Hamiltonian Reduction of Non-Linear Waves

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ABSTRACT

The Faddeev-Jackiw Hamiltonian Reduction approach to constrained dynamics is applied to the collective coordinates analysis of non-linear waves, and compared with the alternative procedure known as symplectic formalism.

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1 Introduction

The analysis of the quantum significance of finite-energy solutions of non-linear classical field theories has deserved much attention in the last two decades [1, 2, 3, 4]. These solutions, which we shall call “non-linear waves” henceforth, cannot be obtained by starting from solutions of the linear part of the field equations and treating the non-linear terms perturbatively. For that reason, non-linear waves probe the non-perturbative regime of the quantum theory and are essential to our understanding of quantum dynamics beyond small perturbations of the vacuum. When non-linear waves (NLW) exist the spectrum of the quantum theory is divided into sectors, corresponding to excitations around the vacuum or around the NLWs [1]. Besides, the NLWs themselves acquire a particle-like status and are stabilized owing to the existence of topological conservation laws; these laws are associated to the boundary conditions imposed on the NLWs.

We shall consider a general boson field theory with fields $\Phi_p$, where $p = 1, \ldots, D$. Use will be made of a collective coordinates decomposition of the field $\Phi$ [2, 3, 4]. This procedure is the first step in some quantisation methods, but we shall not present a full quantisation of NLWs. What will be presented here is a general analysis of the phase space $\Gamma$ of small fluctuations around a NLW. The object of interest is the symplectic two-form $\Omega$ that determines the Lie algebra structure of the phase space. As in most mechanical systems, $\Gamma$ is a cotangent bundle and the symplectic form is exact, $\Omega = d\omega$. The 1-form $\omega$ is called “canonical 1-form” or “symplectic potential”. The output of this analysis is the Poisson brackets of a set of local coordinates in $\Gamma$.

Of course, much is already known about these matters. In most practically oriented approaches [2] the fluctuations about a NLW $\phi$ are restricted from the beginning to be orthogonal in field space to the zero modes of $\phi$. These zero modes span the tangent space of $\mathcal{M}$, the moduli space of NLWs, at a certain point. This restriction on the fluctuations (denoted by $\eta$) about the classical configuration $\phi$ is solved by means of a mode decomposition of $\eta$. The normal modes of $\eta$ are then taken as fundamental variables. Other approaches avoid the mode decomposition by taking the orthogonality of $\eta$ as a constraint and applying Dirac’s analysis of constrained dynamics [6, 7].

Our treatment will make use of Hamiltonian Reduction. This method was put forward by Faddeev and Jackiw [9, 10] as an alternative to Dirac’s analysis of constrained dynamics. As Dirac’s, this method is concerned with the classical phase space of field theories and does not address the problem of quantisation, especially the issue of ordering non-commuting operators. It spite of this is customary to refer to these classical analyses as “quantisation procedures”. An interesting variation on the Faddeev-Jackiw method, known as “symplectic quantisation” is due to Wotzasek, Montani and Barcelos-Neto [11, 12, 13]. By now it is clear that these new approaches to quantise classical systems have superseded Dirac’s, being both simpler and more fundamental. It is thus interesting to see how they work in NLW quantisation. There are differences between the original Faddeev-Jackiw method and symplectic quantisation, which will be illustrated in the main text. Although our emphasis is on the use of these new methods, some repetition of old results is unavoidable.

The main difference between the approach of [9, 10] and [11, 12, 13] is, roughly, that in the former we are asked to solve the constraints and reduce the phase space of the system
to the independent degrees of freedom, while in the latter constraints are no solved but embedded in an extended phase space, in such a way that the constraints are non-dynamical. The alternative between these two versions of Hamiltonian Reduction is reminiscent of the alternative between imposing the constraints before or after quantisation. In the first case we must solve the constraints and quantise the independent degrees of freedom, while in the second case constraints are ignored at first, but eventually we require physical states to be annihilated by them.

This article is organized as follows. A very general classical theory of scalar fields is presented in section 2, along with its Hamiltonian formulation. In section 3 we review the collective coordinates formalism; the total field $\Phi$ is decomposed into a classical part $\phi$ that should be a NLW (sometimes called the “barion”), and a quantum part $\eta$ (the “meson”). The classical part will depend on some parameters $\alpha^a$, $a = 1, \ldots, N$ that are the collective coordinates. The dynamical variables will be the collective coordinates of the NLW and $\eta$. The resulting dynamical system will be shown to be constrained. Section 4 deals with the Hamiltonian Reduction of this constrained system in the original Faddeev-Jackiw version; the constraints are solved by means of a formal mode decomposition of the meson $\eta$ and its canonical momentum. In Section 5 we attack the same problem from the point of view of symplectic quantisation [11, 12, 13], where the constraints are not solved but incorporated into the symplectic potential; the outcome of this analysis is the Poisson brackets of the system. The last section contains our conclusions.

