from one branch to another for this potential where classically forbidden domains exist. We also notice from (17) that the classically forbidden region increases with $m$. This is also reflected in (22) since $R_k$ increases as $m$ increases.

We note that if we would have chosen $V = -V_0 e^{m\bar{\phi}}$ instead, the solutions to the equations of motion along with the constraint equation lead

$$\beta = \frac{k\tau}{\lambda}, \quad \bar{\phi} = \frac{-2}{m-2} \ln\left[\frac{\lambda V_0}{k} \cosh\left(\frac{k(m-2)\tau}{2\lambda}\right)\right].$$

If we proceed as above the function $\psi_k(\bar{\phi})$ would become a linear combination of the modified Bessel functions $K_{\nu}(z)$ and $I_{\nu}(z)$. But again the requirement of the wave function
to be regular in the limit $\bar{\phi} \to \infty$ would fix the wave function to be expressed only in term of $K_\nu(z)$ and thus asymptotically, in the regime $\bar{\phi} \to -\infty$, we would have

$$\lim_{\bar{\phi} \to -\infty} \Psi_k(\bar{\phi}, \beta) = \Psi_k^{(+)} + \Psi_k^{(-)}$$

(23)

where

$$\Psi_k^{(+)} = -\frac{N\pi}{2\sin\left(\frac{2ik\pi}{m-2}\right)} \left( \frac{\lambda\sqrt{V}_0}{m-2} \right)^{\frac{2ik}{m-2}} e^{-ik(\beta-\frac{2\bar{\phi}}{m-2})} \frac{e^{-ik(\beta-\frac{2\bar{\phi}}{m-2})}}{\Gamma(1 + \frac{2ik}{m-2})},$$

and

$$\Psi_k^{(-)} = \frac{N\pi}{2\sin\left(\frac{2ik\pi}{m-2}\right)} \left( \frac{\lambda\sqrt{V}_0}{m-2} \right)^{\frac{-2ik}{m-2}} e^{-ik(\beta+\frac{2\bar{\phi}}{m-2})} \frac{e^{-ik(\beta+\frac{2\bar{\phi}}{m-2})}}{\Gamma(1 - \frac{2ik}{m-2})}.$$

Thus in this case, in the low energy limit, $R_k \to 1$ for all $k$. This was expected since in this example the two branches are smoothly connected at the classical level also. In particular, in terms of cosmic time $t$, near $t = 0$, $\bar{\phi}$ has an expansion

$$\bar{\phi} = -\frac{2}{m} \ln\left[ \frac{\lambda V_0}{k^2} \right] + \frac{mk^2}{4\lambda} t^2 + \left( \frac{m}{4} - 1 \right) \frac{m k^4}{24\lambda^3 V_0} t^4 + ....].$$

(24)

Now we turn our attention to the second scenario where the dilaton potential is set to zero and $q \neq 0$. It is evident from equation (12) that the action is no longer invariant under SFD transformation $\beta \to -\beta$ and $\bar{\phi} \to \bar{\phi}$ and if $q$ does not transform nontrivially under duality. However, we know that the action (4) is $O(d,d)$ and hence duality transformation invariant. We arrived at action (12) after implementing Poincare duality (11). Therefore, in order to maintain duality invariance of action (12) the transformations $(\bar{\phi}, \beta, q^2) \to (\bar{\phi}, -\beta, q^2 e^{-4\sqrt{3}\beta})$ should hold good. We can interpret the presence of the axion term as the effect of a specific potential which carries both $\beta$ and $\bar{\phi}$ dependence. It is not possible to obtain classical solutions and wave function for the WDW equation for a potential with arbitrary $\beta$ and $\bar{\phi}$ dependence. We shall see below that, for the problem at hand, we can define new variables which are linear combinations of $\bar{\phi}$ and $\beta$ such that the resulting classical equations of motion take rather simple form which admit simple solutions.

