Renormalization in words

1. Calculate the rad. corr. to the definition of the bare parameters in terms of "measurements":

\[ m_E^2 = m_\tau^2 + \delta m_E^2 \]
\[ m_W^2 = m_W^2 + \delta m_W^2 \]
\[ \epsilon_0 = \epsilon + \delta \epsilon \]
\[ \delta m_E^2 = \text{Re } \Sigma_{ee}(m_E^2) \]
\[ \delta m_W^2 = \text{Re } \Sigma_{WW}(m_W^2) \]
\[ \delta \epsilon = \frac{1}{\alpha} \Pi_{\epsilon}(0) - \frac{g_\tau}{\epsilon_0} \frac{\Sigma_{ee}(0)}{m_E^2} \]

2. Calculate the prediction for something (cross section, width, asymmetry ....) using the bare parameters.

3. Express the bare parameters in the physical ones, using the counterterms.

4. All infinities cancel!!

Only physical parameters appear!!

Order by order in perturbation theory!!

Only 3 \( \Box \) counterterms needed!!

\textbf{IF the theory is renormalizable}

\[ \bullet \text{ plus mass counterterms for fermions, Higgs...} \]
Extra renormalization?

The physical quantities in the theory are masses and charges. Therefore physical predictions ($\sigma, \Gamma, A_{FB}$ ...) are made finite by mass and charge renormalization.

Some people also want to have unphysical quantities (external lines in diagrams, numerators of propagators) finite. To do this, counterterms for unphysical quantities have to be added: the wave function renormalization.

**PRO w.f.r.**

It makes "building blocks" of diagrams and cross sections finite (by cancelling infinities that would cancel between building blocks) => easier for careful bookkeeping, especially in complicated processes or higher orders.

**CONTRA w.f.r.**

It is irrelevant for physical quantities => conceptually simpler to disregard it!

Propagators are finite (rather than "finite") after w.f.r.
Calculating the renormalized photon self-energy $\Pi(s)$

Dispersion integral for unrenormalized $\Pi_f(s)$:

$$\text{Re} \, \Pi_f(s) = \frac{\alpha}{3\pi} \int_0^\infty dt \, \frac{R_f(t)}{t-s}$$

Log. divergence if $\lim_{t \to 0^+} R_f(t) = Q_f^2$

Put $\text{Re} \, \Pi(s) = \text{Re} \, \Pi_f(s) - \text{Re} \, \Pi_f(0)$:

$$\text{Re} \, \Pi_f(s) = \frac{\alpha}{3\pi} \int_0^\infty dt \, R_f(t) \left( \frac{1}{t-s} - \frac{1}{t} \right)$$

$$= \frac{\alpha s}{3\pi} \int_0^\infty dt \, \frac{R_f(t)}{t(t-s)}$$

better convergence!

This is a once-subtracted dispersion integral

$$R_f(t) = \frac{\sigma(\text{ee} \to \gamma \to ff)}{\sigma(\text{ee} \to \gamma \to \mu\mu)}$$

Using the general results of p. 3 and p. 4:

$$\sigma(\text{ee} \to \gamma \to ff) = \frac{2\pi \alpha^2}{3s} \frac{Q_f^2}{s} \beta(3-\beta^2) \quad \beta = \sqrt{1-4m^2/s}$$

$$R_f(t) = \frac{1}{2} Q_f^2 \beta(t)(3-\beta^2 \beta(t)) \quad \beta(t) = \sqrt{1-4m^2/t}$$
\[ \text{Re } \bar{\Pi}_f(s) = \frac{\alpha G_f^2}{6\pi} \int_0^\infty dt \frac{\beta(t)}{t(t-s)} (3-\beta(t)^2) \]

\[
S \gg 4m^2:
\]
\[ \text{Re } \bar{\Pi}_f(s) = -\frac{\alpha G_f^2}{3\pi} \left\{ \frac{\beta(3-\beta^2)}{2} \ln \left( \frac{1+\beta^2}{1-\beta^2} \right) - \frac{8}{3} + \beta^2 \right\} \]
\[ \beta = \sqrt{1-\frac{4m^2}{s}} \]
\[ \text{Im } \bar{\Pi}_f(s) = \text{Im } \bar{\Pi}_f(s) = \frac{\alpha}{8} \beta(3-\beta^2) \]

\[
S \gg 4m^2:
\]
\[ \text{Re } \bar{\Pi}_f(s) = -\frac{\alpha G_f^2}{3\pi} \left[ \ln \frac{S}{m^2} - \frac{5}{3} \right] \]

\[
S \leq 4m^2:
\]
\[ \text{Re } \bar{\Pi}_f(s) = \frac{\alpha G_f^2}{3\pi} \left\{ -\gamma(3+\gamma^2) \arctan \left( \frac{4}{\gamma} \right) + \frac{8}{3} + \gamma^2 \right\} \]
\[ \gamma = \sqrt{1-\frac{4m^2}{s}} - 1 \]
\[ \text{Im } \bar{\Pi}_f(s) = \text{Im } \bar{\Pi}_f(s) = 0 \]

\[
0 < S \ll 4m^2:
\]
\[ \text{Re } \bar{\Pi}_f(s) = \frac{\alpha G_f^2}{15\pi} \frac{s}{m^2} \]

An important consequence:

If \( m_f^2 >> s \), the contribution to \( \bar{\Pi}_f(s) \) goes away!

(\text{de-coupling theorem})
**Direct diagrammatic calculation of $\bar{\Pi}(s)$**

We can also calculate $\bar{\Pi}(s)$ from the Feynman diagrams:

\[ \Gamma(s) = -i \Sigma_Y(s) \]

A technical (but interesting?) exercise!

Implement Feynman rules for Fermions:

\[ -i \Sigma_Y(s) = (-1) \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left\{ i \frac{k^\mu + q^\mu + m}{(q+k)^2-m^2} \right\} \]

\[ \Sigma_Y = -i e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\text{Tr} \left( (q+k+m) \gamma^\nu (k+m) \gamma^\mu \right)}{(q+k)^2-m^2)(k^2-m^2)} \]

Work out the trace:

\[ \text{Tr} \left( (q+k+m) \gamma^\nu (k+m) \gamma^\mu \right) = 4 \left( q^\mu k^\nu + q^\nu k^\mu + 2 k^\mu k^\nu - (qk) g^\mu^\nu - (k^2-m^2) g^\mu^\nu \right) \]

We have to compute

\[ I(\varnothing) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2-m^2)(q+k)^2-m^2)} \]

with $\varnothing = 1, \ k^\mu$ or $k^\mu k^\nu$
Regularization

In the loop calculations we will meet divergencies. To handle them, use regularization:

\[
\text{(infinite quantity)} = \lim_{\text{some parameter} \to \text{some value}} \left( \frac{\text{finite, parameter-dependent quantity}}{\text{some parameter}} \right)
\]

Most popular nowadays:

\text{DIMENSIONAL REGULARIZATION}

\[
\text{(some parameter)} = D , \text{ the number of dimensions of space time}
\]

\[
\text{(some value)} = 4 \text{ of course!}
\]

The loop integration element:

\[
\int \frac{d^4k}{(2\pi)^4} \rightarrow \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D}
\]

"engineering dimension"

to keep the right power of (GeV)
in the cross section.

It is not the renormalization scale!

Dim. reg. is nice since it does not influence the gauge symmetry.

On the other hand, some technical problems:

\[
4\text{-dim } \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow D\text{-dim } \gamma^5 = ???
\]
Define standard integrals: **SCALAR INTEGRALS**

\[ \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2-m^2} \equiv \frac{i}{16\pi^2} A(m) \]

\[ \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-m^2)(q+k)^2-m^2)} \equiv \frac{i}{16\pi^2} B(q^2, m) \]

\[ C[k] [q+k] \]

The other integrals can be reduced to these:

\[ \int \frac{k^\mu}{[k][q+k]} = F_0 q^\mu : \text{find } F_0! \]

\[ F_0 q^2 = \int \frac{(k.q)}{[k][k+q]} = \int \frac{i}{2} \left( \frac{[k+q]-[k]-q^2}{[k+q][k]} \right) \]

\[ = \frac{i}{2} \left\{ \int \frac{1}{[k]} - \int \frac{1}{[q+k]} - q^2 \int \frac{1}{[q+k][k]} \right\} \]

\[ = \frac{i}{2} \left( A(m) - A(m) - q^2 B(q^2, m) \right) \Rightarrow F_0 = -\frac{i}{2} B(q^2, m) \]
Slightly more complicated:

\[ X^{\mu\nu} = \int \frac{k^{\mu}k^{\nu}}{[k][q+k]} = F_1 g^{\mu\nu} + F_2 q^\mu q^\nu \quad \text{said } F_1, F_2 ! \]

\[ X^\mu = \int \frac{k^2}{[k][q+k]} = \int \frac{[k] + m^2}{[k][q+k]} = A(m) + m^2 B(q^2, m) = D F_1 + q^2 F_2 \]

\[ X^{\mu\nu} q_0 = \int \frac{(p^2)_\nu}{[k][q+k]} = \frac{1}{2} \left\{ \int \frac{(p^2)_\nu}{[k]} - \int \frac{(p^2)_\nu}{[q+k]} - q^2 \int \frac{(q^2)_\nu}{[k][q+k]} \right\} \]

\[ = \frac{1}{2} \left\{ 0 - \int \frac{(q^2 - p^2)_\nu}{[k]} - q^2 \left( \frac{1}{2} q^2 B(q^2, m) \right) \right\} \]

\[ = \frac{1}{2} q^2 A(m) + \frac{1}{4} q^4 B(q^2, m) = q^2 F_1 + q^4 F_2 \]

Solve:

\[ \begin{align*}
&\left\{ \begin{array}{l}
A(m) + m^2 B(q^2, m) = D F_1 + q^2 F_2 \\
\frac{1}{2} A(m) + \frac{1}{4} q^2 B(q^2, m) = F_1 + q^2 F_2
\end{array} \right. \\
\Rightarrow & F_1 = \frac{1}{D-1} \left[ \frac{1}{2} A(m) + (m^2 - \frac{1}{4} q^2) B(q^2, m) \right] \\
& F_2 = ....
\]

Combining all terms:

\[ \Sigma g^{\mu\nu}(s) = g^{\mu\nu} \alpha \left\{ -\frac{2}{3} A(m) + \frac{1}{3} (s + 2m^2) B(s, m^2) \right\} + (g^{\mu\nu} \text{ terms}) \sim \text{drop!} \]
Calculation of $A(m)$ and $B(s,m²)$

$$\frac{i}{16\pi²} A(m) = \frac{\mu^{4-D}}{(2\pi)^D} \int d^Dk \frac{1}{k^2 - m^2}$$

For $B(s,m²)$ the Feynman trick:

$$\frac{1}{a \cdot b} = \int dx \frac{1}{(ax + b(1-x))^2}$$

$$\frac{i}{16\pi²} B(s,m²) = \frac{\mu^{4-D}}{(2\pi)^D} \int_0^1 dx \int d^Dk \frac{1}{[k^2 + (s - x^2)q^2 - m^2]^2} \quad k \rightarrow k + xq$$

A complication:

$$k^2 = (k^0)^2 - k^2; \quad \text{not positive definite}$$

⇒ rewrite in Euclidean coordinates:

$$k^0 \equiv \sqrt{k^0} \quad \{ \quad d^Dk = i d^Dk_E$$

$$k \equiv k_E \quad \{ \quad \quad k^2 = - k_E^2 = - (k_E^0 + k_E^2)$$

$$\Rightarrow \left\{ \begin{align*}
A(m) &= -16\pi² \cdot \frac{\mu^{4-D}}{(2\pi)^D} \int d^Dk_E \frac{1}{k_E^2 + m^2} \\
B(s,m²) &= -16\pi² \cdot \frac{\mu^{4-D}}{(2\pi)^D} \int_0^1 dx \int \frac{1}{[k_E^2 + m^2 + (s-x^2)q^2]^2}
\end{align*} \right.$$
\[
\int \frac{d^D k_E}{(k_E^2 + Q)^n} = \int \frac{k_E^{D-1} dk_E d\Omega_D}{(k_E^2 + Q)^n} \\
= \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dk \frac{k^{D-1}}{(k^2 + Q)^n} = \frac{\pi^{D/2}}{\Gamma(D/2)} \frac{\Gamma(n-D/2)}{\Gamma(n)} Q^{D/2-n}
\]

Use \( n = 1 \) for \( A(m) \)
\( n = 2 \) for \( B(S, m^2) \)
and \( D = 4 - \varepsilon \), \( \varepsilon \to 0 \)

Hankel Relations
\[
\Gamma(-1 + \frac{\varepsilon}{2}) = -\frac{2}{\varepsilon} + \gamma_E - 1 + \ldots
\]
\[
\Gamma(\frac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma_E + \ldots \quad \text{Euler's constant} \quad \gamma_E = 0.577\ldots
\]

Dropping all terms that go to 0 as \( \varepsilon \to 0 \):

\[
A(m) = m^2 \left[ \frac{2}{\varepsilon} - \gamma_E + \ln 4\pi - \ln \frac{m^2}{\mu^2} + 1 \right] + \ldots
\]

\[
B(S, m^2) = \frac{2}{\varepsilon} - \gamma_E + \ln 4\pi - \ln \frac{m^2}{\mu^2}
- \int_0^1 dx \log \left[ \frac{S(x^2 - x) + m^2}{m^2} - i\varepsilon \right]
\]
defines Im part for values \( < 0 \)
Result of the diagram calculation

Contribution from fermion loop

\[ \text{Re } \Pi_f(s) = \Pi_f^{\infty} + \text{Re } \Pi_f(s) \]

\[ \Pi_f^{\infty} = \frac{\alpha}{3\pi} Q_f^2 \left[ \frac{2}{\pi} - \gamma_E + \ln(4\pi) - \ln \frac{m_f^2}{\mu^2} \right] \]

survives if \( s \to 0 \)

\[ \text{Re } \Pi_f(s) = -\frac{\alpha}{3\pi} Q_f^2 \left\{ \frac{\beta(\beta^2 - 3)}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{8}{3} + \beta^2 \right\} \quad \beta = \sqrt{1 - 4m^2/s} \]

for \( s > 4m^2 \)

\[ = -\frac{\alpha}{3\pi} Q_f^2 \left\{ \gamma(3 + \gamma^2) \arctan \left( \frac{\gamma}{3} \right) - \frac{8}{3} - \gamma^2 \right\} \quad \gamma = \sqrt{4m^2/s - 1} \]

for \( s \leq 4m^2 \)

vanishes if \( s \to 0 \)

This is precisely the result derived before!