2 Preliminaries

Let $\Phi(x, t)^p$, with $p = 1, \ldots, D$ be a set of $D$ classical scalar fields in a $1 + d$ dimensional manifold $M$, and consider the Lagrangean density

$$\mathcal{L} = \dot{\Phi}^\dagger K \dot{\Phi} - V(\Phi, \partial_i \Phi; \Phi^\dagger, \partial_i \Phi^\dagger)$$

where dot and $\partial_i$ mean derivative with respect to time $t$ and with respect to some spatial coordinates $x_i$, $i = 1, \ldots, d$ respectively. The dagger in $\Phi^\dagger$ denotes Hermitian conjugation. The object $K^p_q(\Phi)$ is in general a Hermitian operator that does not contain time derivatives of the fields $\Phi_p$. The potential $V$ contains the spatial derivatives of $\Phi$ and $\Phi^\dagger$, which we left unspecified (in particular we do not require Lorentz invariance). The only condition that $V$ must satisfy is that the classical equations of motion have NLW solutions.

A general NLW solution $\phi(x; \alpha^1, \ldots, \alpha^N)$ will depend on a number of real parameters $\alpha^a$ which reflect symmetries of the classical equations of motion. In a simple case like the kink in $\Phi^4$ theory the translational symmetry of the theory leads to static solution of the form $\phi(x - X)$, being $X$ the position of the center of mass of the kink. In general the parameters $\alpha^a$ are local coordinates on the moduli space $\mathcal{M}$ of static NLWs of the theory. The geometry of $\mathcal{M}$ will depend on the potential $V$; we shall not suppose that $\mathcal{M}$ has any particular structure other than being a smooth manifold with local derivatives denoted by $\partial_a$. The vectors $\partial_a \phi$ span $T(\mathcal{M})$, the tangent space of $\mathcal{M}$, at a given point. If $\phi$ is a static NLW, $\partial_a \phi$ also satisfies the static equations of motion; in other words, the vectors $\partial_a \phi$ are the zero modes of the full time-dependent equations of motion.

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We shall denote the integration of a function $f$ over a $d$-dimensional subspace of $M$ by $\int f$, omitting the measure in the integral. This subspace will be fixed and common to all integrations. If the theory under consideration is defined in Minkowski space, this $d$-dimensional subspace will be a maximal space-like submanifold of $M$.

The Hamiltonian density that corresponds to the original Lagrangean density (1) is readily calculated: the canonical momenta are $\Pi = \dot{\Phi}$ and $\Pi^\dagger = \dot{\Phi}$, and the Hamiltonian

$$H = \int \Pi \dot{\Phi} + \int \dot{\Phi}^\dagger \Pi^\dagger - L = \int \Pi K^{-1} \Pi^\dagger + V. \quad (2)$$

The fields $\Phi$ and $\Pi$ are assumed to satisfy the usual Poisson brackets:

$$\{\Pi(x,t), \Pi(y,t)\} = \{\Phi(x,t), \Phi(y,t)\} = 0$$

$$\{\Phi(x,t), \Pi(y,t)\} = \delta(x - y) \quad (3)$$

### 3 Collective Coordinates

Of all the existing procedures to study NLWs we shall be concerned with the collective coordinates method. In this method [2, 3], the semiclassical quantisation of a theory that possesses static NLW solutions starts with the decomposition of the classical field $\Phi(x,t)$ into two parts:

$$\Phi(x,t) = \phi[x; \alpha^a(t)] + \eta[x; \alpha^a(t)]. \quad (4)$$

Generally the field $\phi$ represents a classical solution and $\eta$ the quantum fluctuations about it. The choice of a particular classical solution $\phi$ will always break some of the symmetries of the theory. For example, if the original theory has translation symmetry, a given $\phi$ will have a center-of-mass position which breaks the symmetry (i.e. the theory built with $\phi$ as background will not enjoy translation symmetry). These broken symmetries do not altogether disappear from the theory; for each broken symmetry there will be a collective coordinate in the classical solution $\phi$.

The exact meaning of the fields $\phi$ and $\eta$ will depend on what problem we want to solve, and on what type of classical solutions we know. If we only have a static solution $\phi_0(x; \alpha)$ and we are interested in quantising the theory in the presence of such a classical static field configuration, then $\phi[x; \alpha^a(t)] = \phi_0(x; \alpha^a)$. In this case the $\alpha^a$ in $\phi$ and $\eta$ are time-independent, but we still allow for explicit time dependence in $\eta$, typically in the form of oscillatory exponentials $e^{i\omega_n t}$ where $\omega_n$ are the normal frequencies of the system. In other words, $\eta$ will represent small oscillations about the static background $\phi_0$.

If we are interested in quantising time-dependent field configurations but we do not know any time-dependent classical solution of the equations of motion we can still use $\phi_0$. In this situation $\phi[x; \alpha(t)]$ is the same function as $\phi_0(x, \alpha)$ but with time-dependent parameters $\alpha^a$. It is important to notice that $\phi[x; \alpha(t)]$ is not, in general, a time-dependent solution of the classical equations of motion. Its only relation to $\phi_0(x, \alpha)$ is that we have promoted the parameters $\alpha^a$ to dynamical, time-dependent variables without changing the functional structure of $\phi_0$. When we do this we must expect both quantum corrections, and kinematical corrections due to $\phi[x; \alpha(t)]$ not being a time-dependent classical field configuration.
For example, if our theory has Lorentz symmetry but we take a static solution with time-dependent collective coordinates as our $\phi[x;\alpha(t)]$ we will find corrections that represent a Lorentz contraction of the initial static solution [3].