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1 Similar transformations were used in the context of Brans-Dicke theory or various string theories and M theory to reduce equations of motion of various fields into simple Liouville and/or Toda like equations [8].
Furthermore, we can solve the WDW equation for this case \(^2\). Let us first write the equations of motion in the variables \(\beta\) and \(\bar{\phi}\) in the N=1 gauge. They are

\[
\partial^2_\tau \beta - \frac{\sqrt{3}}{2} q^2 e^{-2(\sqrt{3} \beta + \bar{\phi})} = 0, \tag{25}
\]

\[
\partial^2_\tau \bar{\phi} + \frac{q^2}{2} e^{-2(\sqrt{3} \beta + \bar{\phi})} = 0, \tag{26}
\]

and the constraint is

\[
(\partial_\tau \bar{\phi})^2 - (\partial_\tau \beta)^2 - \frac{q^2}{2} e^{-2(\sqrt{3} \beta + \bar{\phi})} = 0. \tag{27}
\]

Defining \(X = (\sqrt{3} \beta + \bar{\phi})\) and \(Y = \beta + \sqrt{3} \bar{\phi}\), from (25) and (26) we get

\[
\partial^2_\tau X - q^2 e^{-2X} = 0, \quad \partial^2_\tau Y = 0. \tag{28}
\]

Now the constraint equation has the following form

\[
(\partial_\tau X)^2 - (\partial_\tau Y)^2 + q^2 e^{-2X} = 0. \tag{29}
\]

Note that these equations are identical to the corresponding equations of the earlier example if we identify \(X, Y\) and \(q^2\) with \(\bar{\phi}, \beta\) and \(-V\) respectively which may have some interesting significance but we will not proceed further in this direction. The solutions to the above equations are

\[
X = -\frac{1}{2} \ln \left[ \frac{c}{4q^2} \text{sech}^2 \frac{\sqrt{c}}{2} \tau \right], \quad Y = -\frac{\sqrt{c} \tau}{2}. \tag{30}
\]

Using \(X\) in (25) and (26) we get

\[
\beta = -\frac{\sqrt{3}}{4} \ln \left[ \frac{c}{4q^2} \text{sech}^2 \frac{\sqrt{c} \tau}{2} \right] + \frac{\sqrt{c} \tau}{4}, \tag{31}
\]

\[
\bar{\phi} = \frac{1}{4} \ln \left[ \frac{c}{4q^2} \text{sech}^2 \frac{\sqrt{c}}{2} \tau \right] - \frac{\sqrt{3} c \tau}{4}. \tag{32}
\]

The above solutions are to be used for \(\tau > 0\). For the other branch \(\tau < 0\) we have the set of solutions, namely,

\[
\beta = -\frac{\sqrt{3}}{4} \ln \left[ \frac{c}{4q^2} \text{sech}^2 \frac{\sqrt{c} \tau}{2} \right] - \frac{\sqrt{c} \tau}{4}, \tag{33}
\]

\(^2\) The equations of motion of string theory in the presence of the axion field has also been solved in [9], but with a different parametrization. As we will see, in our parametrization, it will be easier to analyse the corresponding WDW equation.
\[
\bar{\phi} = \frac{1}{4} \ln \left[ \frac{c}{4q^2 \sech^2 \left( \frac{\sqrt{c}}{2} \tau \right)} \right] + \frac{\sqrt{3}c\tau}{4} \tag{34}
\]

We can write down the WDW equation after passing over to the Hamiltonian description and this is found to be

\[
[\partial^2_\phi - \partial^2_\beta + \frac{1}{2} q^2 \lambda^2 e^{2(\sqrt{3}\beta + \bar{\phi})}] \Psi(\beta, \bar{\phi}) = 0. \tag{35}
\]

Note that because of the mixed nature of the last term in the square bracket we cannot look for solutions by separation of variables i.e. there is no more conservation of momentum in the \( \beta \)-direction. However, equation (35) can be brought to the following convenient form for further analysis in terms of \( X \) and \( Y \) as defined before.

\[
[\partial^2_X - \partial^2_Y + \frac{1}{2} q^2 \lambda^2 e^{-2X}] \Psi(X, Y) = 0. \tag{36}
\]