- (of course!)

For unequal masses

- internal boson lines

- \( \geq 2 \)-point integrals

things become more complicated but the principles remain the same.
In order to be able to present the forthcoming self energy expressions as compact as possible, without losing any transparency, we are led to introduce a few more shorthand notations (only to be used in this appendix):

\[ z = M_Z, \quad w = M_W, \quad h = M_H, \quad \Delta_i = \Delta_{M_i}. \]  

(1.3)

where \( \Delta_{M_i} \) has been defined in appendix G. By decomposing the scalar 2-point function in appendix G according to

\[ B_{\phi}(s, M_1, M_2) = \frac{1}{2} (\Delta_1 + \Delta_2) + 1 - \frac{M_1^2 + M_2^2}{M_1^2 - M_2^2} \log \left( \frac{M_1}{M_2} \right) + F(s, M_1, M_2), \]

(1.4)

where the function \( F(s, M_1, M_2) \) is symmetric in the masses, the unrenormalized transverse gauge boson self energies \( \Sigma(s) \) occurring in the above expressions can be brought in the form: (the fermion summation also extends over the quark colours)

\[
\begin{align*}
\Sigma^1(s) &= \frac{\alpha}{4\pi} \left\{ \frac{4}{3} \sum_f \left[ s\Delta_f + (s + 2m_f^2) F(s, m_f, m_f) - \frac{5}{3} \right] \\
&\quad - \left[ 3s\Delta w + (3s + 4w^2) F(s, w, w) \right] \right\} \\
\Sigma^2(s) &= \frac{\alpha}{4\pi} \left\{ -\frac{4}{3} \sum_f \left[ s\Delta_f + (s + 2m_f^2) F(s, m_f, m_f) - \frac{5}{3} \right] \\
&\quad + \frac{1}{c_w s} \left[ \left( (3c_w^2 + \frac{1}{6}) s + 2w^2 \right) \Delta w \\
&\quad + \left( (3c_w^2 + \frac{1}{6}) s + (4c_w^2 + \frac{4}{3}) w^2 \right) F(s, w, w) + \frac{5}{3} \right] \right\}
\end{align*}
\]

(1.5)

\[ \Sigma^1(s) \in \phi \bar{\phi}, \quad \Sigma^2(s) \in w^+ w^- \]

from W. Beenakker  
\[ \sum^f(s) = \frac{a}{4\pi} \left[ \frac{4}{3} \sum_{q=m_{-1},-1}^{m_{+1},+1} 2a^2 \left( \Delta_{q} \frac{5}{3} - \log \left( \frac{z + i\epsilon}{m_{q}^2} \right) \right) \right] \{ \text{\( f^f \)}} \\
- \frac{4}{3} \sum_{q=m_{-1},-1}^{m_{+1},+1} \left[ \left( \eta^2 + a_\eta^2 \right) \left( s\Delta_{\eta} + (s + 2m_{\eta}^2)F(s,m_{\eta},m_{\eta}) - \frac{s}{3} \right) \right] \} \{ \text{\( f^f \)}} \\
+ \left[ \left( 3 - \frac{19}{6s_{\eta}^2} + \frac{1}{6s_{\eta}^2} \right) s + (4 + \frac{1}{c_\eta^2} - \frac{1}{s_{\eta}^2})s^2 \right] \Delta w \} \{ \text{\( w^+w^- \)}} \\
+ \left[ \left( -c_\eta^4 (80s + 80w^2) + \left( c_\eta^4 - s_\eta^4 \right)(8w^2 + s) + 12w^2 \right) F(s,w,w) \right] \\
+ \left( 10z^2 - 2h^2 + s + \left( \frac{h^2 - z^2}{s} \right)^2 \right) F(s,h,z) \} \{ \text{\( zH \)}} \\
- 2h^2 \log \left( \frac{h^2}{w^2} \right) - 2z^2 \log \left( \frac{z^2}{w^2} \right) \\
+ \left( 10z^2 - 2h^2 + s \right) \left( 1 - \frac{h^2}{h^2 - z^2} \log \left( \frac{h^2}{z^2} \right) - \log \left( \frac{h^2}{z^2} \right) \right) \\
+ \frac{2}{3} s \left( 1 + \left( c_\eta^4 - s_\eta^4 \right)^2 - 4c_\eta^4 \right) \frac{1}{12s_{\eta}^2 s_{\eta}^2} \} \{ \text{\( \bar{w}^{\bar{g}} \)}} \\
\sum^w(s) = \frac{a}{4\pi} \left[ \sum_{m_{-1},-1}^{m_{+1},+1} \left( \frac{3 - \frac{5}{2} m_{\pm}^2}{2s_{\pm}} - \frac{m_{\pm}^2}{2s_{\pm}} \right) F(s,0,m_{\pm}) + \frac{2}{3} s - \frac{1}{2} m_{\pm}^2 \right] \{ \text{\( \bar{w}^{\bar{g}} \)}} \\
+ \sum_{m_{-1},-1}^{m_{+1},+1} \left[ \frac{\Delta_{\pm}}{2} \left( s - \frac{5}{2} m_{\pm}^2 + \frac{1}{2} m_{\pm}^2 \right) \right] \left( \frac{1}{2} \left( s - \frac{5}{2} m_{\pm}^2 + \frac{1}{2} m_{\pm}^2 \right) \right) \left( \frac{1}{2} \left( s - \frac{5}{2} m_{\pm}^2 + \frac{1}{2} m_{\pm}^2 \right) \right) \} \{ \text{\( w^+w^- \)}} \\
- \left[ \frac{19}{2} s + 3w^2 \left( 1 - \frac{s^2}{c_\eta^2} \right) \right] \Delta w \} \{ \text{\( w^\gamma \)}} \\
+ \left[ 3c_\eta^4 - c_\eta^4 \left( 7z^2 + 7w^2 + 10s - 2 \frac{(z^2 - w^2)^2}{2s} \right) \right] \\
- \frac{1}{2} \left( z^2 + z^2 - \frac{5}{2} \left( z^2 - w^2 \right)^2 \right) F(s,z,w) \} \{ \text{\( w^\gamma \)}} \\
+ \frac{2}{s} \left( -4w^2 - 10a + \frac{2w^4}{s} \right) F(s,0,w) \} \{ \text{\( w^\gamma \)}} \\
+ \frac{1}{2} \left[ 5w^2 - h^2 + \frac{s}{2} + \left( \frac{h^2 - w^2}{2s} \right)^2 \right] F(s,h,w) \} \{ \text{\( wH \)}} \\
+ \left[ c_\eta^2 \left( 3z^2 + 11w^2 + 10s \right) - 3c_\eta^2 z^2 + \frac{1}{2} \left( 2w^2 - \frac{s}{2} \right) \right] \frac{z^2}{z^2 - w^2} \log \left( \frac{z^2}{w^2} \right) \\
- \frac{1}{2} \left( 2w^2 - \frac{s}{4} \right) \frac{h^2}{h^2 - w^2} \log \left( \frac{h^2}{w^2} \right) - c_\eta^2 \left( 7z^2 + 7w^2 + \frac{32}{3} s \right) \\
+ \left[ \frac{3}{2} z^2 + \frac{5}{3} s + 4w^2 - z^2 - h^2 \right] - s_\eta^2 \left( 4w^2 + \frac{32}{3} s \right) \} \{ \text{\( \gamma \)}}
\( M_1 = 0 \)

\[
F(s, 0, M_1) = 1 + \left[ \frac{M_1^2}{s} - 1 \right] \log \left( 1 - \frac{s + i \epsilon}{M_1^2} \right)
\]

(1.6)

\( s \ll M_1^2, M_2^2 \) and \( M_1 \neq M_2 \):

\[
F(s, M_1, M_2) \approx \frac{s}{(M_1^2 - M_2^2)^2} \left[ \frac{M_1^2 + M_2^2}{2} - \frac{M_1^2 M_2^2}{M_1^2 - M_2^2} \log \left( \frac{M_1^2}{M_2^2} \right) \right]
\]

(1.7)

\( s \ll M_1^2 = M_2^2 \):

\[
F(s, M_1, M_1) \approx \frac{s}{6M_1^4} \left[ 1 + \frac{s}{10M_1^2} \right]
\]

(1.8)

\( s \ll M_1^2 < M_2^2 \):

\[
F(s, M_1, M_2) \approx \frac{s}{2M_2^4}
\]

(1.9)

and hence \( F(0, M_1, M_2) = 0 \) provided \( M_1 \) and \( M_2 \) are not both equal to zero.

\( s \gg M_1^2, M_2^2 \):

\[
F(s, M_1, M_2) \approx 1 - \log \left( -\frac{s + i \epsilon}{M_1 M_2} \right) + \frac{M_1^2 + M_2^2}{M_1^2 - M_2^2} \log \left( \frac{M_1}{M_2} \right)
\]

(1.10)

\( M_1^2 < s < M_2^2 \):

\[
F(s, M_1, M_2) \approx \frac{s}{2M_2^4} \left[ 1 + \frac{s}{3M_2^2} \right]
\]

(1.11)

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What happened to $\theta_W$?

The weak mixing angle $\theta_W$ is not a fundamental parameter of the theory. At tree level we had:

$$1 - \frac{M_W^2}{M_Z^2} = s_w^2 = \frac{e^2}{8g_w^2}$$

Two consistent definitions of $s_w^2 = \sin^2\theta_W$.

After loop corrections these two definitions are no longer (necessarily) consistent: we have to take (at most) one of them (note that we can write everything without ever using $\sin^2\theta_W$ anyway!)

The most aesthetic choice: (Veltman, Sirlin, ...)

$$\sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}$$ before and after radiative corrections.

- Therefore we no longer have $s_w^2 = \frac{e^2}{8g_w^2}$ beyond tree level.
- "Wherever you see $\theta_W$, read $\arccos(\frac{M_W}{M_Z})$"
Alternative renormalization schemes

So far we have considered the on-shell scheme: the renormalized mass is defined to be the physical mass:

\[ \delta m_w^2 \equiv \text{Re } \Sigma_w(m_w^2) \]

So that

\[ \text{Re } \Sigma_w(m_w^2) = 0 \]

One might consider alternatives, for instance with

\[ \delta m_w^2 \equiv \text{the infinite part only of } \text{Re } \Sigma_w(m_w^2), \]

proportional to

\[ \frac{2}{\epsilon} - \gamma_E + \ln(4\pi) \]

so that

\[ \text{Re } \Sigma_w(m_w^2) \neq 0 \quad (\text{but of course } < \infty) \]

This is the \( \overline{\text{MS}} \) scheme (customary in QCD): now the pole in the propagator is not necessarily at the value of \( m_w \)

- The on-shell is conceptually simpler
- The \( \overline{\text{MS}} \) scheme is mathematically simpler
A fundamental radiative correction calculation:

\[ \Delta r \]

Remember the starting point of our discussion: \( \mu \) decays, leading to the introduction of the \( W \)

\[ \mu \longrightarrow e + \nu_e + \bar{\nu}_e \]

\[ \frac{\Gamma_\mu}{\Gamma_\mu} = \frac{G^2 m_\mu^5}{192 \pi^3} + O\left(\frac{m_e^2}{m_\mu^2}\right) \]

One can include QED corrections

\[ \Gamma_\mu = \frac{G^2 m_\mu^5}{192 \pi^3} \left[ 1 + \frac{\alpha}{2\pi} \left( \frac{25}{4} - \pi^2 \right) \right] \]

\[ = 0.0042 \]

- Use this formula to define \( G \)
- At tree level, \( G \) is "predicted" to be

\[ \frac{G}{\sqrt{2}} = \frac{e^2}{8 \sin^2 \theta_w m_w^2} \]

- What is the prediction including corrections?
The strategy of "calculating" $G/\sqrt{s}$

1. Calculate weak corrections to the amplitude

$$\nu_m \rightarrow e \rightarrow \nu_e$$

in terms of the bare parameters $e_0, m_0, m_0^2$

\[ \rightarrow \text{infinities in result} \]

2. Express the bare parameters $e_0, m_0, m_0^2$

in the physical ones $e, m_w, m_\tau$ and the counterterms $\delta e, \delta m_w, \delta m_\tau$

\[ \rightarrow \text{extra infinities} \]

3. Truncate to fixed (1st) order

\[ \rightarrow \text{infinities should drop} \]

4. Hopefully the resulting expression for the amplitude is analytically and numerically close to the tree-level one

N.B. Renormalizability guarantees that the radiative corrections are finite, but not that they are small !
Weak correction diagrams

N.B. The photonic ("GED") corrections are already included in the definition of $G$! Avoid double counting!