Finally, if we have a classical time-dependent solution whose time dependence comes from some identifiable collective coordinates $\alpha^a(t)$, we use it as $\phi[x;\alpha(t)]$. No kinematical corrections should arise in this situation.

Whatever the case, we want $\alpha^a$, with $a = 1, \ldots, N$, and $\eta$ to be our new dynamical variables. It is to be noted that we are allowing $\eta$ to depend on the collective coordinates $\alpha^a$, and assuming at the same time that both $\eta$ and $\alpha^a$ are independent dynamical variables. The question if the $\alpha^a$ dependence of $\eta$ is redundant or not will be shown to be irrelevant to our investigation of the symplectic structure of the phase space of NLWs in the Faddeev-Jackiw approach. In the symplectic approach the final Poisson brackets do detect the dependence of $\eta$ on the collective coordinates; for this reason we shall keep this dependence throughout our analysis.

In the rest of this section we shall formulate the theory in terms of the new variables $\alpha, \eta$. Inserting the decomposition (4) into the lagrangean density (1) we find

$$L=\partial_a(\phi^\dagger + \eta^\dagger) \dot{\alpha}^a K \partial_b(\phi + \eta) \dot{\alpha}^b + \partial_t \eta^\dagger K \partial_t \eta$$
$$+ \partial_a(\phi^\dagger + \eta^\dagger) \dot{\alpha}^a K \partial_t \eta + \partial_t \eta^\dagger K \partial_a(\phi + \eta) \dot{\alpha}^a - V,$$

where $\partial_t = \partial/\partial t$. We are interested in the phase space of this new dynamical system; we must therefore go over to the hamiltonian formulation. To this end we define the canonical momenta

$$p_a = \frac{\partial L}{\partial \dot{\alpha}^a} = \int \left[ \partial_a(\phi^\dagger + \eta^\dagger) K \partial_b(\phi + \eta) \dot{\alpha}^b + \partial_a(\phi^\dagger + \eta^\dagger) K \partial_t \eta + \text{h.c.} \right]$$
$$\pi = \frac{\delta L}{\delta \dot{\eta}} = \left[ \partial_t \eta^\dagger + \partial_a(\phi^\dagger + \eta^\dagger) \dot{\alpha}^a \right] K.$$

It is important to observe that the canonical momenta $p_a$ and $\pi$ are not independent but related by

$$p_a - \int \pi \partial_a(\phi + \eta) - \int \partial_a(\phi^\dagger + \eta^\dagger) \pi^\dagger = 0,$$

In Dirac’s terminology the constraint (7) is a first-class constraint that generates the symmetry

$$\delta \phi = \epsilon^a \partial_a \phi,$$
$$\delta \eta = -\epsilon^a \partial_a \phi,$$

with $\epsilon^a$ an arbitrary $N$-dimensional parameter. This symmetry is obviously related to the invariance of the decomposition (4) under shifts of $\phi$ and $\eta$ that leave the total field $\Phi$ invariant. This indicates that the decomposition (4) does not have physical meaning; in particular, physically meaningful Green functions must be formed with the total field $\Phi$ [8].

Besides, the transformation from the initial phase space coordinates $(\Phi, \Pi)$ to the new variables $(\alpha^a, p_a, \eta, \pi)$ does not preserve the canonical 1-form $\omega = \omega_i d\xi^i$, (the $\xi^i$ are local
coordinates in phase space):

\[ \omega_i \dot{\xi^i} = \int (\Pi \dot{\Phi} + \text{h.c.}) = p_a \dot{\alpha^a} + \int (\pi \partial_t \eta + \text{h.c.}) \neq p_a \dot{\alpha^a} + \int (\pi \dot{\eta} + \text{h.c.}). \tag{9} \]

The problem is that \( \eta \) should appear under a total time derivative (because \( \omega \) is a one-form) instead of under a partial time derivative. Also, the non-canonical appearance of \( \omega \) implies that the transformation from \( \Phi, \Pi \) to \( \alpha^a, p_a, \eta, \pi \) is not a canonical transformation\(^1\). Besides, the naïve phase space \((p_a, \alpha^a; \pi, \eta)\) does not correspond to the true phase space of the theory, because of the constraint (7).

These facts were first realised by Tomboulis in the particular case of only one canonical coordinate corresponding to the center-of-mass of the NLW [6], and by Tomboulis and Woo in their general analysis of soliton quantisation [7]. These authors applied Dirac’s approach to constrained dynamics to the system (5) subject to the constraint (7). In the next sections we shall show that such analysis can be bypassed within the Faddeev-Jackiw procedure [9, 10] or its variant [11, 12, 13]. At the same time we shall generalise the results of [7] to include dependence of \( \eta \) on the collective coordinates \( \alpha^a \). Our analysis will also include moduli spaces of NLWs with transitive non-abelian groups of transformations, which are relevant to monopole quantisation [14], especially when the unbroken gauge group is non-abelian [15].