It is obvious that we can separate the variables and the wave function will be factored into an oscillatory part in \( Y \) and a function which is a linear combination of Henkel functions \( H^{(1)} \) and \( H^{(2)} \). We also see that \( X \rightarrow \infty \), for \( \tau \rightarrow \pm \infty \). Thus we can choose \( H^{(2)} \) as a factor of the wave function.

\[
\Psi_k(X, Y) = N H^{(2)}_{ik} \left( \frac{q\lambda}{\sqrt{2}} e^{-X} \right) e^{-ikY}. \tag{37}
\]

Now, expanding for \( X \rightarrow \infty \) we find that the wave function can be written as sum of two terms, \( \Psi^+(\beta, \bar{\phi}) + \Psi^-(\beta, \bar{\phi}) \), as before and in terms of the original variables it has the following form

\[
\Psi_k(\beta, \bar{\phi}) = iN \csc(ik\pi) \left[ \left( \frac{\lambda q}{2\sqrt{2}} \right)^{-ik} e^{-k\pi \frac{e^{-i(k(\sqrt{3}+1)(\beta+\bar{\phi}))}}{\Gamma(1 - ik)}} - \left( \frac{\lambda q}{2\sqrt{2}} \right)^{ik} e^{ik(\sqrt{3}-1)(\beta - \bar{\phi})} \Gamma(1 + ik) \right]. \tag{38}
\]

Notice that the above wave function strongly resembles the solution obtained in [5]. However, it is worthwhile to note a few features. First of all, the quantum number \( k \) appearing in [5] or the one we introduced in our first example is the eigen value of the canonical momentum operator in \( \beta \)-direction. On the other hand, \( k \) appearing here is the eigen value of \( \Pi_Y \), momentum conjugate to \( Y \) which is a linear combination of \( \beta \) and \( \bar{\phi} \). Thus, it is not straightforward to interpret the wave function as linear combination of left and right moving waves as was presented earlier. We feel it is quite interesting, in this case, one would solve the Wheeler-De-Witt equation in the presence of a potential which depends both on \( \beta \) and \( \bar{\phi} \). We must admit that in passing from classical to quantum hamiltonian,
we have not been careful about the ordering ambiguity. In some cases, however, the $O(d, d)$ symmetry uniquely fixes the ordering (see ref [5] for a discussion on this issue).

In this note we have considered string effective action in presence of dilaton potential and the axionic field in two separate cases. When the axion is absent, a generalised form of the dilatonic potential used by [5] is considered and the classical solutions were obtained in terms of the dilatonic time, $\tau$, rather than the cosmic time, $t$. Then we looked for solutions of the Wheeler-De Witt equation for the potential. We chose the boundary condition such that the initial incoming wave starts as left moving mode and after scattering off the barrier, we have the right moving mode which corresponds to the situation of an expanding, decelerating Universe. Subsequently, we analysed the case, where the dilaton potential was set to zero and the antisymmetric field strength was included. Since the axionic charge is conserved, we have a potential term involving the dilaton in presence of the axionic charge. The classical equations of motion were solved exactly after making a suitable coordinate transformation. Subsequently, we solved the WDW equation for this case and found the appropriate wave function satisfying the required boundary conditions.

We may remark that in the presence of arbitrary dilaton potential and the axion, we shall not be able to solve the WDW equation exactly; however, one could resort to suitable approximation method, such as WKB approximation, as is utilised in quantum cosmology. Furthermore, it is recognised that the dilaton-graviton system has close resemblance with the Brans-Dicke theory with the specific choice of the parameter. Indeed, recently dualities in the Brans-Dicke theory have been discussed and the solutions of WDW equation have been analyzed in this frame work [10]. It will be interesting to investigate the solutions of WDW equations and their properties for axion and dilaton in such theories.

Finally, we would like to mention that in type II string theories, there are other antisymmetric rank 3 tensor fields at massless level with different dilaton coupling than the one that has been considered here. Also depending upon the detail of the compactification, there can be other moduli fields. It turns out that in many of the cases, classical equations of motion can be solved exactly [11,8]. Following closely the discussion here, it might be possible to solve and analyse the WDW equations in those cases. We hope to report on it in the future.

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