\[ \text{Born} \quad \text{Self-energy} \quad \text{Vertex, Box} \]

\[ \ell \nu = \ell z \nu + \ell \nu \]

\[ + \frac{\ell \ell \nu}{w} + \frac{\ell z \nu}{w} + \frac{\ell \nu}{w} \]

\[ = \frac{\ell z \nu}{w} + \frac{\ell \nu}{w} + \frac{\ell \nu}{w} \]

\[ \text{e} \nu = \frac{\text{e} \nu \nu}{w} + \frac{\text{e} \nu \nu}{w} + \frac{\text{e} \nu \nu}{w} + \frac{\text{e} \nu \nu}{w} \]

\[ \text{e} \nu \nu \nu \nu \]

\[ \leftarrow \text{this one was added already in the QED corrections} \]
The Born (tree level) relation was

\[ \frac{G}{\sqrt{2}} = \frac{e^2}{8 m_{W}^2 s_{W}^2} \quad s_{W}^2 = 1 - \frac{m_{W}^2}{m_{Z}^2} \]

After calculation of the loop corrections

\[ \frac{G}{\sqrt{2}} = \frac{e_0^2}{8 m_{W}^2 s_{W}^2} \left[ 1 + \frac{\Sigma_w(0)}{m_{W}^2} + \delta_{VB} \right] \quad s_{W}^2 = 1 - \frac{m_{W}^2}{m_{Z}^2} \]

1) Self-energy correction:

\[ \frac{-i}{s-m_{W}^2} \left( -i \Sigma_w(s) \right) \frac{-i}{s-m_{W}^2} \bigg|_{s=0} = \frac{-i}{s-m_{W}^2} \cdot \Sigma_w(0) \]

2) Vertex/box corrections:

\[ \delta_{VB} = \frac{\alpha}{\pi s_{w}^2} \left[ \Delta - \ln \frac{m_{W}^2}{\mu^2} \right] + \frac{\alpha}{4 \pi s_{w}^2} \left( 6 + \frac{7 - 3 s_{w}^2}{2 s_{w}^2} \ln c_{w}^2 \right) \]

Note the correspondence with the non-abelian charge counterterm:

\[ \frac{\alpha}{\pi s_{w}^2} \left[ \Delta - \ln \frac{m_{W}^2}{\mu^2} \right] = \frac{2}{c_{w} s_{w} m_{Z}^2} \Sigma_{w}(0) \]

\[ \Delta \equiv \frac{2}{\epsilon} - \gamma_{E} + \ln(4\pi) \]
Express the bare parameters in the physical ones:

\[ e_0^2 = (e + \delta e)^2 = e^2 \left(1 + \frac{2 \delta e}{\zeta} \right) = e^2 \left(1 + \Pi_y(0) - 2 \frac{S_w}{c_w} \frac{\Sigma_y(0)}{m^2_e} \right) \]

\[ \frac{1}{m^2_{w^2}} = \frac{1}{(m^2_w + 8m^2_e)} = \frac{1}{m^2_w + \text{Re} \Sigma_y(m^2_w)} = \frac{1}{m^2_w} \left(1 - \frac{\text{Re} \Sigma_y(m^2_w)}{m^2_w} \right) \]

\[ S_{w^2}^2 = 1 - \frac{m^2_w}{m^2_e} = 1 - \frac{m^2_w + \text{Re} \Sigma_y(m^2_w)}{m^2_w + \text{Re} \Sigma_e(m^2_e)} = 1 - \frac{m^2_w \left(1 + \frac{\text{Re} \Sigma_y(m^2_w)}{m^2_w} \right)}{m^2_e \left(1 + \frac{\text{Re} \Sigma_e(m^2_e)}{m^2_e} \right)} \]

\[ = 1 - c_w^2 \left[1 + \frac{\text{Re} \Sigma_y(m^2_w)}{m^2_w} - \frac{\text{Re} \Sigma_e(m^2_e)}{m^2_e} \right] \]

\[ = S_w^2 \left(1 - c_w^2 \left(\frac{\text{Re} \Sigma_y(m^2_w)}{m^2_w} - \frac{\text{Re} \Sigma_e(m^2_e)}{m^2_e} \right) \right) \]

Adding everything:

\[ \frac{G}{v^2} = \frac{e^2}{8 m^2_w S_w^2} \left(1 + \Delta r \right) \]

\[ \Delta r = \Pi_y(0) - 2 \frac{S_w}{c_w} \frac{\Sigma_y(0)}{m^2_e} + \frac{2}{c_w S_w} \frac{\Sigma_y(0)}{m^2_e} \]

\[ - \frac{c_w^2}{S_w^2} \left[ \frac{\text{Re} \Sigma_e(m^2_e)}{m^2_e} - \frac{\text{Re} \Sigma_y(m^2_w)}{m^2_w} \right] + \frac{\Sigma_y(0)}{m^2_w} - \frac{\text{Re} \Sigma_y(m^2_w)}{m^2_w} \]

\[ + \frac{\alpha}{4 \pi S_w^2} \left\{ 6 + \frac{7 - 4 s_w^2}{2 s_w^2} \ln c_w^2 \right\} \]

And this should be finite!
Using
\[ \Pi_y(0) = \text{Re} \, \Pi_y(m_e^2) - \text{Re} \, \Pi_y(m_e^2) \]
we can write
\[
\Delta r = -\text{Re} \, \Pi_y(m_e^2) \\
+ \text{Re} \, \Pi_y(m_e^2) + 2 \frac{C_w}{S_w} \frac{\Sigma_{Z}(0)}{m_e^2} \\
- \frac{C_w^2}{S_w^2} \left[ \frac{\text{Re} \, \Sigma_{Z}(m_e^2)}{m_e^2} - \frac{\text{Re} \, \Sigma_{W}(m_w^2)}{m_w^2} \right] \\
+ \frac{\Sigma_{W}(0)}{m_w^2} - \frac{\text{Re} \, \Sigma_{W}(m_w^2)}{m_w^2} \\
+ \frac{\alpha}{4\pi S_w^2} \left\{ 6 + \frac{7 - 45^2}{2S_w^2} \ln C_w^2 \right\}
\]

We can distinguish several contributions:

1) from light fermions (i.e. \( m \to 0 \) when possible)

2) from heavy fermions (\( 2m \gtrsim m_e \) etcetera)

3) from the Higgs

4) from the gauge bosons \( \text{not separately gauge-invariant} \)
The light fermion contribution

Consider an "up", "down" pair (\((u, d)\), \((u, d)\), ...) with \(m_{u, d} \ll m_w, m_Z\) and look at the significant terms in the \(\Delta \sigma\) contribution

\[
\frac{\Sigma_w(\infty)}{m_w^2} \sim O\left(\frac{m_Z^2}{m_w^2}\right) \quad \text{negligible in leading order}
\]

\[
\frac{\Sigma_Z(\infty)}{m_Z^2} \sim O\left(\frac{m_Z^2}{m_w^2}\right) \quad \text{idem}
\]

\[
\frac{\text{Re} \, \Sigma_E(m_Z^2)}{m_Z^2} \sim \frac{\alpha}{3\pi} \cdot \frac{v_u^2 + a_u^2 + v_d^2 + a_d^2}{e^2} \left[ \Delta - \ln \frac{m_Z^2}{\mu^2} + \frac{5}{3} \right]
\]

\[
\frac{\text{Re} \, \Sigma_W(m_Z^2)}{m_W^2} \sim \frac{\alpha}{3\pi} \cdot \frac{2g_w^2}{e^2} \left[ \Delta - \ln \frac{m_W^2}{\mu^2} + \frac{5}{3} \right] \quad \text{shift in scale from } m_W \text{ to } m_Z
\]

\[
\frac{\text{Re} \, \Sigma_Y(m_Z^2)}{m_Z^2} \sim \frac{\alpha}{3\pi} \cdot \frac{Q_u^2 + Q_d^2}{e^2} \left[ \Delta - \ln \frac{m_Z^2}{\mu^2} + \frac{5}{3} \right]
\]
Use the coupling constants:

\[ g_w = \frac{e}{s_w v_\beta} \quad Q_u = Q_d + e \]

\[ a_u = \frac{e}{4 s_w c_w} \quad \nu_u = a_u \left(1 - \frac{Q_u}{e} \cdot s_w^2\right) \]

\[ a_d = -\frac{e}{4 s_w c_w} \quad \nu_d = a_d \left(1 + \frac{Q_d}{e} \cdot s_w^2\right) \]

Then for the light fermion contribution:

\[ \Delta r^{(l.f.)} = -Re \bar{\pi}_y (m_e^2) \]

\[ + \frac{\alpha}{3\pi} \left( \Delta - \ln \frac{m_e^2}{\mu^2} + \frac{5}{3} \right) \left[ Q_u^2 + Q_d^2 - \frac{c_w^2}{s_w^2} \left( v_u^2 + q_u^2 + v_d^2 + q_d^2 - 2 g_w^2 \right) - 2 g_w^2 \right] \frac{1}{e^2} \]

\[ - \frac{\alpha}{3\pi} \cdot \frac{2 g_w^2}{e^2} \left( \frac{c_w^2}{s_w^2} - 1 \right) \ln c_w^2 \quad \text{cancels!} \]

\[ = -Re \bar{\pi}_y (m_e^2) - \frac{\alpha}{3\pi} \frac{c_w^2 - s_w^2}{4 s_w^4} \ln c_w^2 \]

\[ \approx 0.00055 \text{ per } (u,d) \text{ doublet} \]

\[ \frac{\alpha}{3\pi} N_c \left[ Q_u^2 \left( \ln \frac{m_e^2}{m_u^2} - \frac{5}{3} \right) \right. \]

\[ + Q_d^2 \left( \ln \frac{m_e^2}{m_d^2} - \frac{5}{3} \right) \]

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Contributions to $\text{Re} \Pi_y (m_b^2)$

(1) Leptons:

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$m_\ell$ (GeV)</th>
<th>Result (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>0.000511</td>
<td>1.74</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.106</td>
<td>0.92</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1.87</td>
<td>0.47</td>
</tr>
</tbody>
</table>

(2) Lightest quarks: $u, d, s$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$m_q$ (GeV)</th>
<th>Result (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>0.005</td>
<td>1.85</td>
</tr>
<tr>
<td>$d$</td>
<td>0.007</td>
<td>0.45</td>
</tr>
<tr>
<td>$s$</td>
<td>0.150</td>
<td>0.29</td>
</tr>
</tbody>
</table>

- $u$ 0.3 1.01
d 0.3 0.25
s 0.5 0.22

(3) Semi-heavy quarks: $c, b$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$m_q$ (GeV)</th>
<th>Result (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>1.5</td>
<td>0.68</td>
</tr>
<tr>
<td>$b$</td>
<td>5.0</td>
<td>0.11</td>
</tr>
</tbody>
</table>
Selected measurements of $R \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, where the annihilation in the numerator proceeds via one photon or via the Z°. Measurements in the vicinity of the Z° mass are shown in the following figure. The denominator is the calculated QED single-photon process. The horizontal extent of the plot symbols has no significance. The positions of the $J/\psi(1S), \psi(2S)$ and the four lowest $Y$ vector meson resonances are indicated. Two curves are overlaid for $E_{\text{cm}} > 11$ GeV, showing the theoretical prediction for $R$, including higher order QCD corrections. The values are for $5$ flavors in the $\overline{MS}$ scheme and are $A_{\text{QCD}} = 60$ MeV (lower curve) and $A_{\text{QCD}} = 250$ MeV (upper curve). References (including several references to data not appearing in the figure and some references to preliminary data):


Problematic:

- What are the quark masses?
  
  current masses: \( u + d + s \rightarrow 2.59 \% \)
  
  constituent masses: \( 1.48 \% \) an unacceptable difference!

- QCD corrections are expected to be large!

\[ \gamma \rightarrow q \bar{q} \rightarrow \text{QCD} \rightarrow \text{resonances: } \rho, \omega, \phi, \ldots \]

Solution: Back to the dispersion relation!

\[
\text{Re } \Pi_{\text{had}}(m_{\pi}^2) = \frac{\alpha m_{\pi}^2}{3\pi} \int ds \frac{R_{\text{had}}(s)}{s(s-m_{\pi}^2)}
\]

\( R_{\text{had}}(s) \) is built up out of:

1. hadronic resonances \( \Rightarrow \) integrate analytically
2. smooth background \( \Rightarrow \) integrate numerically
A careful integration: \( \text{(H. Burckhardt et al.)} \)

\[- \text{Re } \overline{\Pi}^{\text{had}}(s) = 0.00165 + 0.0030 \ln \frac{s}{\text{GeV}^2} \pm 0.0009 \quad s \geq m_\text{b}^2 \]
\[- \text{Re } \overline{\Pi}^{\text{had}}(t) = 0.008 \ln \frac{|t|}{\text{GeV}^2} \quad t \leq 0 \]

(1) At \( s = m_\text{b}^2 \):
\[- \text{Re } \overline{\Pi}^{\text{had}}(m_\text{b}^2) = 0.0288 \pm 0.0009 \]

The error comes from the exp. error in \( R_{\text{had}} \).