### 4 Hamiltonian Reduction

The starting point of this method is a first-order formulation of dynamical systems. The Legendre transformation

\[ L(q, \dot{q}) = \omega(q, p) - H(q, p) \tag{10} \]

may be understood as a first-order lagrangean in phase space, with the hamiltonian playing the role of a potential. We have denoted the symplectic potential by \( \omega \). The object of interest is the symplectic two-form, defined as the exterior differentiation of the symplectic potential: \( \Omega = d\omega \). At this point it is convenient to indicate that the operation \( d \) depends on the geometry of phase space. The general situation will now be described [17]. Let \( \gamma_1, \gamma_2 \in T(\Gamma) \) be two vector fields in \( \Gamma \). If we denote the natural pairing between a vector \( \gamma \) and a 1-form \( \omega \) by \( \langle \omega, \gamma \rangle \), and the acting of a two-form \( \Omega \) on two vectors by \( \Omega(\gamma_1, \gamma_2) \), we define the operation \( d \) acting on \( \omega \) as

\[ d\omega(\gamma_1, \gamma_2) = \gamma_1 \langle \omega, \gamma_2 \rangle - \gamma_2 \langle \omega, \gamma_1 \rangle - \langle \omega, [\gamma_1, \gamma_2] \rangle. \tag{11} \]

Particularly interesting examples of dynamical systems are those with a configuration space \( Q \) isomorphic to a connected Lie group \( G \). For these systems the vector fields \( \gamma \) restricted to the tangent space of \( Q \) can always be taken as the “push forward” of the Lie algebra of \( G \). In these cases the symplectic form can be written in components as

\[ \Omega_{ij} = \frac{\partial}{\partial \xi^i} \omega_j - \frac{\partial}{\partial \xi^j} \omega_i - F_{ik}^j \omega_k \tag{12} \]

\(^1\)This problem would not have appeared if \( \eta \) had been taken independent of the collective coordinates \( \alpha^a \).
where we have used the structure constants of the algebra of vector fields:

\[ [\gamma_i, \gamma_j] = F_{ij}^k \gamma_k. \] (13)

It should be pointed out that in most early analyses of the phase space of NLWs the last term in (12) was ignored. This restricted definition is valid if and only if the configuration space admits a transitive abelian group \( G \) of transformations. This not being the case in many systems where NLWs exist, we must keep the correct definition 12.

If the symplectic matrix \( \Omega_{ij} \) is non-singular its inverse, denoted by \( \Omega^{ij} \), will exist. In this situation there are no true constraints on the system, the Poisson brackets of two functions \( F(\xi), G(\xi) \) are given by

\[ \{F(\xi), G(\xi)\} = \partial_i F \Omega^{ij} \partial_j G \] (14)

and we have a complete hamiltonian description of the dynamics. When the symplectic matrix is singular the system is constrained. This is due to the existence of zero modes of the symplectic matrix. If we denote these zero modes by \( Z^n \), being \( n \) an index that enumerates the different zero modes, the hamiltonian equations of motion imply that

\[ (Z^n)^i \frac{\partial}{\partial Z^i} H = 0. \] (15)

These equations involve no time derivatives, and therefore correspond to the constraints of the problem. In the Faddeev-Jackiw approach \([9, 10]\) these constraints are simplified by means of a Darboux transformation from the original variables \( \xi^i \) to new ones \( p_k, q^l, z_m \). In the new variables the symplectic potential \( \omega \) takes the form \( p_i dq^i \), while the hamiltonian may also depend on the \( z_m \). Equation (15) will now correspond to the equations of motion for the variables \( z_m \)

\[ \frac{\partial}{\partial z_m} H(p, q, z) = 0, \] (16)

which can be used to evaluate the \( z \)'s in terms of the \( p \)'s and \( q \)'s unless \( H \) depends linearly on some \( z \)'s. If this elimination is performed one is left with an expression linear in the surviving \( z \) variables,

\[ L = p_i \dot{q}^i - H(p, q) - z_m h^m(p, q) \] (17)

and the only true constraints are \( h^m = 0 \). These constraints must be solved and the solution used to reduce the number of degrees of freedom in the lagrangean. If the new, reduced symplectic matrix is still singular the procedure starts again, until one finally arrives at an unconstrained lagrangean. We wish to point out that sometimes the system is given directly in the form (17). This happens, for example, in electrodynamics; Gauss' Law appears as a constraint in the lagrangean from the beginning.

After this brief review of the Faddeev-Jackiw treatment of constrained systems we turn back to our NLWs. We shall first consider the problem of the non conservation of the symplectic potential. We can write \( \omega \) as follows:

\[ \omega_i \dot{\xi}^i = p_a \dot{\alpha}^a + \int (\pi \partial_t \eta + \text{h.c.}) \]
\[ = \left[ p_a - \int (\pi \partial_a \eta + \text{h.c.}) \right] \dot{\alpha}^a + \int (\pi \dot{\eta} + \text{h.c.}) \]
\[ = \bar{p}_a \dot{\alpha}^a + \Re \int \pi \dot{\eta}, \] (18)
where we have defined a new momentum $\tilde{p}$ as

$$\tilde{p}_a = p_a - \int (\pi_a \partial_a \eta + \text{h.c.}).$$

(19)

In the new variables $\tilde{p}_a$, $\alpha^a$, $\pi$ and $\eta$, the symplectic potential $\omega$ is in canonical form. Therefore we should adopt these new variables as local coordinates in the phase space of our system. This is, of course, only an intermediate step, since we still have to deal with the constraint (7).

4.1 Solving the Constraints

Let us write the lagrangean in first-order form:

$$L = \tilde{p}_a \dot{\alpha}^a + \int (\pi \dot{\eta} + \text{h.c.}) - H$$

$$H = \int \pi K^{-1} \pi^\dagger + \int V.$$

The system (20) together with the constraint (7) is already in the form (17), so no Darboux transformation is needed to expose the constraint. The task is therefore to solve the constraint (7). The obvious way to proceed is to eliminate $p_a$ in favour of $\pi$ and the $\alpha$’s,

$$p_a = \int (\pi \partial_a (\phi + \eta) + \text{h.c.}),$$

(20)

but this leads to a canonical set of variables that undoes the decomposition (4), i.e. requires $\Phi$ and $\Pi$ as canonical variables. The reason for this is, roughly, that in the solution (20) $\phi$ and $\eta$ play a symmetric role, so that there is no way to separate them in the resulting lagrangean. The alternative is to eliminate $\pi$ in terms of $p_a$ and $\alpha^a$. It will be convenient to introduce the notation

$$\mu_{ab} = \int \partial_a \phi^\dagger K \partial_b \phi,$$

$$\xi_{ab} = \int \partial_a \phi^\dagger K \partial_b \eta.$$

(21)

The matrix $\mu_{ab}$ is hermitian, and we shall assume that it is also non-singular; the components of its inverse will be denoted by $\mu^{ab}$. Therefore there exists a local basis $\{f_a(x;\alpha) \mid a = 1,\ldots,N; \alpha \in \mathcal{M}\}$ of $T(\mathcal{M})$ that diagonalises $\mu_{ab}$ at each point $\alpha \in \mathcal{M}$. In that basis the vectors $\partial_a \phi$ can be taken proportional to the $f_a$:

$$\partial_a \phi = \|\partial_a \phi\| f_a, \quad a = 1,\ldots,N$$

$$\|\partial_a \phi\| = \int \partial_a \phi^\dagger \partial_a \phi$$

$$\mu_{ab} = \|\partial_a \phi\| \|\partial_b \phi\| \delta_{ab} \quad \text{(no sum)}.$$  

(22)

This fact indicates that the $N$ vectors $f_a$ are part of an orthonormal basis in the Hilbert space $\mathcal{H} = L^2(\mathcal{M})$ of (possibly complex) square-integrable functions over $\mathcal{M}$. This fact will
be used below. Let us decompose the momentum $\pi$ into “transverse” and “longitudinal”
components with respect to the directions $\partial_a \phi$:

$$\pi = \tilde{\pi} + \frac{1}{2} \left[ p_a - \int (\tilde{\pi} \partial_a \eta + \text{h.c.}) \right] (\mu + \xi)^{ab} \partial_a \phi^\dagger K, \quad \text{(23)}$$

where $\tilde{\pi}$ is transverse in the sense that

$$\psi_a \equiv \int \tilde{\pi} \partial_a \phi = 0. \quad \text{(24)}$$

The constraint is now $\psi_a = 0$. It is also important to notice that the transversality of $\tilde{\pi}$ is
relative to $N$ functions $\partial_a \phi$. This contrasts with the situation in electrodynamics, where the
electric field $E$ must also be decomposed in transversal and longitudinal components

$$E = E_T + E_L, \quad \text{(25)}$$

but the transversality of $E_T$ is defined as $\nabla \cdot E_T = 0$ without involving other fields.

Now we must solve the constraint (24). A way to proceed is to choose a base $\mathcal{B}$ of
orthonormal functions in the Hilbert space $H$ that should contain the $N$ vectors $f_a$. Let us
write the elements of this base as

$$\mathcal{B} = \{ f_a; f_i \} \quad a = 1, \ldots, N \quad i = N + 1, \ldots, \infty$$

$$(f_m, f_n) \equiv \int f_m^\dagger K f_n = \delta_{mn} \quad m, n = 1, \ldots, \infty \quad \text{(26)}$$

where $(\cdot, \cdot)$ is the inner product defined in $H$. Having introduced the basis $\mathcal{B}$, we solve the
constraint (24) by restricting $\tilde{\pi}$ to be in the subspace of $H$ that is orthogonal to the $f_a$,

$$\tilde{\pi}(x, t) = \sum_{n=N+1}^\infty c_n(t) f_n^\dagger K, \quad \text{(27)}$$

where the $f_i$ will depend on the collective coordinates $\alpha^a(t)$ while the coefficients $c_n$ will
depend only on time. Using the decomposition (23) and the mode decomposition (27) we
write the symplectic potential, up to a total time derivative, as

$$\omega_i \dot{\xi}^i = \hat{p}_a \dot{\alpha}_a + \sum_{n=N+1}^\infty \int (c_n f_n^\dagger K \dot{\eta} + \text{h.c.}) - \dot{\phi}^a \chi_a - \chi^a \chi^a \dot{\phi}_a^\dagger. \quad \text{(28)}$$

where the following definitions have been used:

$$\hat{p}_a = \tilde{p}_a - \left\{ \frac{1}{2} \left[ p_b - \int (\tilde{\pi} \partial_b \eta + \text{h.c.}) \right] (\mu + \xi)^{bc} \int \partial_c \partial_c \phi^\dagger K \eta + \text{h.c.} \right\},$$

$$\chi_a = \int \partial_a \phi^\dagger K \eta,$$

$$v_a = \frac{1}{2} \left[ p_b - \int (\tilde{\pi} \partial_b \eta + \text{h.c.}) \right] (\mu + \xi)^{ba}. \quad \text{(29)}$$