(2) Can be mimicked for \( s > m_\text{b}^2 \) by
\[- \text{Re } \overline{\Pi}^{\text{had}}(s) = \sum Q_i^2 \left[ \ln \frac{s}{m_i^2} - \frac{s}{3} \right] \left( 1 + \frac{\alpha_s}{\pi} \right) \]

with
\[ m_u = 0.053 \quad m_c = 1.5 \quad m_b = 4.5 \]
\[ m_d = 0.071 \quad m_s = 0.174 \]
\[ \alpha_s = 0.124 \]

but these values do not work for \(-\infty < s < (-20 \text{ GeV})^2\)

The total contribution from leptons and light quarks:
\[- \text{Re } \overline{\Pi}_\gamma(m_\text{b}^2) = -0.0602 \pm 0.0009 \]

...but don't forget the \( \ln c_\alpha^2 \) term!
Heavy fermion contribution

In the MSM there is one heavy fermion: **TOP** with unknown mass $m_t$. Unambiguous limits:

$$m_t > 45 \text{ GeV} \text{ from LEP (assuming "nothing")},$$

$$m_t > 89 \text{ GeV} \text{ from Tevatron (assuming MSM)}.$$  

Again, collect the relevant terms in the $(b,t)$ contribution to $\Delta \sigma$

(keep terms $- \ln m_b^2/m_t^2$ and $- m_t^2/m_b^2$)

$$\Pi^t_\gamma (m_t^2) = \frac{\alpha N_c}{3\pi} \left( \frac{Q^2}{e^2} \right)^2 \Delta t + \ldots$$

$$\Pi^b_\gamma (m_b^2) = \alpha N_c \left( \frac{Q^2}{e^2} \right)^2 \left[ \Delta b - \ln \frac{m_t^2}{m_b^2} + \frac{5}{2} \right] + \ldots$$

$$\frac{1}{m_t^2} \text{Re} \Sigma^t_\gamma (m_t^2) = \frac{\alpha N_c}{3\pi} \frac{1}{e^2} \left[ (v_t^2 + q_t^2 - 6 q_t^2 m_t^2) \Delta t + \ldots \right]$$

$$- \frac{1}{m_b^2} \text{Re} \Sigma^b_\gamma (m_b^2) = \frac{\alpha N_c}{3\pi} \frac{1}{e^2} \left[ (v_b^2 + q_b^2)(\Delta b - \ln \frac{m_t^2}{m_b^2} + \frac{5}{2}) + \ldots \right]$$

$$\frac{1}{m_t^2} \text{Re} \Sigma^t_w (m_t^2) = \frac{\alpha N_c}{3\pi} \frac{1}{4 e^2 s_w^2} \left[ \frac{\Delta t}{2} \left( 1 - \frac{5 m_t^2}{2 m_w^2} \right) + \frac{\Delta b}{2} \left( 1 - \frac{m_b^2}{2 m_w^2} \right) \right. \right. \right. \right. \right.$$  

$$- \left. \left. \left. \left. \left. \frac{m_t^2}{4 m_w^2} + \left( 1 - \frac{m_t^2}{2 m_w^2} \right)(1 - \ln \frac{m_t^2}{m_b^2}) \right) + \ldots \right] \right]$$

$$\frac{1}{m_b^2} \text{Re} \Sigma^b_w (m_b^2) = \frac{\alpha N_c}{3\pi} \frac{1}{4 e^2 s_w^2} \left[ \frac{\Delta b}{2} \left( - \frac{5 m_t^2}{2 m_w^2} \right) + \frac{\Delta b}{2} \left( - \frac{m_b^2}{2 m_w^2} \right) - \frac{m_t^2}{2 m_w^2} \right) \left( 1 - \ln \frac{m_t^2}{m_b^2} \right) + \ldots \right]$$

$$\frac{1}{m_t^2} \Sigma^t_{\gamma^2} (0) = 0$$

$$\frac{1}{m_b^2} \Sigma^b_{\gamma^2} (0) = 0$$

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Check on the part $\alpha \frac{2}{\pi^2} - \kappa_4 + \ln(4\pi)$:

$$\frac{\alpha N_c}{3\pi} \cdot \frac{1}{e^2} \left\{ \frac{Q_e^2 + Q_b^2}{\sin^2 \theta_W} \left[ \frac{v_t^2 + v_b^2 + v_t^2 + v_b^2}{v_t^2 + v_b^2} - 6 \frac{Q_t^2 m_t^2}{m_t^2} - \frac{1}{4 s_W^2} \left[ 1 - \frac{3}{2} \frac{m_t^2}{m_w^2} \right] \right] - \frac{1}{4 s_W^2} \right\} = 0 : \text{finite again!}$$

- The terms without $m_t$ cancel as before, due to $a_t = -a_b = \frac{e}{s_W c_W}$, $v_t = \frac{e}{s_W} (1 - \frac{Q_t^2}{e^2} s_W^2)$, $v_b = \frac{e}{s_W} (1 + \frac{Q_b^2}{e^2} s_W^2)$.

- To make the terms with $m_t^2$ cancel, we also need $m_w^2 = c_w m_z^2 \Rightarrow \text{nontrivial influence of the Higgs sector!}$

For $m_t \gg m_w$, the finite part is dominated by $m_t^2$:

$$\Delta r^{(t,b)} \sim - \frac{c_w^2}{s_w^2} \Delta \rho$$

$$\Delta \rho = \frac{\alpha N_c}{16 \pi s_w^2 c_w^2} \frac{m_t^2}{m_w^2}$$
Some remarks on $\Delta P$

1. We also have

$$\Delta P^t_b = \frac{\sum_{u} t_b(u)}{m^2_u} - \frac{\sum_{W} t_b}{m^2_W}$$

which is precisely the $(t,b)$ contribution to the corrections to the NC/CC ratio of $uN$ scattering.

2. \[ \lim_{m_t \to \infty} \Delta P = m_t^2 \to \infty \]

For $m_t \to \infty$ the rad. corr. diverges!

This is a reminder of the fact that the MSM without $t$ is not renormalizable!

Note: in $b\bar{b} \to W^+W^-$:

\[ \text{unitarity is destroyed if we remove this diagram} \Rightarrow \text{quadratic divergences} \]

This is nowadays called the Non-decoupling theorem although this is a crazy name!
3. \[ \lim_{m_H \to \infty} \Delta \rho = \ln \frac{m_H^2}{m_W^2} \]

This is a much milder divergence. Veltman screening (related to the fact that there is only one, neutral Higgs) "accidental SU(2)"

At 2-loop level, one indeed gets terms \( \propto m_H^2 \alpha^2 \)

4. \( \Delta \rho \) can be considered as a measure for the amount of isospin symmetry breaking

(significant \( \Delta \rho \Rightarrow m_W \) and \( m_Z \) renormalized in significantly different way

\( m_t \to \infty \Rightarrow \text{"no t quark" \Rightarrow b quark all alone!} \Rightarrow \Delta \rho = \infty \)

5. New physics can also contribute to \( \Delta \rho \)

("oblique corrections") for instance

1. new generations with large mass splittings
2. susy partners
3. technicolor, compositeness
4. non-minimal Higgs structure

Note: in this case we already have \( \rho \neq 1 \) at tree level: \( \rho \) is an additional free parameter here! \( \Rightarrow \) influence of large \( m_t \) much more complicated....
Summary on $\Delta r$ in $O(\alpha)$

In the minimal standard model

\[ \Delta r = \Delta \alpha - \frac{c_w^2}{s_w^2} \Delta \rho + \Delta r_{\text{rest}} \]

\[ \Delta \alpha = - \text{Re} \frac{\pi}{g} (m_t^2) \]

\[ \Delta \rho = \frac{\alpha N_c}{16 \pi s_w^2 c_w^2} \frac{m_b^2}{m_z^2} = 0.00255 \frac{m_t^2}{m_z^2} \]

\[ = 0.00255 \]

\[ \Delta r_{\text{rest}} = \Delta r_{\text{rest}}^{(t)} + \Delta r_{\text{rest}}^{(H)} + \Delta r_{\text{rest}}^{\text{rest}} \]

\[ \Delta r_{\text{rest}}^{(t)} = \frac{\alpha}{4 \pi s_w^2} \left( \frac{c_w^2}{s_w^2} - 1 \right) \ln \frac{m_t}{m_z^2} \]

\[ = 0.0028 \]

\[ \Delta r_{\text{rest}}^{(H)} = \frac{\alpha}{16 \pi s_w^2} \frac{11}{3} \left( \ln \frac{m_H^2}{m_w^2} - \frac{5}{6} \right) \]

\[ = 0.0018 \]

\[ \Delta r_{\text{rest}} = \text{very tiny} \]

\[ \Delta r = \Delta r(m_z, m_w, m_t, m_H, \text{known masses}, \alpha, d_s, \ldots) \]

But: there is a constraint!

\[ \text{use} \quad m_z = 91.16 \]

\[ m_w = 80.6 \]

\[ s_w^2 = 0.218 \]

\[ m_t = 137 \]

\[ m_H = 200 \]
Higher orders in $\Delta r$

In QCD:

$$\Delta r = \Delta x - \frac{C_w^2}{S_w} \Delta \phi + \Delta R_{\text{rest}}$$

entering in the relation for $G$ as

$$(1 + \Delta r) \rightarrow \frac{1}{1 - \Delta r}$$

How to include leading higher orders?

- The renormalization group tells us how to include large logarithms to all orders by

$$1 + \Delta x \rightarrow \frac{1}{1 - \Delta x}$$

- ... but the $m_t^2$ terms are not large logs!

An explicit calculation of the two-loop terms

$$(-\frac{t}{b} \frac{t}{b} \frac{t}{b} + \frac{t}{b} \frac{t}{b} + \frac{t}{b} \frac{t}{b} + \cdots)$$

indicates that

$$(1 + \Delta r) \rightarrow \frac{1}{1 - \Delta x} \cdot \frac{1}{1 + \frac{C_w^2}{S_w} \Delta \phi} + \Delta R_{\text{rest}}$$

$$\Delta \phi = N_c \frac{G m_t^2}{8 \pi^2 \sqrt{2}} \left[ 1 + \frac{G m_t^2}{8 \pi^2 \sqrt{2}} (1g - 2m^2) \right]$$

gives all large terms to second order at least.
Self-consistent solutions

\[ \Delta r = \Delta r(m_z, m_w, m_t, m_H) \]

\[ G = \frac{\pi \alpha}{2} \frac{\frac{i}{m_z^2 s_w^2 c_w^2} (1 + \Delta r)}{1 - \frac{m_z^2}{m_w^2}} \]

Therefore, \( m_w \) must satisfy the equation

\[ m_w^2 = \frac{m_z^2}{2} \left\{ 1 + \left[ 1 - 4 \left( \frac{\pi \alpha}{G \sqrt{2}} \right) \frac{1}{m_z^2} (1 + \Delta r(m_z, m_w, m_t, m_H)) \right]^{1/2} \right\} \]

Can easily be solved numerically

input parameters

\[ \alpha = \left( 137.0359895(61) \right)^{-1} \]
\[ G = 1.1663 \times 10^{-5} \text{ GeV}^{-2} \]
\[ M_Z = 91.16 \pm 0.03 \text{ GeV} \]

output parameters

\( m_w, \Delta r, s_w^2 \)
Result for $\Delta r$

$m_Z = 91.15$ GeV

$m_h = 100$ GeV

$\Delta r$ can be experimentally defined as

$$\Delta r = 1 - \frac{\pi \alpha}{\sqrt{2} G} \left( \frac{1}{m_W^2} \left(1 - \frac{m_W^2}{m_Z^2} \right)^2 \right)$$

$\Rightarrow$ measurements of $\alpha_s G, m_W$ and $m_Z$ tell you the value of $\Delta r(\alpha_s G, m_Z, m_W, m_H, m_t \ldots \cdot)$

$\Rightarrow$ obtain info on $m_t, m_H$!
Rough comparison with experiment

\[ m_\tau = 91.15 \pm 0.03 \text{ GeV} \]

\[ 24 \text{ GeV} < m_H < 1000 \text{ GeV} \]

\[ 50 \text{ GeV} < m_t < 250 \text{ GeV} \]

\[ \Delta r \]

\[ \begin{array}{c}
\text{1}\sigma \text{ bounds from} \\
\frac{m_W}{m_\tau} = 0.8829 \pm 0.0055
\end{array} \]

Any conclusions are up to you!
A note on the input parameters

The original Lagrangian (starting from the gauge symmetry principle) has parameters:

\[ g_U, g_\tilde{g}, \lambda, \mu^2, g_f \]

gauge couplings, Higgs potential, Yukawa couplings

Equivalent to the more physical set:

\[ e, m_W, m_Z, m_H, m_f \]

of these are known:

- \( e, m_f \) (except \( m_f \)) and \( m_Z \) "very precisely"
- \( m_W \) poorly
- \( m_t, m_H \) not at all

Actually we can trade \( m_W \) for \( g_\mu \):

\[ e, m_Z, G, m_H, m_f \]

which is the practical on-shell scheme.
The $Z^0$ propagator revisited

A slightly simplified treatment:

\[
\frac{1}{s - m_{Z^0}^2 + \text{Re} \Sigma_z(s)} = \frac{1}{s - m_e^2 + \text{Re} \Sigma_z(s) - \text{Re} \Sigma_e(m_e^2) + i \text{Im} \Sigma_z(s)}
\]

\[
\text{Re} \Sigma_z(s) - \text{Re} \Sigma_z(m_e^2) \propto \frac{s}{s - m_e^2} \left( \frac{\partial^2}{\partial s^2} \text{Re} \Sigma_z(s) \right) + \ldots
\]

\[
\text{Im} \Sigma_z(s) \propto s
\]

\[
\text{Im} \Sigma_e(m_e^2) = m_e^2 \Gamma_e^{(0)}
\]

\[
\Rightarrow \text{Im} \Sigma_z(s) = \frac{s}{m_e^2} \Gamma_e^{(0)}
\]

\[
= \frac{1}{(s - m_e^2)(1 + X) + i \frac{s}{m_e^2} \Gamma_e^{(0)}}
\]

\[
= \frac{1}{1 + X} \cdot \frac{1}{s - m_e^2 + i \frac{s}{m_e^2} \Gamma_e^{\text{phys}}}
\]

\[\text{X becomes finite when combined with vertices (or after wave function renormalization!)}\]

\[\pi \rightarrow \pi^0 = \frac{\text{Re} \Sigma_z(s) - \text{Re} \Sigma_e(m_e^2)}{s - m_e^2}
\]

\[
- \pi^0(0) + \frac{c_w - s_w^2}{s_w} \left( \frac{\text{Re} \Sigma_e(m_e^2) - \text{Re} \Sigma_e(m_w^2)}{m_e^2} - 2s_w \frac{\Sigma_Z(0)}{c_w m_e^2} \right)
\]

\[\Gamma_e^{\text{phys}} = \frac{\Gamma_e^{(0)} + \Gamma_e^{(1)}}{1 + \pi_e(m_e^2)} \quad \text{non-negligible higher orders!}\]
Corrections to the process $ee \to ff$ at LEP

Typical e.w. corrections are of order

$$\alpha \frac{\ln \frac{m_{\ell}^2}{m_f^2}}{\pi}$$

for light fermions

$$\alpha \frac{m_t^2}{m_\ell^2}$$

for heavy top quark

$$1$$

otherwise ($\frac{m_t^2}{m_b^2}, \ln \frac{m_W^2}{m_Z^2}, ...$)

easily comparable to the experimental accuracy!