We can also decompose the field $\eta$ in the base $\mathcal{B}$. At the same time we observe that the
symplectic potential is not in canonical form due to the last two terms in (28). This can
be solved by assuming $\chi_a = 0$. In Dirac’s terminology this is a second-class constraint that corresponds to a “gauge condition”. In this terminology, we are choosing a gauge where $\eta$ is orthogonal to the vectors $\partial_a \phi$:

$$\eta(x, t) = \sum_{n=N+1}^{\infty} q_n(t) f_n(x). \quad (30)$$

where the explicit time dependence lies in the coefficients $q_n$ and the dependence on the collective coordinates is in the functions $f_n$, as in the decomposition of $\tilde{\pi}$. This choice is sometimes called “rigid gauge” [8]. Using the orthonormality of the elements of the base $B$, the symplectic potential is now

$$\omega_i \dot{\xi}^i = \hat{p}_a \dot{\alpha}^a + \sum_{n=N+1}^{\infty} (c_n \dot{q}_n + \text{h.c.}) + \int (\tilde{\pi} \partial_a \eta + \text{h.c.}) \dot{\alpha}_a. \quad (31)$$

It is now clear that a further redefinition of the momentum will render the symplectic potential in Darboux form. The new momentum is

$$\bar{p}_a = \hat{p}_a + \int (\tilde{\pi} \partial_a \eta + \text{h.c.}), \quad (32)$$

which leads to the final form of the symplectic potential:

$$\omega_i \dot{\xi}^i = \bar{p}_a \dot{\alpha}^a + \sum_{n=N+1}^{\infty} (c_n \dot{q}_n + \text{h.c.}). \quad (33)$$

We wish to remark here that the orthonormality of the elements of the base $B$ makes irrelevant the dependence of $\eta$ on the collective coordinates. It is the normal modes $c_n, q_n$ that act as coordinates and momenta, and can always be taken independent of the $\alpha^a$.

### 4.2 Gauge Invariance

The decomposition (30), that corresponds to $\chi_a = 0$, is not the most general way to put the symplectic potential $\omega$ in canonical form. Here we shall show that there is a large class of “gauges” beyond the rigid one that lead to the same $\omega$.

In the discussion leading to (33) our choice was to take $\eta$ orthogonal to the zero nodes $\partial_a \phi$. This choice was motivated by the decomposition of $\pi$ into transversal and longitudinal components with respect to the $\partial_a \phi$. However, as shown in [8], the orthogonality of $\eta$ can be defined with respect to any set of linearly independent vectors $g_a \in T(M)$ with $a = 1, \ldots, N$. In order to see this possibility we must solve again the constraint (7) by means of a more general decomposition of $\pi$.

The vectors $g_a$ can be used to complete a basis $B' = \{g_a; g_i\}$ with $i > N$ and $(g_a, g_i) = 0$. This new basis is not simply a rotation of the original basis $B$; $g_a$ and $\partial_a \phi$ may have different functional form. Let us define now the matrix

$$\Lambda_{ab} = \int g_a^\dagger K \partial_b (\phi + \eta). \quad (34)$$
We need this matrix to be non-singular. Then we denote its inverse by $\Lambda^{ab}$ and write the solution of the constraint (7) as

$$\pi = \tilde{\pi}' + \frac{1}{2} \left[ p_a - \int (\tilde{\pi} \partial_a (\phi + \eta)) + \text{h.c.} \right] \Lambda^{ab} g_b^\dagger,$$

$$\tilde{\pi}' = \sum_{i=N+1}^{\infty} a_i(t) g_i^\dagger K.$$ (35)

The new symplectic potential will be in Darboux form if $\eta$ is constrained to be orthogonal to the $g_a$. This is achieved by means of a new mode decomposition of $\eta$ which should exclude the $g_a$. Thus we have verified that it is possible to define transversality with respect to a general set of $N$ vectors $g_a \in T(M)$ and preserve the canonical structure of the theory at the same time.

An even more general choice would be to allow for a constant longitudinal component of $\eta$, not necessarily zero. In our language this corresponds to the weaker constraint

$$\chi_a = K(x) \quad \text{with} \quad K'(x) = 0.$$ (36)

If this is possible, the symplectic potential will be in Darboux form up to total time derivatives, which can be dropped. In terms of a mode decomposition, (36) amounts to the following condition on the longitudinal modes of the meson field $\eta$:

$$\frac{d}{dt} [g_a(t) \| \partial_a \phi \|] = 0 \quad \text{or} \quad \frac{\dot{g}_a}{g_a} = -\frac{1}{\| \partial_a \phi \|} \frac{d}{dt} \| \partial_a \phi \|. \quad (37)$$

If we insist that the coefficients $g_a(t)$ of the mode decomposition depend explicitly on $t$ and not on $\alpha^a$ we must conclude that the norms $\| \partial_a \phi \|$ must be independent of the collective coordinates $\alpha^a$. In the more general basis $B'$ the condition is that the norms $\| g_a \|$, $a = 1, \ldots, N$ should be independent of time. If this cannot be satisfied in a particular model, we must take the restricted condition (30). Most models of interest, however, do satisfy this extra condition. If the collective coordinates correspond to the center of mass of a soliton, the dependence on the $\alpha^a$ will be of the form $\phi(x^a - \alpha^a)$ and therefore the integration over the coordinates $x^a$ will eliminate the dependence on $\alpha^a$ in $\| \partial_a \phi \|$. This also happens if $M$ is an abelian group manifold; the collective coordinates appear as parameters in the exponential map of the group and cancel in $\| \partial_a \phi \|$. When the weak condition (36) can be taken the final symplectic potential $\omega$ will be equivalent to the symplectic potential arising from the strong constraint $\chi_a = 0$, since the difference amounts to a total time derivative.