(of course one of the main reasons behind LEP....)

The corrections fall into two "classes":

1. QED corrections (next lecture)

2. "purely weak" corrections (this lecture)

Remarks

(1) This split-up is gauge-invariant, but only unambiguous at 1 loop: \(\cdots\)

(2) not gauge invariant for charged-current processes (cf. $\mu$ decay!)

(3) not so easy e.g. for $ee \to W^+W^-$ (LEP 200!)

(4) similar results, but quantitatively different, for Bhabha scattering: $e^+e^- \to e^+e^-$
The tree level result

Two diagrams (for \( f = \mu, \tau, \text{quarks} \))

\[
\begin{align*}
A_\gamma^0 \equiv & \quad i \frac{1}{s} Q_e Q_f \\
\bar{e}(p_2) f(q_2) & \quad \times \left[ \bar{\nu}(p_1) \gamma^\mu u(p_2) \otimes \bar{u}(q_1) \gamma_\mu \nu(q_2) \right] \\
\end{align*}
\]

\[
A_\ell^0 \equiv i \frac{1}{s - m_\ell^2 + im_\ell^2} \\
\times \left[ \bar{\nu}(p_1)(\nu_e + a_\ell \gamma^5)\gamma^\mu u(p_2) \otimes \bar{u}(q_1)(\gamma + a_\ell \gamma^5) \gamma_\mu \nu(q_2) \right]
\]

Kinematics:

\[
\begin{align*}
p_1^+ & = \textbf{E}(1,0,0,1) & \text{e}^+ \\
p_2^- & = \textbf{E}(1,0,0,-1) & \text{e}^- \\
q_1^- & = \textbf{E}(1,-\sin\theta\sin\phi, -\sin\theta\cos\phi, -\cos\theta) & \mu^- \\
q_2^+ & = \textbf{E}(1,\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta) & \mu^+
\end{align*}
\]

Assume a degree of RH polarization \( \mathcal{P} \) for the \( \text{e}^+ \):

\[
\sum_{\text{spins}} \nu(p_1) \bar{\nu}(p_1) = \mathcal{P} (1 + P \gamma^5) \quad \sum_{\text{spins}} u(p_2) \bar{u}(p_2) = \mathcal{P}
\]
Very explicitly:

\[
<|M|^2> = \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{A_{yy}}{S^2} + \frac{A_{zz}}{(S-m_e^2)^2+\frac{m_e^2}{B_e^2}} + \frac{(S-m_e^2) A_y z}{S((S-m_e^2)^2+\frac{m_e^2}{B_e^2})}
\]

\[A_{yy} = S^2 Q_e^2 Q_f^2 (1+\cos^2\theta)\]

\[A_{zz} = S^2 \left[ (v_e^2+a_e^2)(v_f^2+a_f^2)(1+\cos^2\theta) + 8v_e v_f a_e a_f \cos\theta \right] + 2S^2 \left[ v_e a_e (v_f^2+a_f^2)(1+\cos^2\theta) + 2v_f (v_e^2+a_e^2) \cos\theta \right]
\]

\[A_{yz} = 2S^2 Q_e Q_f [v_e v_f (1+\cos^2\theta) + 2a_e a_f \cos\theta]
+ 2S^2 Q_e Q_f [a_e v_f (1+\cos^2\theta) + 2v_e a_f \cos\theta]
\]

\[\frac{d\sigma}{d\cos\theta} = \frac{1}{32 \pi S} <|M|^2> \]

\[
\sigma_{\text{tot}} = \int_{-1}^{1} \frac{d\sigma}{d\cos\theta} \ d\cos\theta \bigg|_{P=0}
\]

\[A_{FB} = \left[ \int_{0}^{1} \frac{d\sigma}{d\cos\theta} d\cos\theta - \int_{-1}^{0} \frac{d\sigma}{d\cos\theta} d\cos\theta \right] / \sigma_{\text{tot}}
\]

\[A_{LR} = \left[ \int_{P=1}^{1} \frac{d\sigma}{d\cos\theta} d\cos\theta - \int_{-1}^{P=-1} \frac{d\sigma}{d\cos\theta} d\cos\theta \right] / \sigma_{\text{tot}}
\]
\[ \sigma_{\text{tot}} = \frac{1}{12 \pi s} \left\{ \frac{s^2}{(s-m_e^2)^2 + m_e^2 l_2^2} \left( v_e^2 + a_e^2 \right) \left( v_f^2 + a_f^2 \right) \right\} \]
\[ + \frac{Q_e^2 Q_f^2}{s} \]
\[ + \frac{(s-m_e^2)}{(s-m_e^2)^2 + m_e^2 l_2^2} \left( 2 Q_e Q_f v_e v_f \right) \]
\[ \text{for } s < m_e^2 \text{ neglect the } \gamma \text{ channel:} \]
\[ A_{FB} = 3 \frac{v_e a_e v_f a_f}{(v_e^2 + a_e^2)(v_f^2 + a_f^2)} \equiv \frac{3}{4} A_e A_f \]
\[ A_{LR} = \frac{2 v_e a_e}{(v_e^2 + a_e^2)} \equiv A_e \]

other possibilities:
\[ A_{\text{pol}}^e = -A_\tau = A_{LR} \]
\[ A_{\text{pol}}^{FB} : \text{polarized forward-backward asymmetry} \]

Note: These, the usual expressions for the asymmetries, are only approximate, with \( m_\gamma = 0 \) and no photon. In actual practice one of course uses the full thing!
Unitarity notation for $\sigma_{\text{tot}}$

Analogous to the $W$ case: $Z \rightarrow ff$ decay

\[ \begin{align*}
\begin{array}{c}
Z \\
\rightarrow \end{array} \begin{array}{c} \ell_1 \\
\ell_2 \end{array} \\
M = i \varepsilon_{\mu} \bar{u}(p_1)(q_\ell + q_\ell' q_\ell'^* + q_\ell q_\ell'^*) u(p_2) \\
\frac{1}{3} \varepsilon_1 M_1^2 = \frac{4}{3} (\gamma_1^2 + Q^2) m_Z^2 \\
\end{array}
\end{align*} \]

\[ \Gamma(Z \rightarrow ff) = \frac{4}{12 \pi} (\gamma_1^2 + Q^2) m_Z^2 \Rightarrow (\gamma_1^2 + Q^2) = 12 \pi \frac{\Gamma_Z}{m_Z^2} \]

The total cross section (neglecting the $\gamma$) can therefore be written as

\[ \sigma_{\text{tot}}(s) = 12 \pi \left( \frac{\Gamma_{ee}}{m_Z^2} \right) \left( \frac{\Gamma_{ff}}{m_Z^2} \right) \frac{s}{(s-m_Z^2)^2+m_Z^2 \Gamma_Z^2} \]

At the peak:

\[ \sigma_{\text{tot}}(m_Z^2) = \frac{12 \pi (m_Z^2)}{m_Z^2} \left( \frac{\Gamma_{ee}}{m_Z^2} \right)\left( \frac{\Gamma_{ff}}{m_Z^2} \right) \]

\[ \uparrow \quad \uparrow \quad \text{branching ratio for final state} \]

\[ \text{branching ratio for initial state} \]

\[ \text{unitarity limit for } J=1 \text{ channel} \]

- At the peak the cross section is essentially the maximum that is physically possible

- \[ \Rightarrow \] At the peak, higher order effects will be either very small or negative! (cf. $t\bar{b} - Z$ interference)
Applying radiative (weak) corrections in $e^e \rightarrow ff$

The usual strategy:

1. Calculate the one-loop corrections using the bare parameters
   - self-energies
   - vertices
   - boxes
   - no virtual $\gamma$ loops!

2. Express the bare parameters in the physical ones, using the counterterms

3. Truncate to desired (1st) order

4. Publish the (now finite) result.
A note on the box diagrams

The box diagrams are UV finite!

For large $k^+ \alpha$ the loop integral is like

\[ \int d^4 k \: \frac{k}{k^2} \frac{1}{k^2} \frac{k}{k^2} \frac{1}{k^2} \sim O(\int d^4 k \: \frac{1}{k^6}) \: \text{o.k.}! \]

\[ \Rightarrow \text{boxes do not enter in our renormalization considerations} \]

\[ \text{in fact, at the resonance the box diagrams give very small contributions} \]

**BUT**

This only holds if indeed the $W$ propagator goes like $\frac{1}{k^2}$: gauge dependent!

**Feynman gauge:**

\[ \frac{W}{k} \: \frac{m}{k} = -i \frac{g_{\mu\nu}}{k^2-m^2} \sim \frac{1}{k^2} : \text{o.k.} \]

**Unitary gauge:**

\[ \frac{W}{k} \: \frac{m}{k} = -i \frac{g_{\mu\nu} \: \frac{k^\mu k^\nu}{m^2}}{k^2-m^2} \sim \frac{k^\mu k^\nu}{k^2} : \text{not o.k.} \]

In the unitary gauge the boxes are **QUADRATICALLY DIVERGENT** !??!
The optical theorem again!

Consider $e^+e^- \rightarrow W^+W^-$: we needed 3 diagrams:

1. \( e \leftarrow W \) (1)
2. \( W \leftarrow \gamma \) (2)
3. \( W \leftarrow Z \) (3)

Unitary gauge longitudinal $W$'s

If we take only the first one, the cross section diverges!

Now remember that the cross section is related to the (imaginary part of) the higher-order diagram:

\[ (1^2) \quad \text{the box diagram!} \]

\[ (2^2) \quad \gamma \text{ self energy} \]

\[ (3^2) \quad Z \text{ self energy} \]

\[ 2 \times 3 \quad (\gamma W)(\gamma W) \star \leftrightarrow \gamma Z \quad \gamma Z \text{ mixing} \]

\[ 1 \times 2 \quad (\gamma Z)(\gamma W) \star \leftrightarrow \gamma \text{ vertex correction} \]

\[ 1 \times 3 \quad (\gamma W)(\gamma W) \star \leftrightarrow Z \text{ vertex correction} \]

- Only the complete set is gauge-invariant!
- In unitary gauge, quadratic divergences from the box are cancelled by self-energies/vertices \( \Rightarrow \) no need for additional renormalization
- In the Feynman gauge the separate contributions are finite (but \( \exists \) extra ghost diagrams)
A reflection on the self energies

Apparently the $W^+W^-$ loop contributions to the $\gamma$ self energy are not gauge invariant.

BUT

What about the Dyson summation?

We know how to resum 2-point diagrams

but not how to do this for 3-point/4-point diagrams!

Solution adopted in practice:

1) Resum only fermionic loops but keep the bosonic ones only to $O(\alpha)$

OR

2) Don't worry: since in the Feynman gauge the $W^+W^-$ loops are small anyway, resumming them or not is practically irrelevant.
Dressed amplitudes

If we can neglect the boxes the corrected amplitudes can be written in a form much like the Born one for the photon:

\[ i \frac{Q_e Q_f}{s} \frac{1}{1 + \Pi_y(s)} \cdot \bar{v}(p_1) \left[ (1 + F_{V}^{Ye}) + F_A^{Ye} \gamma^5 \right] \gamma^\mu u(p_2) \]

\[ \cdot \bar{u}(q_1) \left[ (1 + F_{V}^{Yf}) + F_A^{Yf} \gamma^5 \right] \gamma_\mu v(q_2) \]

- \( \Pi_y(s) \): Renormalised vacuum polarization \( \approx 6 \% \)
- \( F_{V,A}^{Yf}(s) \): "form factors" \( = 0 \) for \( s = 0 \)
  - \( \text{Re} \, F(m_e^2) \approx 10^{-3} \)
  - \( \text{Im} \, F(m_e^2) \) even smaller

The renormalization group tells us that we can sensibly use a running QED coupling

\[ \alpha^2(s) = \frac{e^2}{1 + \Pi_y(s)} > e^2 \text{ since } \Pi < 0 \]
For the $Z$:

$$A_2 = i \frac{\Gamma}{s-m_e^2+i \frac{s}{m_e^2} \Gamma} \sum_q \overline{u}(q_1)(\gamma^\mu \gamma^5) \gamma^\mu u(q_2)$$

$$\hat{a}_f^2 = \sqrt{2} G m_e^2 \hat{s}_f \quad \text{(was } \sqrt{2} G m_e^2 \text{)}$$

$$\frac{\hat{v}_f^2}{\hat{a}_f^2} = 1 - \left| \frac{Q_f}{e} \right|^2 \cdot 4 G^2 \hat{s}_f$$

Again $s_f$ and $k_f$ are form factors.

They contain universal ($f$-independent) and non-universal ($f$-dependent) contributions:

$$s_f = 1 + \Delta s_u + \Delta s_{Nu}^f$$

$$k_f = 1 + \Delta k_u + \Delta k_{Nu}^f$$
More explicitly:

\[ \Delta p_{\nu} = -\Delta r - \Pi^2(s) \]

\[ \Delta p_{N\mu}^f = \frac{1}{\alpha_f} F_A^{2f}(s) \]

\[ \Delta s_{\nu} = -\frac{c_w}{s_w} \Pi^2(s) \]

\[ \Delta s_{N\mu}^f = -\frac{1}{2s_w^2 \alpha_f} \left[ F_V^{2f}(s) - \frac{\nu_f}{\alpha_f} F_A^{2f}(s) \right] \]

Usually the non-universal parts are small as before:

\[ \Pi^2(s) = \frac{\text{Re} E_2(s) - \text{Re} E_2(m^2_{\tau})}{s - m^2_{\tau}} - \Pi^2(0) \]

\[ + \frac{c_w^2 - s_w^2}{s_w^2} \left[ \frac{\text{Re} E_2(m^2_{\tau})}{m^2_{\tau}} - \frac{\text{Re} E_2(m^2_{\tau})}{m^2_{\tau}} - 2 \frac{s_w}{c_w} \frac{\Sigma^2(0)}{m^2_{\tau}} \right] \]

\[ \Pi^2(s) = \frac{\Sigma^2(0) - \Sigma^2(0)}{s} - \frac{c_w}{s_w} \left( \frac{\text{Re} E_2(m^2_{\tau})}{m^2_{\tau}} - \frac{\text{Re} E_2(m^2_{\tau})}{m^2_{\tau}} \right) + \frac{\Sigma^2(0)}{m^2_{\tau}} \]

These are both finite.
Leading behaviour of $\Delta p_u$ and $\Delta x_u$

The dominant behaviour of the universal parts:

$$\Delta p_u = \Delta p + \ldots$$
$$\Delta x_u = \frac{c_w^2}{s_w^2} \Delta p + \ldots$$

And one can see why from the counterterms:

$$Q_f^2 = \frac{e^2}{16 s_w^2 c_w^2}$$

at Born

$$= \frac{e_0^2}{16 s_w^2 c_w^2} = \frac{e^2}{16 s_w^2 c_w^2} \left[ 1 + 2 \frac{\Delta e}{e} - \frac{c_w^2 - s_w^2}{s_w^2} \left( \frac{\text{Re} \Phi_0(m_E^2)}{m_E^2} - \frac{\text{Re} \Phi_2(m_W^2)}{m_W^2} \right) \right]$$

$$= v_e G m_e^2 \left[ 1 + \left( \frac{\text{Re} \Phi_0(m_E^2)}{m_E^2} - \frac{\text{Re} \Phi_2(m_W^2)}{m_W^2} \right) + \ldots \right]$$

as far as the $m_f^2$-terms are concerned.

$$s_w^2 \to s_w^0 = s_w^2 \left[ 1 + \frac{c_w^2}{s_w^2} \left( \frac{\text{Re} \Phi_0(m_E^2)}{m_E^2} - \frac{\text{Re} \Phi_2(m_W^2)}{m_W^2} \right) \right] + \ldots$$
Effective $\sin^2 \theta_W$: \( \overline{S_W} \)

After expressing the $a_\epsilon, a_\gamma$ in $G, m_t^2$, the only place where $\sin^2 \theta_W$ that appears is in $V_f/a_f$: corrected $S_W \rightarrow S_W + c_\omega \Delta \rho$

This can be understood from the diagrams:

- $\overline{S_W}$ vs. $S_f$: \( \Delta \rho \) at tree level.
- $S_f$ overall correction $\Delta \rho$, but no change in $V_f/a_f$
- No $S_f$ diagram! \( \Rightarrow \) ratio is changed!

Apparently we have an effective mixing angle $\overline{S_W} \equiv S_W + c_\omega \Delta \rho$

- $S_W$ depends on $m_t$ \}
- $\overline{S_W}$ depends on $m_t$ \} but....
\( S_w^2 \) depends much less on \( m_t \) than \( S_w^2 \)!

⇒ The major \( m_t \) dependence in the \( Zf\bar{f} \) couplings is by way of \( S_\rho_{\gamma f} \sim 1 + \Delta \rho \)

![Graph showing the dependence of \( S_w^2 \) and \( S_w^2 \) on \( m_t \)]

\( m_H = 91.15 \)

\( 24 < m_H < 1000 \text{ GeV} \)

Also: if \( \rho_0 \neq 1 \) at tree level, \( \Delta \rho_0 = \rho_0 - 1 \) would act the same way everywhere!

Even if we measure \( \Delta \rho(0) \), we would not know the top mass.

Except in one case ....
\[ \mathcal{Z} \to b\bar{b} \text{ decays} \]

The top quark appears naturally in \( \mathcal{Z} \to b\bar{b} \): "universal"

\[ \begin{array}{c}
\text{\begin{figure}\begin{center}
\includegraphics[width=.5\textwidth]{diagram.png}
\end{center}\end{figure}}
\end{array} \]

"non-universal"

Remember that we argued non-renormalizability for \( m_t \to \infty \) because of unitarity violation in \( b\bar{b} \to WW \):

- Quadratic divergence \( \propto m_t^2 \Rightarrow \propto \Delta \rho \)

\[
\Delta \rho_{N.u.} = -\frac{4}{3} \Delta \rho - \frac{\alpha}{4\pi s_W^2} \left( \frac{8}{3} + \frac{1}{6c_W^2} \right) \ln \frac{m_t^2}{m_W^2}
\]

\[
\Delta \kappa_{N.u.} = -\frac{1}{2} \Delta \rho_{N.u.}
\]

This over-compensates the universal part!

Can we measure \( \Gamma(\mathcal{Z} \to b\bar{b}) \) to 5 MeV?
### $S_W^2$ values

$m_t = 91.15$ GeV

```
"universal"
```

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**Table 1:** Effective mixing angles on resonance for $M_H = 91.15$ GeV
Fighting about the "best" $\sin^2 \theta_W$

- We started by adopting
  \[ S^2_W = 1 - \frac{m^2_W}{m^2_Z} \]
  to all orders.
  This appears to be quite dependent on $m_t$ (or new physics).
  \[ \Delta R = \Delta \alpha - \frac{\Delta \alpha}{S^2_W} \Delta \rho \]
  not big!

- $ee\gamma\gamma$ physics turns out not to be so dependent on $m_t$ (or N.R.).
  \[ S^2_W, \rho = 1 + 1.2 \Delta \rho \]
  not big!

- Since $S^2_W$ is anyway only a bookkeeping device and not a fundamental parameter,
  why not change to another equivalent bookkeeping device that also is not too
  $m_t$ dependent?

- Good idea!
  \[ \bar{S}^2_W = S^2_W + c_{SW} \Delta \rho \]
  is such an effective thing, and
  it has much more to do with the ratio of couplings than with the ratio of masses!
  \[ \Rightarrow \text{back to the original introduction of } S^2_W \]
  in our derivation of the Lagrangian
A number of alternative $\hat{S}_w^2$ definitions exist

$$\hat{S}_w^2 \quad \text{"Hollik"}$$

$$S_w^* \quad \text{"Lynn"}$$

$$S_w^{**2} \quad \text{"Lynn"}$$

$$\hat{S}_w^2 (m_\tau^2) \quad \text{"Sirlin"}$$

$$\hat{S}_w^2 (m_W^2) \quad \text{"Sirlin"}$$

$$S_w^2 \quad \text{"Lynn"}$$

To avoid making enemies: this order is not chronological nor preferential.

All these alternatives incorporate the IDENTICAL leading terms ($\Delta p$).
None of them can incorporate all loop effects. The question of which alternative is better is (COMPLETELY) IRRELEVANT!

Since the left-over terms are truly small.
The improved Born approximation

Since we understand well the dominant leading corrections in $e^+e^-\rightarrow f$ we can take them into account as follows:

(1) Take the Born amplitudes

(2) Replace

\[
\begin{align*}
\epsilon^2 & \rightarrow \epsilon^2(s) \quad \text{in the photon graph } A_y^0, \\
S_W & \rightarrow S_W \quad \text{in the ratio } v_{es}/\alpha_{es}, \\
\alpha_{es} & \rightarrow \left[\sqrt{2}Gm_e^2s \right]^{1/2} \quad \text{in the } Z \text{ graph } A_Z^0, \\
\Gamma_2 & \rightarrow \frac{s}{m_Z^2}\Gamma_2 \quad \text{in the } \gamma \text{ propagator}
\end{align*}
\]

(3) Publish the result.

It is typically good to a few $\%$ around $m_Z$!
Improved Born approximations for $\hat{S}_{\text{tot}}(s)$

These typically read

$$\hat{S}(s) = \frac{12\pi}{m_e^2} \frac{S \Gamma e \Gamma f^2}{(s-m_e^2)^2 + \frac{S^2}{m_e^2} \Gamma^2} + \frac{4\Pi \alpha^2(s)}{3s} N_c (1 + \delta_{QCD})$$

+ (interference term)

Such formulae are VERY NICE because

- compact, transparent, easy to understand
- incorporate all large radiative corrections if you stick to the MSM
- contain $\Gamma e$, $\Gamma f$, $\Gamma$, $m_e$ as independent free parameters:
  - good for fitting
  - good to go beyond the standard model!

As a function of time, the weak corrections have become smaller!

Because we have learned how to write the Born expression
"Mass shift" due to weak corrections

Neglecting the \( \gamma \) channel:

\[
S(s) \propto \frac{s}{(s-m_{Z}^{2})^{2} + \frac{s^{2}}{m_{Z}^{2}} \Gamma_{Z}^{2}}
\]

The peak is no longer precisely at \( m_{Z} \)!

Two competing effects:

1. \( s \) in numerator pulls "mass" up:

\[
\Rightarrow \text{peak} = m_{Z}^{2} \rightarrow m_{Z}^{2} [1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}]^{-2} = m_{Z}^{2} + 17 \text{ MeV}
\]

2. \( s \)-dependent width pulls "mass" down:

\[
(s-m_{Z}^{2})^{2} + \frac{s^{2}}{m_{Z}^{2}} \Gamma_{Z}^{2} = s^{2}(1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}) - 2m_{Z}^{2}s + m_{Z}^{4}
\]

\[
= (1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}})[s - \frac{2m_{Z}^{2}}{1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}} \left( \frac{m_{Z}^{2}}{1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}} \right)^{2} + m_{Z}^{4} - \frac{m_{Z}^{4}}{1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}}
\]

\[
= (1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}})(s - \frac{m_{Z}^{2}}{1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}})^{2} + \frac{m_{Z}^{2} \Gamma_{Z}^{2}}{1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}}
\]

\[
\Rightarrow \text{peak} = m_{Z}^{2} \rightarrow m_{Z}^{2} [1 + \frac{\Gamma_{Z}^{2}}{m_{Z}^{2}}]^{-1} = m_{Z}^{2} - 35 \text{ MeV}
\]

Net effect: a change of \( -17 \text{ MeV} \)

(without \( s \)-dependent width would be \( +17 \text{ MeV}! \))
Other observables

The most important classes:

1. \[ A_f = \frac{2\nu^{-2}}{\nu^2 + \Delta_f^2} \quad A_L = A_e \]
   \[ A_R = A_e \quad A_{\text{pol}} = A_\tau \]
   \[ A_{FB} = \frac{3}{4} A_e A_f \]

   essentially only dependent on \( \sin^2 \theta \)

2. \[ \Gamma_f = \Gamma (e^+ \rightarrow f^+ f^-) = \frac{m^2}{12\pi} \left( \frac{2\nu^{-2}+\Delta_f^2}{\nu^2 + \Delta_f^2} \right) \left( 1 + \delta_{QCD} \right) \left( 1 + \delta_{QED} \right) \]

   \[ m_f = 0 : \]
   \[ \delta_{QED} = 1 + \frac{3}{4} \frac{\alpha}{\pi} \quad -1.0017 \]
   \[ \delta_{QCD} = 1 + \frac{\alpha_s}{\pi} \quad \alpha_s = 0.11 \pm 0.01 \]

   mainly dependent on \( \Delta_f \)

   \( (\Delta \Gamma_{\text{tot}}) \sim 12 \text{ MeV from the uncertainty in } \alpha_s \)
Total Z decay width

\[ \Gamma_Z \text{ (GeV)} \]

\[ m_t \text{ (GeV)} \rightarrow \]

- MSM prediction
- 1σ bounds of measured \( \Gamma_Z \)

Do you find a limit on \( m_t \)?
Strategies of mt searches at LEP1 (so far)

1. Measure $m_t \rightarrow \bar{s}_w^2$ as a function of $m_t$ for fixed $m_t$
2. Measure $\Gamma, A \rightarrow \bar{s}_w^2$ directly
3. Measure $m_w \rightarrow s_w^2$ directly $\Rightarrow$ $\bar{s}_w^2$ as a function of $m_t$

Find the overlap region!

Typical result (Dydykh, Singapore '90):
Towards $m_t$ and $m_H$

We can determine minimal-model allowed $(m_t, m_H)$ ranges.

\[ m_H = 200 \]

\[ m_t = 137 \pm 33 \pm 3 \pm 20 \text{ GeV} \]

\[ m_W = 80.15 \pm 0.25 \leftarrow \text{face up to LEP200!} \]
"Δρ- free physics"

Results that contain some Δρ can usually be fitted to the data by invoking large \( m_t \) or New Physics:

⇒ find some quantities in which Δρ is much suppressed or absent.

⇒ the Linear Combinations Game

- a poor man's way of doing a global fit
- leads to some understanding as well?