### 4.3 Poisson Brackets

After eliminating the unphysical degrees of freedom introduced by the decomposition (4), the phase space is spanned by $(\bar{p}_a, \alpha^a, c_m, d_n, c'_m, d'_m)$. The explicit form of the symplectic matrix will depend on the geometry of the phase space, which we have not specified. More precisely, we need to know the algebra of vector fields on $\Gamma$. This is where particular cases must be considered separately. A very general situation, an one of great interest, is that of
a configuration space admitting a transitive non-abelian group $G$ of transformations. The symplectic matrix $\Omega$ is easily found to be

$$\Omega_{ij} = \begin{pmatrix}
0 & I_{ac} & 0 & 0 \\
-I_{db} & f_{ab}^c \bar{p}_c & 0 & 0 \\
0 & 0 & 0 & I_{mn} \\
0 & 0 & -I_{pq} & 0
\end{pmatrix}. \tag{38}$$

The notations $I$ and $\mathcal{I}$ stand, respectively, for the $N \times N$ unit matrix and for the $\aleph_0 \times \aleph_0$ unit matrix. The inverse of $\Omega$ exists and determines the fundamental Poisson brackets between phase space coordinates:

$$\begin{align*}
\{\alpha^a, \alpha^b\} &= 0, \\
\{\alpha^a, \bar{p}_b\} &= \delta^a_b, \\
\{\bar{p}_a, \bar{p}_b\} &= f_{ab}^c \bar{p}_c \\
\{q_n, c_m\} &= \mathcal{I}_{mn}, \\
\{q_n, q_m\} &= \{c_n, c_m\} = 0. \\
\{\bar{p}_a, c_n\} &= 0 \\
\{\bar{p}_a, q_n\} &= 0
\end{align*} \tag{39}$$

Canonical Quantisation of this reduced dynamical system would proceed along the usual lines. Thus the Faddeev-Jackiw method applied to the quantisation of non-linear waves reproduces the usual decomposition in longitudinal and transverse modes with respect to the background configuration $\phi[x; \alpha(t)]$ that appears in [1, 7], although our discussion has been more general. In the next section we shall apply a different version on the Faddeev-Jackiw method applied to the same problem.

## 5 Symplectic analysis

Let us reconsider the general constraint equation (15). In order to solve these constraints we must be able to construct the Darboux transformation to canonical coordinates. If that is possible, we still have to solve explicitly the new constraints (16), which means that some variables have to be written in terms of a reduced set of coordinates.

When that direct approach is not feasible we can resort to Dirac’s procedure, or adopt the method proposed in [11, 12]. The main idea in this method is to include the “unsolvable” constraints, that we shall call $C_k$, into the lagrangean by means of Lagrange multipliers that are velocities. This has two effects on the theory:

1. The constraints are now part of a new, enlarged symplectic potential. We are then introducing a new symplectic matrix that, if all constraints are taken into account, will be regular.

2. The Poisson brackets defined in this enlarged phase space are Dirac brackets [11, 12, 13].
3. We are shifting the constraints to the tangent space of the phase space of the dynamical system. In other words, the Lagrange equations of motion of the Lagrange multipliers ensure the stability of the constraints under time evolution:

\[ \dot{C}_k = 0. \]  

(41)

We should also impose the initial condition \( C|_{t=0} = 0 \) in order to make equivalent the dynamics of the new system to that of the system before introducing velocity Lagrange multipliers. It is to be noted, however, that in the original Faddeev-Jackiw method one is solving the constraints without requiring that they should be preserved under time evolution. In all known cases it is not necessary to introduce that stability condition by hand; the dynamical system, if consistent, preserves automatically the constraints under time evolution [16]. The question of under what conditions that is true deserves further examination but will not be addressed here.

We shall, in this section, apply this idea of enlarging (rather than reducing) the symplectic potential with velocity Lagrange multipliers to the collective coordinates approach to NLWs. In the previous section the constraint (24) was solved by choosing a particular basis \( B \) of functions. If we do not want to choose any basis, or if we do not know how to define it in a specific situation, we can still apply the modified procedure just described to quantise NLWs. The new, enlarged symplectic potential, denoted by \( \omega' \), is

\[ \omega' = \hat{p}_a \dot{\alpha}_a + \left[ \int \tilde{\pi} \dot{\eta} + \chi_a \dot{\lambda}^a + \psi_a \dot{\sigma}^a + \text{h.c.} \right], \]