1. \[ R = \frac{\Gamma_{\text{had}}}{\Gamma_{\ell \ell}} \] quite independent of \( m_t \) (partly fundamental, partly coincidence)

2. both AFB and ALR are \( m_t \) dependent, but...
Fig. 13: $A_{pol}$ or $A_{LR}$ in the minimal model. $M_Z = 91.15$ GeV; $M_H = 25 (\cdots), 100 (- - -), 1000 (\cdots)$ GeV

Fig. 14: $\tau$-polarization versus forward-backward asymmetries for muons (-----), c-quarks (-----), b-quarks (-----)
QED corrections to $e^-e^+ \to ff$

So far we have only considered:
- $\gamma$ and $Z$ self-energy diagrams
- vertex corrections with $W$ or $Z$
- boxes with $W$ or $Z$
- fermion self-energies

Now: QED corrections

= all diagrams obtained by adding 1 photon to either the $\gamma$ ($A_\gamma$) or $Z$ ($A_Z$) graph.

Classes:

1) Fermionic self-energies

2) Vertex corrections

3) Box diagrams ($\gamma\gamma$ or $\gamma Z$)

4) Real photon emission:

\[\text{BREMSSTRALUNG}\]
General remarks on the QED corrections

1) Adding a virtual photon in all possible ways is gauge invariant
   ⇒ also separately renormalizable
   (a theory with only fermions + Z° + γ
    is renormalizable!)

2) If W exchange would be involved instead of Z
   exchange (as in ud → W+ν→ν, μν→μ+eν, etc.)
   we would also have to include
   \( W - \rightarrow W \)
   ⇒ affects renormalizability, etc.

3) The Bremsstrahlung diagrams describe a
   different final state: not ff̅ but ffγ

- virtual and real photon effects are
  connected by way of the

INFRARED PROBLEM
An introduction to Bremsstrahlung

(1) Single Bremsstrahlung phase space

\[ d\sigma = \frac{1}{2s} \frac{\langle 1M^2 \rangle d(LIPS)}{(2\pi)^3 n-4} \]

\[ d(LIPS) = d^{4q_1} \delta(q_1^2-m^2) \quad d^{4q_2} \delta(q_2^2-m^2) \quad d^4k \delta(k_2^2) \]
\[ \times \delta^4(p_1 + p_2 - q_1 - q_2 - k) \]

Several variable choices possible:

(1) \( k^0, \vec{Q}_k \) in lab frame, \( \vec{S}_{q_f}^* \) in \( ff \) CM frame:

(\( ff \) back-to-back !)

\[ d(LIPS) = d^4k \delta(k_2^2) \delta^4(p_1 + p_2 - Q) \]
\[ \quad \times d^{4q_1} \delta(q_1^2-m^2) \quad d^{4q_2} \delta(q_2^2-m^2) \quad \delta^4(Q-q_1-q_2) \]

\[ = \frac{1}{16} k_0 dk_0 d\Omega_k \cdot \beta_{q_f} d\vec{S}_{q_f}^* \]
\[ \quad \beta_{q_f} = \sqrt{1 - 4m^2/s} \]
\[ s' = Q^2 = (q_1 + q_2)^2 \]

(2) \( q_1^0, q_2^0, \vec{S}_q, \varphi_{q} (= \text{azimuthal angle of } \vec{q}_2 \text{ around } \vec{q}_1) \) in lab:

\[ d(LIPS) = \frac{1}{8} dq_1^0 dq_2^0 d\Omega_q d\varphi_{q} \]
(3) $k^0, \Omega_k, \Omega_1 \neq \Omega_2$ since no longer back-to-back in lab

$$\alpha(LIPS) = \frac{1}{16} \frac{q_i^2}{E(E-k^0)} \delta(k^0 \Omega_k \Omega_1), \quad E = \frac{1}{2} \sqrt{3}$$

$$q_i^o = \frac{2E(E-k^0)}{2E-k^0 + k^0 \cos \theta q_i, q_i, \Omega_k}$$

very simplified form! ($m=0$)

A dilemma?

- The simple phase space formulations have inadequate variables if you want to impose cuts
- The most "cut-friendly" formulation (3) has a complicated Jacobian and is still not good enough:

$$q_i^o(k^0, \Omega_k, \Omega_1) = \frac{2E(E-k^0)}{2E-k^0 + k^0 \cos \theta q_i, \Omega_k}$$

$$q_i^o = 2E - k^0 - q_i^o(k^0, \Omega_k, \Omega_1)$$

$$\cos \theta_2 = -\frac{k^0 \cos \theta_k + q_i^o(k^0, \Omega_k, \Omega_1)}{2E-k^0 - q_i^o(k^0, \Omega_k, \Omega_1)}$$

→ cuts on $q_i^o, \Omega_2$ are extremely difficult!

Simplification in limit $k^0 \to 0$:

- $\Omega_2^0$ (in $ff$ CM frame) $\to \Omega_2$ (in lab frame)

(4) $[\alpha(LIPS)] \sim \frac{1}{16} k^0 dk^0 d\Omega_1$

$k^0$ small
Multileg amplitudes with Soft bremsstrahlung

If we consider a process with > 1 external legs:

- \( e^+(p_1) e^-(p_2) \rightarrow f(q_1) \bar{f}(q_2) \)
  amplitude \( M_0 \)

- \( e^+(p_1) e^-(p_2) \rightarrow f(q_1) \bar{f}(q_2) \gamma(k) \quad k^0 \rightarrow 0 \)
  amplitude \( M_1 = M_0 \left[ -Q_{\pi^+} \frac{P_{\pi^+} \cdot E}{E_{\pi^+} - k} + Q_{\pi^-} \frac{P_{\pi^-} \cdot E}{E_{\pi^-} - k} - Q_{\gamma} \frac{P_{\gamma} \cdot E}{E_{\gamma} - k} + Q_{\pi^0} \frac{P_{\pi^0} \cdot E}{E_{\pi^0} - k} \right] \)

- Check current conservation:
  \[ M_1 \mid_{E^\mu \rightarrow k^\mu} = M_0 \left[ -Q_e Q_e - Q_f + Q_{\pi^-} \right] = 0 \]
  (in general, current conserved if total e.m. charge conserved)

- Bremsstrahlung from internal lines does not contribute! At least, not in the leading \( 1/k^0 \) terms. Example: in \( \mu \rightarrow \nu e^+ e^- \) diagram \( \gamma \) is not dominant.

- Form of soft-photon amplitude only depends on external charge flows, not on internal lines.

Physical picture: as \( k^0 \rightarrow 0 \), \( \lambda_{\text{photon}} \rightarrow \infty \). Long-wavelength photons can not resolve the hard scattering, but only the long-distance charge distribution.
The soft-photon cross section

Photon spin sum: \( \sum_{\text{spins}} e^{\mu} e^{\mu*} = -g^{\mu\nu} \Rightarrow \sum_{\mu} \left( \frac{p_i^\mu}{p_i \cdot k} \right) \left( \frac{q_j^\mu}{q_j \cdot k} \right) = -\frac{R_{\text{infra}}}{(k \cdot W_{\eta, k})} \)

1. \( d\sigma = \frac{1}{2s} \frac{1}{(2\pi)^5} \langle |M_L|^2 \rangle \ d(\text{LIPS}) \)

2. \( \langle |M_L|^2 \rangle \sim \langle |M_0|^2 \rangle \cdot e^2 R_{\text{infra}} \)

\[ R_{\text{infra}} \equiv \left[ \frac{s}{(p_1 \cdot k)(q_1 \cdot k)} - \frac{m_e^2}{(p_1 \cdot k)^2} - \frac{m_e^2}{(q_1 \cdot k)^2} \right. \\
+ \left. \frac{s}{(p_2 \cdot k)(q_2 \cdot k)} - \frac{m_f^2}{(p_2 \cdot k)^2} - \frac{m_f^2}{(q_2 \cdot k)^2} \right. \\
+ \frac{2p_1 \cdot q_1}{(p_1 \cdot k)(q_1 \cdot k)} - \frac{2p_1 \cdot q_2}{(p_2 \cdot k)(q_1 \cdot k)} - \frac{2p_2 \cdot q_1}{(p_1 \cdot k)(q_2 \cdot k)} + \frac{2p_2 \cdot q_2}{(p_2 \cdot k)(q_2 \cdot k)} \]

3. \( d(\text{LIPS}) \sim \frac{1}{s} k^o dk^o \cdot \frac{1}{8} d\Omega_f \)

1+2+3 \Rightarrow \frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega} \cdot \frac{\alpha}{4\pi^2} R_{\text{infra}} k^o dk^o d\Omega_k

Cross section factorizes in the soft-photon limit!
(2) **Soft-photon approximation**

Not only phase space but also matrix elements become simple in the limit \( k^0 \to 0 \).

Consider a process with an outgoing fermion:

\[
M_0 = \bar{u}(p) A(p) \xrightarrow{\text{rest of diagram}}
\]

Now attach Bremsstrahlung:

\[
M_1 = -Q \bar{u}(p) \frac{\gamma + m + k}{(p \cdot k)^2 - m^2} A(p+k)
\]

\[\xrightarrow{\text{neglect } k^0 \text{ where possible}}\]

\[\xrightarrow{\text{anticommutate and use Dirac eqn.}}\]

\[= -Q \frac{k \cdot p}{p \cdot k} \bar{u}(p) A(p)\]

\[= \left(-Q \frac{k \cdot p}{p \cdot k}\right) M_0\]

- The lowest-order amplitude \( X \) a simple factor!

- Amplitude scales as \( \frac{1}{k^0} \) for \( k^0 \to 0 \)
Integration of the soft-photon cross section

Concentrate on the terms \( \frac{S}{(p_{1k})(p_{2k})} - \frac{m_e^2}{(p_{1k})^2} - \frac{m_e^2}{(p_{2k})^2} \) in \( R_{\text{infra}} \)

write

\[
\begin{align*}
&\beta = \frac{p_{\perp k}}{E_k} = \sqrt{1 - \frac{m_e^2}{E_k^2}} \\
&c = \cos \theta (p_{\perp k}, k) \\
\end{align*}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma^0}{d\Omega} \cdot \alpha \frac{1}{\pi \Sigma} \left[ \frac{1}{1 - \beta^2 c^2} - \frac{m_e^2/s}{(1 - \beta c)^2} - \frac{m_e^2/s}{(1 + \beta c)^2} \right] \frac{1}{k^0} \frac{dk^0}{d\phi} d\alpha d\phi
\]

- Bremsstrahlung spectrum: \( \sim \frac{1}{k^0} \frac{dk^0}{d\phi} \)

- Tremendous angular peaks

\[
\frac{1}{\Sigma} \leq \frac{1}{1 + \beta^2 c^2} \leq \frac{\Sigma^2}{m_e^2} \sim 10^{-10}
\]

at LEP1

for \( k_{\perp} = p_{\perp 1}, p_{\perp 2}, \vec{p}_{\perp 1}, \vec{p}_{\perp 2} \)

- Integrated over \( \gamma \) angles:

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma^0}{d\Omega} \cdot \frac{2\alpha}{\pi} \left[ \ln \frac{S}{m_e^2} - 1 \right] \frac{1}{k^0} \frac{dk^0}{d\phi}
\]

"classical radiator factor" \( \sim 0.11 \)

at LEP1

- Total Bremsstrahlung cross section: simply integrate over \( k^0 \)!

BUT....
The infrared divergence

Soft bremsstrahlung spectrum: \(\int \frac{dk^0}{k^0}\)

- Upper limit provided by kinematics: \(k^0 \leq E(1 - \frac{4m^2}{s})\) \(\Rightarrow\) \(E\)
- Lower limit = 0!

* The total bremsstrahlung cross section is infinite

* The divergence comes from the region \(k^0 = 0\),
  i.e. zero-mass, zero-energy photons

* Are such photons real photons?

* A physically sensible answer can only be expected
  when we combine contributions from real
  with those from virtual photons

This is not renormalization!
No redefinition of parameters involved

Regularization: give the photon a small finite mass \(m_y\)

The spectrum integral becomes

\[\int_0^E \frac{dk^0}{k^0} \rightarrow \int_{m_y}^E \frac{dk^0}{k^0} - \ln \frac{E}{m_y}\]

(The real calculation is a bit more complicated
since if \(m_y \neq 0\), \(|k^0| \neq k^0\) in \((p, k), (p', k')\) \)]

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Some results for real and virtual photon corrections

After renormalization to get rid of the UV divergencies:

$$\delta_{\text{virt}} = \frac{2\alpha}{\pi} \left\{ \left( \ln \frac{S}{m_e^2} - 1 \right) \ln \frac{m_Y}{m_e} - \frac{1}{4} \ln^2 \frac{S}{m_e^2} + \frac{3}{4} \ln \frac{S}{m_e^2} + \frac{\pi^2}{3} - 1 \right\}$$

$$\delta_{\text{soft}} = \frac{2\alpha}{\pi} \left\{ \left( \ln \frac{S}{m_e^2} - 1 \right) \ln \frac{k_{\text{max}}}{m_Y} - \frac{1}{4} \ln^2 \frac{S}{m_e^2} + \frac{1}{2} \ln \frac{S}{m_e^2} - \frac{\pi^2}{6} \right\}$$

where \( k_{\text{max}} \ll E \)

is the upper bound on what you still want to call soft bremsstrahlung (note that the soft-photon approximation assumes \( k \ll E, q_1, q_2, \ldots \))

$$\delta^{\text{VS}} = \delta_{\text{virt}} + \delta_{\text{soft}} = \frac{2\alpha}{\pi} \left\{ \left( \ln \frac{S}{m_e^2} - 1 \right) \ln \frac{k_{\text{max}}}{E} + \frac{3}{4} \ln \frac{S}{m_e^2} + \frac{\pi^2}{6} - 1 \right\}$$

• IR infinities have cancelled!
• no terms with \( \ln^2(S/m_e^2) \) left
• if \( k_{\text{max}} \to 0 \) then \( \delta^{\text{VS}} \to -\infty \) again.
Additional Remarks on the IR cancellation

Up to now: initial-state radiation
Similar result for final-state radiation

\[ \delta^{\text{VS}} \text{final} = \frac{2\alpha}{\pi} \left( \frac{Q_F}{e} \right)^2 \left\{ \left( \ln \frac{S}{m_F^2} - 1 \right) \ln \frac{k_{\text{max}}}{E} + \frac{3}{4} \ln \frac{S}{m_F^2} + \frac{\pi^2}{6} - 1 \right\} \]

Also similar (but more complicated) for the interference between initial- and final-state radiation

Characteristic IR divergent terms:

Initial: \[ \frac{2\alpha}{\pi} \left( \ln \frac{S}{m_F^2} - 1 \right) \frac{dk^0}{k^0} \]

Final: \[ \frac{2\alpha}{\pi} \left( \ln \frac{S}{m_F^2} - 1 \right) \frac{dk^0}{k^0} \cdot \left( \frac{Q_F}{e} \right)^2 \]

Interference: \[ \frac{2\alpha}{\pi} \ln \left[ \frac{(p_1 - q_1)^2 (p_1 - q_2)^2}{(p_1 - q_2)^2 (p_1 - q_1)^2} \right] \frac{dk^0}{k^0} \cdot \left( -\frac{Q_F}{e} \right) \]

\[ 2 \ln \tan \frac{\theta}{2} \]

angle-indepedent

\[ s = (p_1 + p_2)^2 \]

\[ s = (q_1 + q_2)^2 \]

angle-dependent!

but not large
Understanding the IR cancellation from the optical theorem.