(42)

where in the Lagrange multiplier \( \dot{\lambda}_a \) has absorbed the \( -\dot{v}_a \) present in (28). The new variables \( \sigma \) and \( \lambda \) have transferred the constraints to the symplectic potential. We can now verify that the new extended symplectic matrix is regular, and then define Poisson brackets between the coordinates of the new phase space without having to resort to Dirac’s method. The new symplectic matrix is

\[
\Omega_{ij} = \begin{pmatrix}
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-I & -i f_{abc} \hat{p}_c & 0 & 0 & 0 & \Xi^T & \Delta & \Xi & \Delta^T \\
0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & \partial \phi & 0 \\
0 & 0 & -\delta & 0 & 0 & 0 & \partial \phi^T K & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & \partial \phi^T \\
0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 & 0 \\
0 & -\Xi & 0 & -\partial \phi^T K & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\Delta^T & -\partial \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\Xi^* & 0 & 0 & 0 & 0 & -K \partial \phi & 0 & 0 & 0 \\
0 & -\Delta^* & 0 & 0 & 0 & -\partial \phi^* & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(43)

where we have eliminated subindexes (as in \( \partial \eta \)) in order to simplify the notation, and \( \Delta \) and \( \Xi \) are \( N \times N \) matrices with elements

\[
\Delta_{ab} = \int (\tilde{\pi} \partial_a \phi + \partial_a \tilde{\pi} \partial_b \phi) ,
\]

\[
\Xi_{ab} = \xi_{ab} + \int \partial_b \partial_a \phi^T K \eta .
\]

(44)
As announced, the extended $\Omega_{ij}$ is non-singular. The lines separate the contributions from the physical variables $\tilde{p}, \alpha, \tilde{\pi}$ and $\eta$ from the contributions due to the constraints and Lagrange multipliers. The inverse of $\Omega_{ij}$ provides the basic Poisson brackets between the canonical variables of $L'$,

$$\{\hat{p}_a, \hat{p}_b\}_D = -i f_{ab} \hat{c}_c + \left[ \Delta \mu^{-1} T \Xi - \Xi^T \mu^{-1} \Delta^T + \Delta^* \mu^{-1} T \Xi^* - \Xi^T \mu^{-1} \Delta^T \right]_{ab},$$

$$\{\alpha^b, \hat{p}_a\}_D = \delta_a^b,$$

$$\{\hat{p}_a, \tilde{\pi}\}_D = -\partial_a \tilde{\pi} + \Delta_{ab} \mu^{cb} \partial_c \phi \hat{K},$$

$$\{\hat{p}_a, \eta\}_D = -\partial_a \eta + \Xi_{ba} \mu^{cb} \partial_c \phi,$$

$$\{\hat{p}_a, \sigma^b\}_D = -\Xi_{ca} \mu^{cb},$$

$$\{\hat{p}_a, \lambda^b\}_D = \Delta_{ac} \mu^{cb},$$

$$\{\tilde{\pi}(x), \eta(y)\}_D = -\delta(x - y) + \partial_a \phi(x) \hat{K} \mu^{ab} \partial_b \phi(y),$$

$$\{\tilde{\pi}(x), \sigma_a\}_D = -\partial_a \phi(x) \hat{K} \mu^{ba},$$

$$\{\eta(x), \lambda^a\}_D = -\mu^{ab} \partial_b \phi,$$

$$\{\lambda^a, \sigma^b\}_D = -\mu^{ab}. \tag{45}$$

The subindex $D$ indicates that these are Dirac brackets. The rest of the Poisson brackets are zero or can be obtained by hermitian conjugation. It is easy to check that the constraints $\xi_a$ and $\psi_a$ have vanishing brackets with all the variables, which ensures that the constraints do not evolve in time.

6 Conclusions

It was the purpose of this article to apply the modern approach to constrained systems pioneered by Faddeev and Jackiw [9, 10] to non-linear waves. The alternative method due to Wotzasek, Montani and Barcelos-Neto [11, 12, 13] has also been applied to the same problem.

The original Faddeev-Jackiw method leads to a reduced phase space with only the “true” degrees of freedom. Moreover, after a Darboux transformation the symplectic form, and therefore the Poisson brackets, will be canonical. Use of this approach leads to the necessity of introducing a formal mode decomposition of the meson $\eta$ and the canonical momentum $\tilde{\pi}$. The final structure of the dynamical system is a generalisation of the Christ-Lee version of the collective coordinate formalism [2].

In the modified procedure we do not solve the constraints; these are absorbed into the symplectic potential by means of velocity Lagrange multipliers. The total phase space must now include the Lagrange multipliers. The new, expanded symplectic structure will be regular if all constraints are taken into account. The inverse of the symplectic matrix determines the Poisson (Dirac) structure of the phase space. When applied to the collective coordinate analysis of NLWs, we find results that reduce to those of [6, 7] when the meson $\eta$ does not depend on the collective coordinates and the configuration space has a transitive group of transformations that is abelian.

From a practical point of view it is clear that the simplest approach is the original Faddeev-Jackiw Hamiltonian Reduction in the sense that it leads to fewer degrees of freedom,
which furthermore are the physical ones. Its only disadvantage is that it relies on a mode
decomposition of the meson field and its canonical momentum. This approach will therefore
loose manifest covariance in field space. It may also happen that the elimination of the
unphysical longitudinal degrees of freedom obscures some symmetries of the system. The
symplectic approach is much less suitable for practical calculations but does not require a
particular choice of basis to decompose the meson field. It is therefore more useful when we
want to investigate general properties of the dynamics of small fluctuations around NLWs,
like symmetries and global aspects of the phase space.

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