Lowest order diagram: \( \langle \ell \rangle \)

Lowest order cross section: \( \langle \ell \rangle ^2 \rightarrow \langle \ell \rangle \)

Add photonic (QED) corrections to this self-energy:

\[
\langle \ell \rangle + \langle \ell \rangle + \langle \ell \rangle
\]

This sum is UV finite (after renormalization) and also IR finite.

Cut again to see which diagrams give the cross section

\[
\langle \ell \rangle \leftrightarrow (\langle \ell \rangle)(\langle \ell \rangle)^*
\]

Only the sum is IR finite!

(similar for initial-state radiative and interference exercise)

\[ + \ldots \]
Hard photon effects

We have cut the soft photons off at $k^0 = k^{\text{max}} \ll E$
but this is either

- arbitrary ⇒ have to add a piece with $k^0 \geq k^{\text{max}}$
- not a good model of an experimental set-up

⇒ we also have to account for the cross section
from $k^0 > k^{\text{max}}$

This cross section is UV finite and IR finite but:

- matrix element becomes terrible! strong peaks curving through IR's
- phase space formulation becomes awful!
- experimental constraints become horrible!

The only ways of attacking this are:

1) use extremely simplified (or no) cuts
   and work (semi) analytically

2) use
   
   **MONTIE CARLO**
   
   ↓
   
   This is worth an academic
   training course by itself

Here, we have to be content with just a few
qualitative remarks.

For more details consult the "Z Physics at LEP1"
Yellow Books
Remarks on initial state radiation at LEP 1

In the soft-photon approximation, for initial state radiation:

\[
\frac{2}{\partial k^0} \sigma^*(s) = \frac{2\alpha}{\pi} \left( \ln \frac{s}{m_e^2} - 1 \right) \frac{1}{k^0} \sigma^0(s) \quad k^0 < k^{max} \ll E
\]

Including hard-photon effects:

\[
\frac{2}{\partial k^0} \sigma^*(s) = \frac{2\alpha}{\pi} \left( \ln \frac{s}{m_e^2} - 1 \right) \frac{1+(1-k^0/E)^2}{2k^0} \sigma^0 \left( \frac{s(1-k^0/E)}{E} \right)
\]

This can be understood diagrammatically:

\[Q^2 = (p_1 + p_2 - k)^2 = s(1-k^0/E)\]

The shift in energy \(s \rightarrow s(1-k^0/E)\) is important for resonant cross sections. Qualitatively:

1) at resonance: resonance "disappears" for \(s(1-k^0/E) \leq m_e^2 - \Gamma^2\)  
   \(\Rightarrow\) "natural" cutoff on \(k^0\) of order \(k^{max} \sim \Gamma^2\) 
   \(\Rightarrow\) \(\delta_{\text{initial}} \sim \frac{2\alpha}{\pi} \left( \ln \frac{s}{m_e^2} - 1 \right) \ln \frac{\Gamma^2}{m_e^2} \sim -30\) \(^2\)

2) above resonance: resonance "reappears" if \(k^0\) is such that \(s(1-k^0/E) = m_e^2\) 
   \(\Rightarrow\) large radiative tail!
The Z line shape

\[ \sigma(s) \]

\[ \sqrt{s} \text{ (GeV)} \]

\[ M_Z \]

0(\alpha) correction

Born or Improved Born

First-order correction \( \approx -30\% \) is huge!
Final-state radiation and interference

Final-state radiation:

The KLN theorem says that the total correction is not singular as $m_f \to 0 \Rightarrow$ no terms $\sim \ln s/m_f^2$!

$$\delta_{\text{final}} \sim \frac{3}{4} \frac{\alpha}{\pi} \sim 0.17 \%$$

If you have strict cuts, can have $\delta_{\text{final}} = \pm \text{few } \%$

Interference

At resonance, no cuts: very small! $\delta \sim 10^{-3}$

Physical: on resonance, the $Z$ is produced with a non-negligible life-time $\Rightarrow$ 'wavefunctions' for production (with initial-state rad) and decay (with final-state rad) have small overlap.

Away from the resonance, or with strict cuts, again have $\delta_{\text{interference}} = \pm \text{few } \%$ like at PETRA/PEP/Tristan
Higher order effects: exponentiation

- If the old correction is $-30\%$, we have to worry about higher orders !!!
- The correction is $-(many)\%$ because $k^{max}$ is small (about $\Gamma_3/m^2$): What if $\Gamma_3 \rightarrow 0$? $\delta \leq -100\%$ ???

Exponentiation (simplified form) $V =$ virtual + soft photon $H =$ hard photon

1. $O(x)$ corrected cross section
   $\sigma^V = \sigma_0 (1 + \beta \ln \Delta + \ldots)$
   $\sigma^H = \sigma_0 \beta \ln \frac{k}{\Delta} + \ldots$
   $\Rightarrow \sigma^{(1)} = \sigma_0 (1 + \beta \ln k + \ldots)$ non-negligible but finite terms

2. $O(x^2)$ corrected cross section
   $\sigma_{V_1 V_2} = \sigma_0 (1 + \beta \ln \Delta + \frac{1}{2} \beta^2 (\ln \Delta)^2 + \ldots)$
   $\sigma_{V_2 H_1 + V_2 H_1} = \sigma_0 \beta \ln \frac{k}{\Delta} (1 + \beta \ln \Delta + \ldots)$
   $\sigma_{H_1 H_2} = \frac{1}{2} \sigma_0 \beta^2 \ln^2 \frac{k}{\Delta} + \ldots$
   $\Rightarrow \sigma^{(2)} = \sigma_0 (1 + \beta \ln k + \frac{1}{2} \beta^2 \ln^2 k + \ldots)$

exercise: guess the $O(x^3)$ term
guess the $O(x^4)$ term
guess the $O(x^\infty)$ term
Dominant behaviour summed to all orders:

\[ \sigma^{(\infty)} = \sigma_0 \left[ \exp(\beta \ln k) + \ldots \right] \]

\[ = \sigma_0 (K^\beta + \ldots) \quad \text{"EXPORENTIATION" obviously} \]

1. The real treatment is more complicated (Yennie–Frautschi–Suura)

2. If \( k \to 0 \): \( K^\beta \to 0 \Rightarrow \sigma^{(\infty)} \to 0! \)
   "There is no scattering without radiation" (Bloch–Nordsieck)

3. Modified Bremsstrahlung spectrum

\[ \mathcal{E} \frac{d\sigma^{(\infty)}}{dk_0} = \left[ \frac{\partial}{\partial K} \sigma^{(\infty)} \right] \quad \text{\( \sim K = k_0/E \)} \]

\[ = \beta \frac{E}{k_0} \sigma^{(s)} \cdot \left( \frac{k_0}{E} \right)^\beta. \]

Regulating factor

Result from \( O(\omega) \)

Rule of thumb:

if \( \sigma^{(1)} = -A < 0 \), \( A \) not small

then \( \sigma^{(2)} \sim +\frac{3}{2} A^2 \)

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Structure functions for the line shape

Total initial-state corrected cross section to \( O(x) \):

\[
\sigma = \sigma_0 (1 + \delta^V) + \int_{k_{\text{max}}}^{k} \frac{1}{2k} \frac{(1-k)^2}{\beta} \sigma_0(s') \quad s' = s(1-k/E)
\]

Exponentiation takes virtual and hard photon effects together and allows to write

\[
\sigma = \int dk \ F(s,k) \sigma_0(s') \quad k \approx 1-\frac{s}{5} \text{ now!}
\]

Flux function

or, when keeping separate contributions for radiation from the \( e^+ \) and from the \( e^- \) (which is only possible in leading log level):

\[
\sigma = \int dx_1 F_{e^+}(x_1) \ F_{e^-}(x_2) \sigma(s,x_1,x_2) \quad x_i = \text{energy of } e^+ \text{ after radiation}
\]

Structure functions

This looks a lot like QCD (of course!)

historically: \( \overset{\text{QED}}{\downarrow} \overset{\text{QCD}}{\downarrow} \overset{\text{QED}}{\downarrow} \)

The flux function is known very precisely to \( \sim 0.1 \% \)
From the results of these calculations it becomes clear that certain terms can be resummed, and the division in \( \delta(1 - z) \) and \( \theta(1 - z - \varepsilon) \) terms is not necessary anymore.

A number of cases is listed below. The first case reads [22]

\[
F(z, k) = G_A(z) = \beta(1 - z)^{s - 1} \delta^{\psi + s} + \delta^H,
\]

with

\[
\beta = \frac{2\alpha}{\pi}(L - 1),
\]

\[
\delta^{\psi + s} = 1 + \delta^{\psi + s} + \delta^{\psi + s}_1,
\]

\[
\delta^H = \delta^H_1 + \delta^H_2,
\]

\[
\delta^{\psi + s}_1 = \frac{\alpha}{\pi} \left( \frac{3L}{2} + 2\zeta(2) - 2 \right),
\]

\[
\delta^{\psi + s}_2 = \left( \frac{\alpha}{\pi} \right)^2 \left[ \left( \frac{9}{8} - 2\zeta(2) \right)L^2 + \left( \frac{45}{16} + \frac{11}{2}\zeta(2) + 3\zeta(3) \right)L 
- \frac{6}{5}\zeta(2)^2 - \frac{9}{2}\zeta(3) - 6\zeta(3)\ln 2 + \frac{3}{8}\zeta(2) + \frac{19}{4} \right],
\]

\[
\delta^H_1 = \frac{\alpha}{\pi}(1 + z)(L - 1),
\]

\[
\delta^H_2 = \left( \frac{\alpha}{\pi} \right)^2 \left\{ X - (1 + z) \left[ 2\ln(1 - z)(L - 1)^2 
+ (L - 1) \left( \frac{3L}{2} + 2\zeta(2) - 2 \right) \right] \right\},
\]

\[
X = \left( \frac{1 + x^2}{1 - x} \right) \ln x + \frac{1}{2} \ln x + (1 + z) \frac{1}{2} \ln x + z - 1 \right) L^2
+ \left[ \frac{1 + x^2}{1 - x} \left( Li_2(1 - z) + \ln x \ln(1 - z) + \frac{7}{2} \ln x - \frac{1}{2} \ln^2 x \right) 
+ (1 + z) \frac{1}{4} \ln^2 x - \ln x + \frac{7}{2} - 3z \right] L
+ \frac{1 + x^2}{1 - x} \left( -\frac{1}{6} \ln^3 x + \frac{1}{2} \ln x \ln^2 1 - z \right) + \frac{1}{2} \ln^3 x \ln(1 - z) 
- \frac{3}{2} Li_2(1 - z) - \frac{3}{2} \ln x \ln(1 - z) + \zeta(2) \ln x - \frac{17}{6} \ln x - \ln^2 x 
+ (1 + z) \left( \frac{3}{2} Li_2(1 - z) - 2S_1(1 - z) - \ln(1 - z) Li_2(1 - z) - \frac{1}{2} \right) 
- \frac{1}{4}(1 - 5z) \ln^2(1 - z) + \frac{1}{2}(1 - 7z) \ln x \ln(1 - z) - \frac{25}{6} x Li_2(1 - z) 
+ (-1 + \frac{13}{3}) \zeta(2) + \frac{3}{2} \ln(1 - z) + \frac{1}{6}(1 + 10z) \ln x 
+ \frac{2}{1 - x^2} \ln^2 x - \frac{25}{11} \ln x \ln^2 x - \frac{2}{3} \ln x + \frac{1}{(1 - x^2) \ln^2 x} \right) 
\]

In these definitions the polylogarithms \( Li_n(x) \) and \( S_{n,\alpha}(x) \) have been introduced (cf.

refs. [28] and [29]) and the Riemann zeta function \( \zeta(2) = \pi^2/6 \) and \( \zeta(3) \approx 1.202 \).

The terms \( \delta^{\psi + s}_1, \delta^{\psi + s}_2 \) originate from first and second order virtual and soft photon corrections. Similarly \( \delta^H_1 \) and \( \delta^H_2 \) originate from single and double hard bremsstrahlung.
CONCLUSION

- The QED correction $\sim -28\%$ is huge!
- But: it is known VERY PRECISELY!
- We can unfold them to isolate the Improved Born!
- We can do precision measurements!
- We will find the top! the Higgs! New Physics!

⇒ Good Luck!